CSE541 Practice Midterm 1 SOLUTIONS

QUESTION 1

(1) Prove the following

Dedekind Theorem A set A is INFINITE iff there is a set proper subset B of the set A such that $|A| = |B|$.

Solution Exercise0 solutions

(2) Find a function f that is $1-1$ and maps closed interval $[1, 2]$ *onto* open interval $(1, 2)$.

Solution Look at Exercise0 solutions for general case.

Here take for example (might be different!) two $1-1$ sequences $a_n = \{1 + \frac{1}{n+1}\}_{n\geq 2}$ and $b_n = \{\ 2 - \frac{1}{n+1}\}_{n\geq 2}$ of elements of $[1, 2]$. Observe that the condition $n \geq 2$ guarantee that sequences a_n and b_n are disjoint, i.e $a_n \neq b_n$ for all $n \geq 2$ and f defined below is $1-1$.

The required function is

$$f(1) = a_2, \quad f(a_n) = a_{n+1}, \quad \text{all} \quad n \geq 2,$$

$$f(2) = b_2 \quad f(b_n) = b_{n+1} \quad \text{all} \quad n \geq 2,$$

$$\text{and} \quad f(x) = x \quad \text{for} \quad \text{all} \quad \text{other} \quad x \in [1, 2].$$

QUESTION 2 Let S be the following set of formulas

$$S = \{(\neg a \Rightarrow (a \cup b)), (a \cup \neg b), (a \Rightarrow (\neg b \Rightarrow a)), ((a \cap b) \Rightarrow b)\}$$

(1) Write a formal propositional language to which all formulas of S belongs, i.e. a language determined by S.

Solution $\mathcal{L}_{\{\Rightarrow,\neg,\cup,\cap\}}$

Definition We say that a set S of formulas **has a model under semantics** M iff there a variable assignment v, such that

$$v \models_M A \quad \text{for} \quad \text{all} \quad \text{formulas} \quad A \in S.$$

We denote it

$$v \models_M S).$$

2. Determine if S has a classical model. Use the shorthand notation.

Solution Observe that $\models (a \Rightarrow (\neg b \Rightarrow a))$ and $\models ((a \cap b) \Rightarrow b)$, so the model is determined by the 2 other formulas. We need to find v such that (shorthand) $(\neg a \Rightarrow (a \cup b)) = T$ and $(a \cup \neg b) = T$. Take $a = T$, then $(a \cup \neg b) = T$ for any logical value of b, and $\neg T \Rightarrow (a \cup b)) = T$ for any logical value of b. We have found two (restricted) MODELS: $v(a) = T, v(b) = T$ and $v(a) = T, v(b) = F$.

(4) Determine whether S has an **H** model. Use a shorthand notation.

Reminder: We define **H** semantics operations $\cup$ and $\cap$ as follows

$$a \cup b = max\{a, b\}, \quad a \cap b = min\{a, b\}.$$

The Truth Tables for Implication and Negation are:

H-Implication

$\Rightarrow$	F	$\perp$	T
F	T	T	T
$\perp$	F	T	T
T	F	$\perp$	T

H Negation

$\neg$	F	$\perp$	T
	T	F	F

Solution Observe that TT for H semantics restricted to $\{T, F\}$ are identical with classical Truth Tables. The Models from previous question:$v(a) = T, v(b) = T$ and $v(a) = T, v(b) = F$ are also H- models.

QUESTION 3 Let H be the following proof system:

$$H = (\ \mathcal{L}_{\{\Rightarrow, \neg\}},\ \mathcal{F},\ AX = \{A1, A2, A3, A4\},\ \ MP\ )$$

A1 $(A \Rightarrow (B \Rightarrow A))$,

A2 $((A \Rightarrow (B \Rightarrow C)) \Rightarrow ((A \Rightarrow B) \Rightarrow (A \Rightarrow C)))$,

A3 $((\neg B \Rightarrow \neg A) \Rightarrow ((\neg B \Rightarrow A) \Rightarrow B)))$

A4 $(\neg A \Rightarrow (A \Rightarrow B))$

MP (Rule of inference)

$$(MP)\ \frac{A\ ;\ (A \Rightarrow B)}{B}$$

(1) Justify that H is SOUND under classical semantics.

