CSE541 MAKE-UP Midterm 1 SOLUTIONS
Spring 2011

QUESTION 1 Prove that any infinite set contains a countably infinite subset.

Solution We first prove that if A is infinite, then we can define a 1-1 sequence of elements of A. All elements of this sequence form a required countably infinite subset of A. The construction of such sequence was a first STEP in the proof of the Dedekind Theorem.

Let A be infinite, We define a sequence

$$a_1, \dots, a_n, \dots$$

as follows.
1. Observe that $A \neq \emptyset$,because if $A = \emptyset$, A would be finite. contradiction. So there is an element of $a \in A$. We define

$$a_1 = a$$

2. Consider a set $A - \{a_a\} = A_1$. $A_1 \neq \emptyset$ because if $A = \emptyset$, then $A - \{a_1\} = \emptyset$ and A is finite. Contradiction. So there is an element $a_2 \in A - \{a_1\}$ and $a_1 \neq a_2$.
We hence have defined

$$a_1, a_2.$$

Assume now that we have defined

$$a_1, a_2, \dots, a_n \quad \text{for} \quad a_1 \neq a_2 \neq \dots \neq a_n.$$

Consider a set $A_n = A - \{a_1, \dots, a_n\}$.
The set $A_n \neq \emptyset$ because if $A - \{a_1, \dots, a_n\} = \emptyset$, then A is finite. Contradiction. So there is an element a_{n+1} such that

$$a_{n+1} \in A - \{a_1, \dots, a_n\} \text{ and } a_{n+1} \neq a_n \neq \dots \neq a_1.$$

By mathematical induction, the element a_n is defined for all $n \in N$ and we have defined a 1-1 sequence

$$a_1, a_2, \dots, a_n, \dots$$

elements of A.

The set

$$B = \{a_1, a_2, \dots, a_n, \dots\}$$

is a countably infinite subset of A and we have proved its existence.

QUESTION 2 H is the following proof system:

$$H = (\ \mathcal{L}_{\{\Rightarrow,\neg\}},\ \mathcal{F},\ AX = \{A1, A2, A3\},\quad MP\ )$$

A1 $(A \Rightarrow (B \Rightarrow A))$,

A2 $((A \Rightarrow (B \Rightarrow C)) \Rightarrow ((A \Rightarrow B) \Rightarrow (A \Rightarrow C)))$,

A3 $((\neg B \Rightarrow \neg A) \Rightarrow ((\neg B \Rightarrow A) \Rightarrow B)))$

A4 $(((A \Rightarrow B) \Rightarrow A) \Rightarrow A)$

MP (Rule of inference)

$$(MP)\ \frac{A\ ;\ (A \Rightarrow B)}{B}$$

1 Justify whether H is complete or not.

Solution Yes, it is a complete proof system. It is SOUND because a new axiom $A4$ added to a sound and complete system H_2 with axioms $A1 - A3$ is SOUND i.e is a classical tautology (must verify!- it is not a basic tautology).

2 Prove that

$$(A \Rightarrow B), (B \Rightarrow C) \vdash_H (A \Rightarrow C)$$

Solution Observe that the Deduction Theorem holds fir the systen=m H as it contains $A1, A@$ needed for its proof. Applying Deduction Theorem we get

$$(A \Rightarrow B), (B \Rightarrow C) \vdash_H (A \Rightarrow C) \text{ iff } A \Rightarrow B), (B \Rightarrow C), A \vdash_H C.$$

The following is a proof of C from $A \Rightarrow B), (B \Rightarrow C), A$.

$B_1 =$ $(A \Rightarrow B)$, (hyp)

$B_2 =$ $(B \Rightarrow C)$, (hyp)

$B_3 =$ A, (hyp)

$B_4 =$ B, B_1, B_3 and MP

$B_4 =$ C. B_2, B_4 and MP

3. Here are consecutive steps $B_1, ..., B_5$ in a proof of $(B \Rightarrow \neg\neg B)$ in H.

Complete the steps

$$B_1, ..., B_5$$

of the proof by writing all details in the space provided below each step of the proof.

