

Questions on Lecture 9

April 24, 2015

Question 1. *List all formulas that are assumed to be provable in S and used in the first proof (the constructive proof) of the completeness theorem.*

Solution. 1. $(A \Rightarrow \neg\neg A)$,

2. $(A \Rightarrow (B \Rightarrow A))$,

3. $(A \Rightarrow (\neg B \Rightarrow \neg(A \Rightarrow B)))$,

4. $(\neg A \Rightarrow (A \Rightarrow B))$,

5. $((A \Rightarrow B) \Rightarrow ((\neg A \Rightarrow B) \Rightarrow B))$

6. $((\neg B \Rightarrow \neg A) \Rightarrow ((\neg B \Rightarrow A) \Rightarrow B))$,

7. and finally the deduction theorem, which assumes in addition

a) $(A \Rightarrow A)$,

b) $((A \Rightarrow (B \Rightarrow C)) \Rightarrow ((A \Rightarrow B) \Rightarrow (A \Rightarrow C)))$.

□

Remark 1. *Formula 7(a) can actually be deduced from formula 2 and 7(b), i.e the axioms of proving system H_1 .*

Remark 2. *In any system, the axioms needed to prove the deduction theorem are formula 2 and 7(b).*

Question 2. Prove the completeness theorem for $\mathcal{L}_{\{\Rightarrow, \cup, \neg\}}$ by adding necessary axioms.

Solution. We define $H_3 = (\mathcal{L}_{\{\Rightarrow, \cup, \neg\}}, \mathcal{F}, A, MP)$, where A is the set of axioms and MP is the MP rule. In particular, A consists of the following formulas

1. $(A \Rightarrow \neg\neg A)$,
2. $(A \Rightarrow (B \Rightarrow A))$,
3. $(A \Rightarrow (\neg B \Rightarrow \neg(A \Rightarrow B)))$,
4. $(\neg A \Rightarrow (A \Rightarrow B))$,
5. $((A \Rightarrow B) \Rightarrow ((\neg A \Rightarrow B) \Rightarrow B))$,
6. $(A \Rightarrow A)$,
7. $((A \Rightarrow (B \Rightarrow C)) \Rightarrow ((A \Rightarrow B) \Rightarrow (A \Rightarrow C)))$,
8. $((\neg B \Rightarrow \neg A) \Rightarrow ((\neg B \Rightarrow A) \Rightarrow B))$,
9. $(\neg A \Rightarrow (B \Rightarrow (A \cup B)))$,
10. $(\neg A \Rightarrow (\neg B \Rightarrow \neg(A \cup B)))$,
11. $(A \Rightarrow (B \Rightarrow (A \cup B)))$

Note the last three axioms are newly added to those in H_2 .

To prove the completeness of H_3 , we first need to show

$$\text{If } \vdash_{H_3} A \text{ then } \models A.$$

This can be shown by proving the system is sound, i.e. all axioms are tautologies and inference rules are sound. Actually we only have to verify formula 9, 10 and 11 are sound since others have already been proven in slides.

For formula 9, given any truth assignment ν and its extension ν^* , it would falsify 9 only if $\nu^*(A_1) = F$ and $\nu^*(A_2 \Rightarrow (A_1 \cup A_2)) = F$. The later implies $\nu^*(A_2) = T$ and $\nu^*(A_1 \cup A_2) = F$, but since $\nu^*(A_2) = T$, $\nu^*(A_1 \cup A_2)$ must be true. Thus we get a contradiction, so ν cannot falsify 9. Therefore 9 is a tautology.

For formula 10, given any truth assignment ν and its extension ν^* , it would falsify 10 only if $\nu^*(A_1) = F$ and $\nu^*(\neg A_2 \Rightarrow \neg(A_1 \cup A_2)) = F$. The later implies that $\nu^*(A_2) = F$ and $\neg(A_1 \cup A_2) = F$. But since $\nu^*(A_1) = F$ and $\nu^*(A_2) = F$ we must have $\neg(A_1 \cup A_2) = T$, a contradiction. Therefore ν cannot falsify 10, i.e 10 is a tautology.

