CSE 541 - Logic in Computer Science

Solutions for Selected Exercises on Model Checking

Exercise 3.7.1. Let S be the set $\{1, 2, 3, 4, 5, 6, 7, 8, 9, 10\}$ and H_1, H_2, and H_3 be functions of type $\mathcal{P}(S) \to \mathcal{P}(S)$ defined by:

$$H_1(Y) = Y - \{1, 4, 7\}$$
$$H_2(Y) = \{2, 5, 9\} - Y$$
$$H_3(Y) = \{1, 2, 3, 4, 5\} \cap (\{2, 4, 8\} \cup Y)$$

for all $Y \subseteq S$.

(a) First note that, for all sets X, Y, and Z, if $X \subseteq Y$ then $X \cup Z \subseteq Y \cup Z$ and $X \cap Z \subseteq Y \cap Z$. Consequently, H_1 and H_3 are monotone functions (as both can be defined via set union and intersection, e.g., $H_1(Y) = Y \cap \{2, 3, 5, 6, 7, 9, 10\}$).

The function H_2 is not monotone. For instance, $\emptyset \subseteq \{2, 5, 9\}$, but $H_2(\emptyset) = \{2, 5, 9\} \not\subseteq \emptyset = H_2(\{2, 5, 9\}$.

(b) Since H_3 is monotone, we can use the Fixed Point Theorem to infer that $H_3^{10}(\emptyset)$ and $H_3^{10}(S)$ are least and greatest fixed points, respectively, of H_3. The caluclation of these fixed points yields $H_3^{10}(\emptyset) = \{2, 4\}$ and $H_3^{10}(S) = \{1, 2, 3, 4, 5\}$.

(c) The function H_2 has no fixed points. It can easily be seen that for all sets Y with $Y \subseteq S$, $2 \in Y$ if, and only if, $2 \notin H_2(Y)$. Therefore $H_2(Y) \neq Y$, for all sets $Y \subseteq S$.

Exercise 3.7.3.

We label the states of the given transition system by $s_0, \ldots, s_6$ beginning on the upper left and proceeding clockwise.

a. The formula $EF\,p$ is equivalent to $E[\top\, U\, p]$. By Theorem 3.26, the set $[\![E(\top\, U\, p)]\!]$ is the least fixed point of $H_{\top,p}$, where

$$H_{\top,p}(X) = [\![p]\!] \cup ([\![\top]\!] \cap J(X)) = \{s_4\} \cup J(X)$$

and

$$J(X) = \{s \in S : s \to s' \text{ for some } s' \text{ in } X\}.$$

Furthermore, Theorem 3.20 indicates that we can compute this least fixed point by repeated applications of $H_{\top,p}$. More specifically, we have:

$$\begin{array}{rcl}
H_{\top,p}(\emptyset) & = & \{s_4\} \cup J(\emptyset) = \{s_4\} \\
H^2_{\top,p}(\emptyset) & = & \{s_4\} \cup J(\{s_4\}) = \{s_3, s_4\} \\
H^3_{\top,p}(\emptyset) & = & \{s_4\} \cup J(\{s_3, s_4\}) = \{s_2, s_3, s_4\} \\
H^4_{\top,p}(\emptyset) & = & \{s_4\} \cup J(\{s_2, s_3, s_4\}) = \{s_1, s_2, s_3, s_4\} \\
H^5_{\top,p}(\emptyset) & = & \{s_0, s_1, s_2, s_3, s_4\} \\
H^6_{\top,p}(\emptyset) & = & \{s_0, s_1, s_2, s_3, s_4, s_5\} \\
H^7_{\top,p}(\emptyset) & = & H^6_{\top,p}(\emptyset)
\end{array}$$

In sum, $[\![EF\, p]\!] = \{s_0, s_1, s_2, s_3, s_4, s_5\}$.

b. By Theorem 3.25, the set $[\![EG\, q]\!]$ is the greatest fixed point of the function G_q, where

$$G_q(X) = [\![q]\!] \cap J(X) = \{s_1, s_5, s_6\} \cap J(X)$$

and J is as defined above. In this case the fixed point can be computed by repeated applications of G_q, but neginning with $S = \{s_0, \ldots, s_6\}$ as first argument. We have:

$$\begin{array}{rcl}
G_q(S) & = & \{s_1, s_5, s_6\} \cap J(S) = \{s_1, s_5, s_6\} \cap S = \{s_1, s_5, s_6\} \\
G^2_q(S) & = & \{s_1, s_5, s_6\} \cap J(\{s_1, s_5, s_6\}) = \{s_1, s_5, s_6\} = G_q(S)
\end{array}$$

We may conclude that $[\![EG\, q]\!] = \{s_1, s_5, s_6\}$.