You have to write down **the proper substitutions and formulas** used at each step of the proof.

$B_1 = ((\neg\neg\neg B \Rightarrow \neg B) \Rightarrow ((\neg\neg\neg B \Rightarrow B) \Rightarrow \neg\neg B)), \quad A3$

$B_2 = (\neg\neg\neg B \Rightarrow \neg B)$
Already proved fact

$B_3 = ((\neg\neg\neg B \Rightarrow B) \Rightarrow \neg\neg B), \quad B_1, B_2$ and MP

$B_4 = (B \Rightarrow (\neg\neg\neg B \Rightarrow B)), \quad A1$

$B_5 = (B \Rightarrow \neg\neg B)$
$(A \Rightarrow B), (B \Rightarrow C) \vdash_H (A \Rightarrow C)$ applied to B_4, B_3.

QUESTION 3 Consider a propositional language $\mathcal{L}_{\{\neg,\cup,\cap,\Rightarrow\}}$ with a set $\mathcal{F}$ of formulas.

Let $\mathbf{T} \subseteq \mathcal{F}$ be the set of all propositional TAUTOLOGIES under the classical semantics.

Let S be a **COMPLETE Hilbert proof system** with for a classical propositional logic with the language $\mathcal{L}_{\{\neg,\cup,\cap,\Rightarrow\}}$, i.e.

$$\mathbf{T} = \{A \in \mathcal{F} : \vdash_S A\}.$$

Prove that for any $A, B \in F$,

$$Cn(\{A\}) \cap Cn(\{B\}) = Cn(\{(A \cup B)\}),$$

where for any Let $X \subseteq F$ we define

$$Cn(X) = \{A \in F : \ X \vdash_S A\}.$$

Solution

Observe that the system S is Hilbert system i.e. it must have a MP is its rule of inference. It is also complete. We have the following fact.

Fact: Deduction Theorem holds for S.

Proof of the Fact: observe that the two axioms

$$A1: \ (A \Rightarrow (B \Rightarrow A)),$$

$$A2: \quad ((A \Rightarrow (B \Rightarrow C)) \Rightarrow ((A \Rightarrow B) \Rightarrow (A \Rightarrow C)))$$

that are needed to prove Deduction Theorem are classical tautologies, hence by completeness of S, they are provable (by means of MP) in S. Consequently the proof of Deduction Theorem for our system H_1 holds for the system S.

Assume that $C \in Cn(\{A\})$ and $C \in Cn(\{B\})$, we want to prove that $C \in Cn(\{(A \cup B)\})$.

By definition and **Deduction Theorem**, we know (assume) that $\vdash_S (A \Rightarrow C)$ and $\vdash_S (B \Rightarrow C)$, and we want to prove $(A \cup B) \vdash_S C$, i.e. that (by Deduction Theorem) $\vdash_S ((A \cup B) \Rightarrow C)$. We know that

$$\models (((A \Rightarrow C) \Rightarrow ((B \Rightarrow C)) \Rightarrow ((A \cup B) \Rightarrow C))).$$

By the fact that S is complete we have that

$$\vdash_S (((A \Rightarrow C) \Rightarrow ((B \Rightarrow C)) \Rightarrow ((A \cup B) \Rightarrow C))).$$

S is a Hilbert system i.e. it has MP as a rule of inference. From the the assumptions $\vdash_S (A \Rightarrow C)$, $\vdash_S (B \Rightarrow C)$ and MP applied twice to the $\vdash_S (((A \Rightarrow C) \Rightarrow ((B \Rightarrow C)) \Rightarrow ((A \cup B) \Rightarrow C)))$ we get the proof of $((A \cup B) \Rightarrow C)$, what proves that $C \in Cn(\{(A \cup B)\})$.

