Validity and Provability

Recall that a sentence (i.e., a formula without free variables) ϕ is *valid* if it is true in all models.

Theorem

> A sentence ϕ is valid if, and only if, it is provable, i.e.,
>
> $$\models \phi \text{ iff } \vdash \phi.$$

There are many proof systems for predicate logic, such as natural deduction, where the syntactic notion of provability in the calculus accurately captures the semantic concept of validity.

We will next discuss a proof method that is representative of a *refutational* approach in establishing the validity of a sentence by showing that its negation is unsatisfiable. (A sentence is called *unsatisfiable* if it is false in all models.)

Proposition

> A sentence ϕ is valid if, and only if, its negation $\neg\phi$ is unsatisfiable.

Logical Equivalence

Two formulas ϕ and ψ are said to be (logically) *equivalent*, written $\phi \sim \psi$, if they are true for the same models and environments, i.e.,

$$\mathcal{M} \models_l \phi \quad \text{if and only if} \quad \mathcal{M} \models_l \psi,$$

for all models $\mathcal{M}$ and environments l.

By this definition, two sentences are equivalent if they are true in the same models.

For example, the sentences

$$\forall x(\neg(x = 0) \rightarrow \exists y(x \cdot y = 1))$$

and

$$\forall x \exists y(\neg(x = 0) \rightarrow (x \cdot y = 1))$$

are equivalent.

Proposition

> Two sentences ϕ and ψ are equivalent if, and only if, the sentence $\phi \leftrightarrow \psi$ is valid.

Quantifier Equivalences

The following equivalences will be useful in simplifying predicate logic formulas.

Theorem

1. (a) $\neg\forall x\phi \sim \exists x\neg\phi$

   (b) $\neg\exists x\phi \sim \forall x\neg\phi$

2. Assuming that ψ contains no free occurrences of x:

   (a) $(\forall x\phi) \wedge \psi \sim \forall x(\phi \wedge \psi)$

   (b) $(\forall x\phi) \vee \psi \sim \forall x(\phi \vee \psi)$

   (c) $(\exists x\phi) \wedge \psi \sim \exists x(\phi \wedge \psi)$

   (d) $(\exists x\phi) \vee \psi \sim \exists x(\phi \vee \psi)$

   (e) $(\forall x\phi) \rightarrow \psi \sim \exists x(\phi \rightarrow \psi)$

   (f) $\psi \rightarrow (\forall x\phi) \sim \forall x(\psi \rightarrow \phi)$

3. (a) $(\forall x\phi) \wedge (\forall x\psi) \sim \forall x(\phi \wedge \psi)$

   (b) $(\exists x\phi) \vee (\exists x\psi) \sim \exists x(\phi \vee \psi)$

4. (a) $\forall x\forall y\phi \sim \forall y\forall x\phi$

   (b) $\exists x\exists y\phi \sim \exists y\exists x\phi$

Replacement

Replacement of subformulas by equivalent formulas preserves equivalence.

Replacement Theorem

> If ϕ, ψ and ψ' are formulas such that $\psi \sim \psi'$ and ϕ' is obtained from ϕ by replacing an occurrence of ψ by ψ', then $\phi \sim \phi'$.

The theorem can be proved by induction on the structure of formulas.

Propositional equivalences carry over to predicate logic in the following sense:

Proposition

> If α and β are propositional formulas with variables $p_1, \ldots, p_n$, which are equivalent (in a propositional sense), then
>
> $$\alpha[\phi_1/p_1, \ldots, \phi_n/p_n] \sim \beta[\phi_1/p_1, \ldots, \phi_n/p_n).$$
>
> for all predicate logic formulas $\phi_1, \ldots, \phi_n$.

Renaming Bound Variables

A formula ψ is said to be a *conjugate* of a formula ϕ if it can be obtained from ϕ by renaming *bound* occurrences of variables, subject to the following conditions:

> (i) no free occurrences of variables become bound as a result of the renaming and
>
> (ii) distinct bound variables are renamed to distinct variables.

Examples.

> Let ϕ be the formula $\forall x\,(R(x,y) \vee \exists z\, R(x,z))$. Which formulas are conjugates of ϕ:
>
> $\forall w\,(R(w,y) \vee \exists w\, R(x,w))$?
> $\forall x\,(R(x,y) \vee \exists x\, R(x,x))$?
> $\forall y\,(R(y,y) \vee \exists z\, R(y,z))$?
> $\forall u\,(R(u,y) \vee \exists v\, R(u,v))$?

Lemma.

> If ψ is a conjugate of ϕ, then $\psi \sim \phi$.

Substitution

Substitution of terms for (free occurrences of) variables also preserves equivalence.

Substitution Theorem

> If ϕ and ψ are equivalent formulas containing variables $x_1, \ldots, x_n$, and $t_1, \ldots, t_n$ are terms, then $\phi'[t_1/x_1, \ldots, t_n/x_n] \sim \psi'[t_1/x_1, \ldots, t_n/x_n)$ where ϕ' and ψ' are conjugates of ϕ and ψ, respectively, which share no common variables with the terms $t_1, \ldots, t_n$.

Prenex Forms

A formula is said to be in *prenex form* if it is of the form

$$Q_1 x_1 \dots Q_n x_n \, \phi$$

where $Q_1, \dots, Q_n$ are quantifiers and ϕ contains no quantifier.

In other words, no quantifiers are in the scope of propositional connectives.

Using the replacement and substitution theorems in combination with the quantifier equivalences listed earlier, we obtain:

Theorem

> Every predicate logic formula is equivalent to a formula in prenex form.

To obtain a prenex form of a formula proceed in two steps:

1. Rename bound variables, so that (i) no variable occurs both bound and free and (ii) no two quantifiers bind the same variable.

2. Use the quantifier equivalences to move quantifiers out of the scope of propositional connectives.

Skolemization

Skolemization is a technique for eliminating existential quantifiers.

A sentence is said to be *universal* if it is in prenex form and contains no existential quantifiers, i.e., of the form

$$\forall x_1 \ldots \forall x_n \, \phi,$$

where ϕ is quantifier-free.

For every predicate logic sentence one can effectively construct a universal sentence such that satisfiability of the original formula is preserved.

Lemma

> Let $\phi = \forall x_1 \ldots \forall x_n \exists y \, \psi$ be a predicate logic sentence where ψ contains no bound occurrences of $x_1, \ldots, x_n$ and y. Moreover, let $\mathcal{L}'$ be the extension of the given language $\mathcal{L}$ by an n-ary function symbol f (not contained in $\mathcal{L}$). Then the sentence ϕ is satisfiable if, and only if, $\forall x_1 \ldots \forall x_n \, \psi[f(x_1, \ldots, x_n)/y]$ is satisfiable.

The new symbol f is called a *Skolem function* or *constant*. (If $n = 0$, then f denotes a constant.)

A *Skolemized form* of a sentence in prenex form is obtained by successively eliminating existential quantifiers via the above Lemma.

For example, from the sentence

$$\forall x \exists y \forall z \exists w \, (\neg P(a, w) \vee Q(f(x), y))$$

we obtain

$$\forall x \forall z \, (\neg P(a, f_w(x, z)) \vee Q(f(x), f_y(x))).$$

Theorem

> For every predicate logic sentence ϕ there exists a universal sentence ψ such that ϕ is satisfiable iff ψ is satisfiable.