Formula 11 is similar to formula 9. It could only be false when both A and B is true and $(A \cup B)$ is false. But this cannot be true. Thus it is a tautology.

Now as in the setting of the proof of the completeness theorem for S , we define for any formula $A = A(b_1, b_2, \dots, b_n)$ its corresponding formula, where b_1, b_2, \dots, b_n are all variables appear in A , in the following way:

$$A' = \begin{cases} A, & \text{if } v^*(A) = T, \\ \neg A, & \text{if } v^*(A) = F, \end{cases} \quad (1)$$

and

$$B_i = \begin{cases} b_i, & \text{if } v^*(b_i) = T, \\ \neg b_i, & \text{if } v^*(b_i) = F. \end{cases} \quad (3)$$

Lemma 1. For any formula $A = A(b_1, b_2, \dots, b_n)$ and any truth assignment v , if A', B_1, B_2, \dots, B_n are corresponding formulas, then

$$B_1, B_2, \dots, B_n \vdash A'.$$

Proof. The proof is done by induction on the degree of formula A .

In the base case, we have $n = 0$, which means that A contains no connectives, i.e $A \in VAR$. We need to show $\models A'$. This is obvious by the definition of A , since if $v^*(A) = T$ then $A' = A$ and thus $v^*(A') = T$. If on the other hand $v^*(A) = F$, then $A' = \neg A$ and thus $v^*(A') = v^*(\neg A) = \neg v^*(A) = T$.

In the inductive step, we first assume the statement of the lemma is true for any $k < m$, and then we need to show that it also holds for $k = m$, where k is the degree of formula A . We discuss case by case according to the form of $A = A(b_1, b_2, \dots, b_n)$.

1. $A = \neg A_1$. This is the same with the proof given in lecture for H_1 . Actually In this case $A_1 = A_1(b_1, b_2, \dots, b_n)$ and if we denote the degree of any formula A by $deg(A)$, then we have $deg(A_1) = m - 1$. By induction hypothesis,

$$B_1, B_2, \dots, B_n \vdash A'_1 \quad (5)$$

where

$$A'_1 = \begin{cases} A_1, & \text{if } v^*(A_1) = T, \\ \neg A_1, & \text{if } v^*(A_1) = F. \end{cases} \quad (6)$$

In the first case (6) we have $v^*(A) = \neg v^*(A_1) = F$ and so

$$A' = \neg A = \neg \neg A_1 \quad \text{and} \quad A'_1 = A_1. \quad (8)$$

Thus by (5) we have

$$B_1, B_2, \dots, B_n \vdash A_1 \quad (9)$$

Recall that in our system we have the axiom

$$\models (A \Rightarrow \neg\neg A)$$

where $A \in \mathcal{F}$. Replace A by A_1 and using the monotonicity property we have

$$B_1, B_2, \dots, B_n \vdash (A_1 \Rightarrow \neg\neg A_1). \quad (10)$$

According to (8) and (10) we have

$$B_1, B_2, \dots, B_n \vdash (A_1 \Rightarrow \neg A), \quad (11)$$

and together with (9) we have by MP rule

$$B_1, B_2, \dots, B_n \vdash \neg A. \quad (12)$$

Therefore

$$B_1, B_2, \dots, B_n \vdash A' \quad (13)$$

since $A' = \neg A$.

In the other case (7) we have $v^*(A) = \neg v^*(A_1) = T$ so

$$A' = A = \neg A_1, \quad (14)$$

and

$$B_1, B_2, \dots, B_n \vdash \neg A_1 \quad (15)$$

by (5). Therefore we have

$$B_1, B_2, \dots, B_n \vdash A \quad (16)$$

and so

$$B_1, B_2, \dots, B_n \vdash A' \quad (17)$$

2. $A = (A_1 \Rightarrow A_2)$. This case is also the same to the proof for S given in the lecture. So I'll omit it.
3. $A = (A_1 \cup A_2)$, where $A = A(b_1, b_2, \dots, b_n)$, $A_1 = A_1(c_1, c_2, \dots, c_l)$, $A_2 = A_2(d_1, d_2, \dots, d_m)$ and $\{b_1, b_2, \dots, b_n\} = \{c_1, c_2, \dots, c_l\} \cup \{d_1, d_2, \dots, d_m\}$. Note that neither $\{c_1, c_2, \dots, c_l\}$ nor $\{d_1, d_2, \dots, d_m\}$ can be empty. Since both $\text{deg}(A_1)$ and $\text{deg}(A_2)$ is less than m , we have by induction hypothesis