Assume now that $C \in Cn(\{(A \cup B)\})$, i.e. $\vdash_S ((A \cup B) \Rightarrow C)$. We use above and the tautologies

$$(((A \cup B) \Rightarrow C) \Rightarrow (A \Rightarrow C)),\ (((A \cup B) \Rightarrow C) \Rightarrow (B \Rightarrow C))$$

to get a proofs of $(A \Rightarrow C)$ and $(B \Rightarrow C)$, and hence we get that $C \in Cn(\{A\})$ and $C \in Cn(\{B\})$, what proves that $C \in (Cn(\{A\}) \cap Cn(\{B\})$.

QUESTION 4 Let S be the proof system from Question 3.

We define **two binary relations** on $\mathcal{F}$ as follows. For any $A, B \in \mathcal{F}$,

$A \leq_S B$ if and only if $\vdash_S (A \Rightarrow B)$ and

$A \leq_{\mathbf{T}} B$ if and only if $\models (A \Rightarrow B)$.

1. PROVE that $\leq_S = \leq_{\mathbf{T}}$.

SOLUTION For any $A, B \in \mathcal{F}$, $(A, B) \in \vdash_S$ iff (by definition of $\vdash_S$) $\vdash_S (A \Rightarrow B)$ iff (by completeness theorem for S) $\models (A \Rightarrow B)$ iff (by definition of $\leq_{\mathbf{T}}$ $(A, B) \in \leq_{\mathbf{T}}$.

2. Prove that $\leq = \leq_S = \leq_{\mathbf{T}}$ is a quasi order relation (reflexive and transitive).

SOLUTION From PART 1 we have that $\leq_S = \leq_{\mathbf{T}}$, so we can chose to carry the proof for any of the two relations $\leq_S$ or $\leq_{\mathbf{T}}$. I choose here $\leq_{\mathbf{T}}$.

$\leq_{\mathbf{T}}$ is reflexive because $\models (A \Rightarrow A)$, hence by definition, $A \leq_{\mathbf{T}} A$.

I you choose $\leq_S$, the proof for it goes as follows.

We know that $\models (A \Rightarrow A)$ and S is complete, hence $\vdash_S (A \Rightarrow A)$, i.e. $A \leq_S A$.

Now we are going to prove transitivity of $\leq_{\mathbf{T}}$ only. In order to prove it we need the following Lemma (we will use it later as well).

LEMMA (Modus Ponens for tautologies)

For any $A, B \in \mathcal{F}$,

if $\models A$ and $\models (A \Rightarrow B)$, then $\models B$.

PROOF (by contradiction). Assume that $\models A$ and $\models (A \Rightarrow B)$ and $\not\models B$. It means that there is a truth assignment v, such that $v(B) = F$. $\models A$ means that for all $v, v(A) = T$ and $(v(A) \Rightarrow v(B)) = (T \Rightarrow v(B)) = T$ if and only if $v(B) = T$. Contradiction.

Proof of TRANSITIVITY of $\leq_{\mathbf{T}}$. Assume $A \leq_{\mathbf{T}} B$ and $B \leq_{\mathbf{T}} C$. This means that $\models (A \Rightarrow B)$ and $\models (B \Rightarrow C)$.

BY LEMMA applied to $\models (A \Rightarrow B)$ and $\models ((A \Rightarrow B) \Rightarrow ((B \Rightarrow C) \Rightarrow (A \Rightarrow C)))$ we get that $\models ((B \Rightarrow C) \Rightarrow (A \Rightarrow C))$. Applying again the LEMMA to $\models (B \Rightarrow C)$ and $\models ((B \Rightarrow C) \Rightarrow (A \Rightarrow C))$ we get that $\models (A \Rightarrow C)$, what ends the proof.

Observe that the proof for $\leq_S$ uses the above tautology completeness of S, the above tautology and MP.