Solution All axioms are basic tautologies.

(2) Does Deduction Theorem holds for H? Justify shortly your answer.

Solution Yes. We needed only two first axioms to prove it.

(3) Justify the fact that H is COMPLETE with respect to all classical semantics tautologies.

Solution The system H ia an extension of a complete system H_2 as the first 3 axioms of H are axioms of H_2. Observe that By the Completeness of H_2 the axiom $A4$ is provable form the first 3 axioms.

(4) All classical tautologies include for example de Morgan Laws

$$(\neg(A \cup B) \Rightarrow (\neg A \cap \neg B)), \quad (\neg(A \cap B) \Rightarrow (\neg A \cup \neg B))$$

Explain what does it mean that they are provable in H.

Solution We proved that $\mathcal{L}_{\{\Rightarrow,\neg\}} \equiv \mathcal{L}_{\{\Rightarrow,\neg,\cup,\cap\}}$ so the Morgan Laws (as any other formula) expressed in the language $\mathcal{L}_{\{\Rightarrow,\neg\}}$ as an equivalent formula, and then proved.

(5) Prove that the system H in NOT COMPLETE under the Heyting semantics **H** restricted to the language $\mathcal{L}_{\{\Rightarrow,\neg\}}$.

Solution System H is not SOUND under H semantics. For example Axiom A3 has a counter-model $A = B = \perp$.

QUESTION 4 Prove that the following Lemma holds for H. Give the full proof; use Deduction Theorem.

LEMMA For any $A, B, C \in \mathcal{F}$,

(a) $(A \Rightarrow B), (B \Rightarrow C) \vdash_H (A \Rightarrow C)$,

(b) $(A \Rightarrow (B \Rightarrow C)) \vdash_H (B \Rightarrow (A \Rightarrow C))$.

Proof of (a): $(A \Rightarrow B), (B \Rightarrow C) \vdash_H (A \Rightarrow C)$ is equivalent by Deduction theorem applied twice to $(A \Rightarrow B), (B \Rightarrow C), A \vdash_H (C$. Here is a (formal) proof

B_1 $(A \Rightarrow B)$ (Hyp.)

B_2 A (Hyp.)

B_3 B MP to B_1, B_2

B_4 $(B \Rightarrow C)$ (Hyp.)

B_5 C MP to B_3, B_4

Proof of (b): $(A \Rightarrow (B \Rightarrow C)) \vdash_H (B \Rightarrow (A \Rightarrow C))$ is equivalent by Deduction theorem applied twice to $(A \Rightarrow (B \Rightarrow C)), B, A \vdash_H C$. Here is a proof

B_1 $(A \Rightarrow (B \Rightarrow C))$ (Hyp.)

B_2 A (Hyp.)

B_3 $(B \Rightarrow C)$ MP to B_1, B_2

B_4 B (Hyp)

B_5 C MP to B_3, B_2

QUESTION 5 Complete the proof sequence (in H)

$$B_1, ..., B_{12}$$

of the formula

$$((A \Rightarrow B) \Rightarrow ((\neg A \Rightarrow B) \Rightarrow B)).$$

by providing comments how each step of the proof was obtained.