$$\begin{aligned} C_1, C_2, \dots, C_l &\vdash A'_1 \\ D_1, D_2, \dots, D_m &\vdash A'_2. \end{aligned} \quad (18)$$

We need to show $C_1, C_2, \dots, C_l, D_1, D_2, \dots, D_m \vdash A'$. We discuss case by case according to the truthfulness of A_1 and A_2 .

- $v^*(A_1) = T$ and $v^*(A_2) = T$. Then $v^*(A) = V^*(A_1 \cup A_2) = V^*(A_1) \cup v^*(A_2) = T$. So $A'_1 = A_1$, $A'_2 = A_2$ and $A' = A = (A_1 \cup A_2)$. In this case (18) implies

$$\begin{aligned} C_1, C_2, \dots, C_l &\vdash A_1 \\ D_1, D_2, \dots, D_m &\vdash A_2. \end{aligned} \quad (19)$$

Note that by definition of our proving system, in particular axiom 11, and monotonicity,

$$C_1, C_2, \dots, C_l, D_1, D_2, \dots, D_m \vdash (A_1 \Rightarrow (A_2 \Rightarrow (A_1 \cup A_2))). \quad (20)$$

Combining this and (19), using MP rule, we have

$$C_1, C_2, \dots, C_l, D_1, D_2, \dots, D_m \vdash A \quad (21)$$

By the definition of $C_1, C_2, \dots, C_l, D_1, D_2, \dots, D_m$ and the fact that $A = A'$, we get

$$B_1, B_2, \dots, B_n \vdash A' \quad (22)$$

- $v^*(A_1) = F$ and $V^*(A_2) = T$. Under this condition we have $A'_1 = \neg A_1$ and $A'_2 = A_2$. Also since $v^*(A) = v^*(A_1 \cup A_2) = T$ we have $A' = A = (A_1 \cup A_2)$. Because both $deg(A_1)$ and $deg(A_2)$ are strictly less than m , by induction hypothesis we have

$$\begin{aligned} C_1, C_2, \dots, C_l &\vdash A'_1 \\ D_1, D_2, \dots, D_m &\vdash A'_2. \end{aligned} \quad (23)$$

Plugging in the definition of A'_1 and A'_2 we have

$$\begin{aligned} C_1, C_2, \dots, C_l &\vdash \neg A_1 \\ D_1, D_2, \dots, D_m &\vdash A_2. \end{aligned} \quad (24)$$

By the definition of our proving system, in particular axiom 9, and monotonicity, we have

$$C_1, C_2, \dots, C_l, D_1, D_2, \dots, D_m \vdash (\neg A_1 \Rightarrow (A_2 \Rightarrow (A_1 \cup A_2))). \quad (25)$$

Combining this with (24), by monotonicity and MP rule, we get

$$C_1, C_2, \dots, C_l, D_1, D_2, \dots, D_m \vdash (A_1 \cup A_2). \quad (26)$$

Finally since under this case $A' = A = (A_1 \cup A_2)$, and the definition of C_1, C_2, \dots, C_l and D_1, D_2, \dots, D_m , we will have

$$B_1, B_2, \dots, B_n \vdash A'. \quad (27)$$

- $v^*(A_1) = T$ and $v^*(A_2) = F$. This case is the same as the previous one due to symmetry.