3. Define a non-classical logic semantics M for $\mathcal{L}_{\{\neg,\cup,\cap,\Rightarrow\}}$ such that the binary relation $\leq_M$ on $\mathcal{F}$ defined as

$$A \leq_M B \quad \text{if and only if} \quad \models_M (A \Rightarrow B)$$

is NOT a quasi order.

$\models_M A$ reads: a formula A is a tautology under semantics M.

SOLUTION For example, let M be any 2 valued semantics such that its implication is defined by the table below.

$\Rightarrow$	F	T
F	T	T
T	T	F

Obviously, $\not\models (A \Rightarrow A)$.

There are, of course many other two or many valued semantics with the same property.

EXTRA CREDIT Let $\leq = \leq_S = \leq_{\mathbf{T}}$, we define a binary relation $\approx$ on $\mathcal{F}$ as follows.

$$A \approx B \quad \text{iff} \quad A \leq B \text{ and } B \leq A.$$

1. PROVE that $\approx$ ia an equivalence relation (reflexive, symmetric and transitive).

SOLUTION Reflexivity follows from $\models (A \Rightarrow A)$. Symmetry follows directly from the definition of $\approx$. Transitivity follows simply from transitivity of $\leq$ proved in Question 4.

2. Find $[(a \cup \neg a)]$, $[(a \cap \neg a)]$ and $[a]$, where $[A]$ denotes an equivalence class of $\approx$ with a representant A.

$$[(a \cup \neg a)] = \{B : \models ((a \cup \neg a) \Rightarrow B) \ \ and \ \ \models (B \Rightarrow (a \cup \neg a))\}$$

$$= \{B : \models ((a \cup \neg a) \Rightarrow B)\} \cap \{B : \models (B \Rightarrow (a \cup \neg a))\}.$$

Observe that $\models ((a \cup \neg a) \Rightarrow B)$ iff $\models B$ and $\models (B \Rightarrow (a \cup \neg a))$ for any formula B.

Hence we get

$$[(a \cup \neg a)] = \{B : \models ((a \cup \neg a) \Rightarrow B)\} \cap \{B : \models (B \Rightarrow (a \cup \neg a))\}$$
$$= \{B : \models B\} \cap \mathcal{F} = \mathbf{T} \cap \mathcal{F} = \mathbf{T}.$$

By definition,

$$[(a \cap \neg a)] = \{B : \models ((a \cap \neg a) \Rightarrow B)\} \cap \{B : \models (B \Rightarrow (a \cap \neg a))\}.$$

Observe that $\models ((a \cap \neg a) \Rightarrow B)$ iff $(F \Rightarrow B) = T$ for all v, what is true for all $B \in \mathcal{F}$.

Similarly, $(B \Rightarrow (a \cap \neg a))$ iff $(B \Rightarrow F) = T$ for all v iff $v(B) = F$ for all v iff B is a CONTRADICTION, i.e. $B \in \mathbf{CONTR}$.

Now we evaluate

$$[(a \cap \neg a)] = \{B : \models ((a \cap \neg a) \Rightarrow B)\} \cap \{B : \models (B \Rightarrow (a \cap \neg a))\}$$

$$= \mathcal{F} \cap \mathbf{CONTR} = \mathbf{CONTR}.$$

Observe that

$$\models \ (A \Leftrightarrow B) \quad \text{iff} \quad \models (A \Rightarrow B) textrmand \quad \models (B \Rightarrow A)$$

and hence

$$A \approx B \quad \text{iff} \ \ (A \Leftrightarrow B) \quad \text{iff} \ \ A \equiv B.$$

We evaluate

$$[a] = \{B \in \mathcal{F} : \; a \equiv B\} = \{a, \neg\neg a, (a \cap a), (a \cup a), \neg\neg\neg\neg(a \cap a), ...\}.$$

We can also use the above to obtain simpler solution for previous equivalence classes, as well for description of $[A]$ for any formula A.