$B_1 =$ $(A \Rightarrow B)$
Hyp

$B_2 =$ $(\neg A \Rightarrow B)$
Hyp

$B_3 =$ $((A \Rightarrow B) \Rightarrow (\neg B \Rightarrow \neg A))$
ALREADY PROVEN

$B_4 =$ $(\neg B \Rightarrow \neg A)$
B_1, B_3 and MP

$B_5 =$ $((\neg A \Rightarrow B) \Rightarrow (\neg B \Rightarrow \neg\neg A))$
Substitute $A = \neg A$ in already proven $((A \Rightarrow B) \Rightarrow (\neg B \Rightarrow \neg A))$

$B_6 =$ $(\neg B \Rightarrow \neg\neg A)$

B_2, B_5 and MP

$B_7 =$ $((\neg B \Rightarrow \neg\neg A) \Rightarrow ((\neg B \Rightarrow \neg A) \Rightarrow B)))$
A3

$B_8 = \; ((\neg B \Rightarrow \neg A) \Rightarrow B)$

B_6, B_7 and MP

$B_9 = \; B$
B_4, B_8 and MP

$B_{10} = \; (A \Rightarrow B), (\neg A \Rightarrow B) \vdash B$
$B_1 - B_9$

$B_{11} = \; (A \Rightarrow B) \vdash ((\neg A \Rightarrow B) \Rightarrow B)$
Ded. Thm

$B_{12} = \; ((A \Rightarrow B) \Rightarrow ((\neg A \Rightarrow B) \Rightarrow B))$

Ded. Thm

QUESTION 6 Proof 1 of the Completeness Theorem for the system S defines a method of efficiently combining $v \in V_A$ while constructing the proof of A that is a tautology.. Let consider the following tautology $A = A(a, b, c)$

$$((\neg a \Rightarrow b) \Rightarrow (\neg(\neg a \Rightarrow b) \Rightarrow c).$$

Write down all steps of the construction of the proof of A as described in the Proof 1.

Solution HERE is the MOST DETAILED Solution!

By the Main Lemma and the assumption that $\models A(a, b, c)$ any $v \in V_A$ defines formulas B_a , B_b, B_c such that

$$B_a, B_b, B_c \;\vdash\; A. \tag{1}$$

The proof is based on a method of using all $v \in V_A$ (there is 16 of them) to define a process of elimination of all hypothesis B_a, B_b, B_c in (1) to construct the proof of A in S i.e. to prove that $\vdash \; A$.

Step 1: elimination of B_c.

Observe that by definition, B_c is c or $\neg c$ depending on the choice of $v \in V_A$. We choose two truth assignments $v_1 \neq v_2 \in V_A$ such that

$$v_1|\{a, b\} = v_2|\{a, b\} \tag{2}$$

and $v_1(c) = T$ and $v_2(c) = F$.

Case 1: $v_1(c) = T$, by definition $B_c = c$. By the property (2), assumption that $\models A$, and the Main Lemma applied to v_1

$$B_a, B_b, c \;\vdash\; A.$$

By Deduction Theorem we have that

$$B_a, B_b \;\vdash\; (c \Rightarrow A). \tag{3}$$

Case 2: $v_2(c) = F$ hence by definition $B_c = \neg c$. By the property (2), assumption that $\models A$, and the Main Lemma applied to v_2

$$B_a, B_b, \neg c \;\vdash\; A.$$

By the Deduction Theorem we have that

$$B_a, B_b \;\vdash\; (\neg c \Rightarrow A). \tag{4}$$

By the assumed provability of the formula A8 for $A = c, B = A$ we have that

$$\vdash ((c \Rightarrow A) \Rightarrow ((\neg c \Rightarrow A) \Rightarrow A)).$$

By monotonicity we have that

$$B_a, B_b \;\vdash\; ((c \Rightarrow A) \Rightarrow ((\neg c \Rightarrow A) \Rightarrow A)). \tag{5}$$

Applying Modus Ponens twice to the above property (5) and properties (3), (4) we get that

$$B_a, B_b \;\vdash\; A. \tag{6}$$

and hence we have eliminated B_c.

Step 2: elimination of B_b from (6). We repeat the Step 1.

As before we have 2 cases to consider: $B_b = b$ or $B_b = \neg b$. We choose from V_A two truth assignments $w_1 \neq w_2 \in V_A$ such that

$$w_1|\{a\} = w_2|\{a\} = v_1|\{a\} = v_2|\{a\} \tag{7}$$

and $w_1(b) = T$ and $w_2(b) = F$.