- $v^*(A_1) = F$ and $v^*(A_2) = F$. Under this case we have $A'_1 = \neg A_1$ and $A'_2 = \neg A_2$. Also $v^*(A) = v^*(A_1 \cup A_2) = v^*(A_1) \cup v^*(A_2) = F$, thus $A' = \neg A = \neg(A_1 \cup A_2)$. Because both $deg(A_1)$ and $deg(A_2)$ are strictly less than m , by induction hypothesis we have

$$\begin{aligned} C_1, C_2, \dots, C_l &\vdash A'_1 \\ D_1, D_2, \dots, D_m &\vdash A'_2. \end{aligned} \quad (28)$$

Plugging in the definition of A'_1 and A'_2 we have

$$\begin{aligned} C_1, C_2, \dots, C_l &\vdash \neg A_1 \\ D_1, D_2, \dots, D_m &\vdash \neg A_2. \end{aligned} \quad (29)$$

By the definition of our proving system, in particular axiom 10, and monotonicity, we have

$$C_1, C_2, \dots, C_l, D_1, D_2, \dots, D_m \vdash (\neg A_1 \Rightarrow (\neg A_2 \Rightarrow \neg(A_1 \cup A_2))). \quad (30)$$

Combining this with (29), by monotonicity and MP rule, we get

$$C_1, C_2, \dots, C_l, D_1, D_2, \dots, D_m \vdash \neg(A_1 \cup A_2). \quad (31)$$

Finally since in this case $A' = \neg A = \neg(A_1 \cup A_2)$, and by the definition of C_1, C_2, \dots, C_l and D_1, D_2, \dots, D_m , we get

$$B_1, B_2, \dots, B_n \vdash A'. \quad (32)$$

□

Now we have proven the main lemma for our new proving system. Using this lemma we can show that if $\models A$ then $\vdash A$ for any formula A . The proof can be transplanted directly from the proof of S given in the lecture. Therefore the completeness theorem holds for our new proving system. □

Question 3. Prove that the system RS1 is strongly sound.

Solution. To prove it's strongly sound, we only need to show that given any inference rule of that system, the conclusion is true if and only if all its premises are true. Actually, RS differs from RS1 only by changing the position of literal sequence and formula sequence, which do not affect the truthfulness of their union, since the disjunction operation is commutable with respect to its operands and the action of truth assigning commutes with the operation of connectives. □

Question 4. Define for any formula A , the decomposition tree T_A in RS1.

Solution. In order to give the definition of a decomposition tree we need some definitions and observations first.

Definition 1. A sequence of formulas is called a **indecomposable sequence** if it only contains literals.

Definition 2. A formula is called a **decomposable formula** if it is not a literal.

Definition 3. A sequence of formulas is called a **decomposable sequence** if it contains decomposable formulas.

Observation 1. Given an decomposable formula, we can use one and only one decomposition rule to its main connective. And after the decomposition, the original formula can be transformed to one formula or a sequence of two formulas with lower degrees, according to whether the main connective is unary or binary.

Observation 2. Given an decomposable formula, if its main connective is unary, i.e negation, then we go into its second connective, which is unique, and determine which decomposition rule we adopt. On the other hand, if the main connective is binary, then we can directly use the appropriate decomposition rule to it.

Observation 3. Given a sequence of formulas, we can start from the farthest decomposable formula to the right and decompose it into at most two formulas with smaller degrees. And then we again decompose the right most decomposable formula. Since the degree of all formulas are finite, the decomposition of the right most formula into literals can be done in finite steps. Since there are only finite formulas in the sequence, we can decompose all formulas in finite steps. In the end we can get an indecomposable sequence.

Now we can construct the tree in following steps.

step 1. Decompose the formula by its main connective, or together with its second connectives, depending on whether the main connective is binary or unary.

step 2. Traverse from right to left the sequence and decompose the first decomposable formula we meet.

step 3. Repeat step 2 until all decomposable formulas are decomposed. □

Question 5. Prove the completeness of the proving system for Intuitionistic Propositional Logic $I' = (\mathcal{L}_{\{\neg, \cup, \cap, \Rightarrow\}}, \mathcal{F}, \{A1, \dots, A11, A12\}, (MP))$ with the additional axiom

$$A12 \quad (A \cup \neg A).$$