Case 1: $w_1(b) = T$, by definition $B_b = b$. By the property (7), assumption that $\models A$, and the Main Lemma applied to w_1

$$B_a, b \;\vdash\; A.$$

By Deduction Theorem we have that

$$B_a \;\vdash\; (b \Rightarrow A). \tag{8}$$

Case 2: $w_2(c) = F$ hence by definition $B_b = \neg b$. By the property (7), assumption that $\models A$, and the Main Lemma applied to w_2

$$B_a, \neg b \ \vdash \ A.$$

By the Deduction Theorem we have that

$$B_a \ \vdash \ (\neg b \Rightarrow A). \tag{9}$$

By the assumed provability of the formula A8 for $A = b, B = A$ we have that

$$\vdash ((b \Rightarrow A) \Rightarrow ((\neg b \Rightarrow A) \Rightarrow A)).$$

By monotonicity we have that

$$B_a \ \vdash \ ((b \Rightarrow A) \Rightarrow ((\neg b \Rightarrow A) \Rightarrow A)). \tag{10}$$

Applying Modus Ponens twice to the above property (10) and properties (8), (9) we get that

$$B_a \ \vdash \ A. \tag{11}$$

and hence we have eliminated B_b.

Step 3: elimination of B_a from (11). We repeat the Step 2.
As before we have 2 cases to consider: $B_a = a$ or $B_a = \neg a$. We choose from V_A two truth assignments $g_1 \neq g_2 \in V_A$ such that

$$g_1(a) = T \text{ and } g_2(a) = F. \tag{12}$$

Case 1: $g_1(a) = T$, by definition $B_a = a$. By the property (12), assumption that $\models A$, and the Main Lemma applied to g_1

$$a \ \vdash \ A.$$

By Deduction Theorem we have that

$$\vdash \ (a \Rightarrow A). \tag{13}$$

Case 2: $g_2(a) = F$ hence by definition $B_a = \neg a$. By the property (12), assumption that $\models A$, and the Main Lemma applied to g_2

$$\neg a \ \vdash \ A.$$

By the Deduction Theorem we have that

$$\vdash \ (\neg a \Rightarrow A). \tag{14}$$

By the assumed provability of the formula (24) for $A = a, B = A$ we have that

$$\vdash ((a \Rightarrow A) \Rightarrow ((\neg a \Rightarrow A) \Rightarrow A)). \tag{15}$$

Applying Modus Ponens twice to the above property (15) and properties (13), (14) we get that

$$\vdash \ A. \tag{16}$$

and hence we have eliminated B_a, B_b and B_c.

System S

Let

$$S = (\ \mathcal{L}_{\{\Rightarrow,\neg\}},\quad AX,\quad MP\ )$$

be **a sound** proof system with a set of logical axioms AX such that all **formulas listed below are provable** in S.

$$(A \Rightarrow (B \Rightarrow A)), \tag{17}$$

$$((A \Rightarrow (B \Rightarrow C)) \Rightarrow ((A \Rightarrow B) \Rightarrow (A \Rightarrow C))), \tag{18}$$

$$((\neg B \Rightarrow \neg A) \Rightarrow ((\neg B \Rightarrow A) \Rightarrow B)), \tag{19}$$

$$(A \Rightarrow A), \tag{20}$$

$$(B \Rightarrow \neg\neg B), \tag{21}$$

$$(\neg A \Rightarrow (A \Rightarrow B)), \tag{22}$$

$$(A \Rightarrow (\neg B \Rightarrow \neg(A \Rightarrow B))), \tag{23}$$

$$((A \Rightarrow B) \Rightarrow ((\neg A \Rightarrow B) \Rightarrow B)), \tag{24}$$

$$((\neg A \Rightarrow A) \Rightarrow A), \tag{25}$$