CSE 541 - Logic in Computer Science
Sample Natural Deduction Proofs

Exercise 1.2.1(i). Proof of $(p \to r) \wedge (q \to r) \vdash p \wedge q \to r$, first in sequent notation

1	$p \wedge q \vdash p \wedge q$	axiom
2	$(p \to r) \wedge (q \to r) \vdash (p \to r) \wedge (q \to r)$	axiom
3	$p \wedge q \vdash p$	$\wedge e_1$ 1
4	$(p \to r) \wedge (q \to r) \vdash p \to r$	$\wedge e_1$ 2
5	$(p \to r) \wedge (q \to r), p \wedge q \vdash r$	$\to$e 3,4
6	$(p \to r) \wedge (q \to r) \vdash p \wedge q \to r$	$\to$i 5

and then using boxes

1	$(p \to r) \wedge (q \to r)$	premise
2	$p \wedge q$	assumption
3	p	$\wedge e_1$ 2
4	$p \to r$	$\wedge e_1$ 1
5	r	$\to$e 3,4
6	$p \wedge q \to r$	$\to$i 2-5

Exercise 1.2.1(j). Proof of $q \to r \vdash (p \to q) \to (p \to r)$:

1	$q \to r$	premise
2	$p \to q$	assumption
3	p	assumption
4	q	$\to$e 3,2
5	r	$\to$e 4,1
6	$p \to r$	$\to$i 3-5
7	$(p \to q) \to (p \to r)$	$\to$i 2-6

Exercise 1.2.1(q). Proof of $\vdash q \to (p \to (p \to (q \to p)))$:

1	q	assumption
2	p	assumption
3	p	assumption
4	q	assumption
5	p	copy 3
6	$q \to p$	$\to$i 4-5
7	$p \to (q \to p)$	$\to$i 3-6
8	$p \to (p \to (q \to p))$	$\to$i 2-7
9	$q \to (p \to (p \to (q \to p)))$	$\to$i 1-8

Exercise 1.2.2(e). Proof of $p \to q \vee r, \neg q, \neg r \vdash \neg p$ without using Modus Tollens:

1	$p \to q \vee r$	premise
2	$\neg q$	premise
3	$\neg r$	premise
4	p	assumption
5	$q \vee r$	$\to$e 4,1
6	q	assumption
7	$\bot$	$\neg$e 6,2
8	r	assumption
9	$\bot$	$\neg$e 8,3
10	$\bot$	$\vee$e 5,6-7,8-9
11	$\neg p$	$\neg$i 4-10

Exercise 1.2.3(b). Proof of $\neg p \vdash p \to q$, first in sequent notation

1	$p \vdash p$	axiom
2	$\neg p \vdash \neg p$	axiom
3	$p, \neg p \vdash \bot$	$\neg$e 1,2
4	$p, \neg p \vdash q$	$\bot$e 3
5	$\neg p \vdash p \to q$	$\to$i 4

and then using boxes

1	$\neg p$	axiom
2	p	assumption
3	$\bot$	$\neg$e 1,2
4	q	$\bot$e 3
5	$p \to q$	$\to$i 2-4

Exercise 1.2.3(e). Proof of $\neg(p \to q) \vdash q \to p$:

1	$\neg(p \to q) \vdash \neg(p \to q)$	axiom
2	$p, q \vdash q$	axiom
3	$q \vdash (p \to q)$	$\to$i 2
4	$\neg(p \to q), q \vdash \bot$	$\neg$e 1,3
5	$\neg(p \to q), q \vdash p$	$\bot$e 4
6	$\neg(p \to q) \vdash q \to p$	$\to$i 5

Exercise 1.2.3(o). Proof of $\neg(\neg p \vee \neg q) \vdash p \wedge q$:

1	$\neg(\neg p \vee \neg q)$	premise
2	$\neg p$	assumption
3	$\neg p \vee \neg q$	$\vee i_1$ 2
4	$\perp$	$\neg e$ 1,3
5	p	PBC 2-4
6	$\neg q$	assumption
7	$\neg p \vee \neg q$	$\vee i_2$ 6
8	$\perp$	¬e 1,7
9	q	PBC 6-8
9	$p \wedge q$	∧i 5,9

Exercise 1.2.3(q). Proof of $\vdash (p \to q) \vee (q \to r)$:

1	$q \vee \neg q$	LEM
2	q	assumption
3	p	assumption
4	q	copy 2
5	$p \to q$	→i 3-4
6	$(p \to q) \vee (q \to r)$	∨i 5
7	$\neg q$	assumption
8	q	assumption
9	$\perp$	¬e 7,8
10	r	⊥e 9
11	$q \to r$	→i 8-10
12	$(p \to q) \vee (q \to r)$	∨i 11
13	$(p \to q) \vee (q \to r)$	∨e 1,2-6,7-12

Exercise 1.2.3(u). Proof of $(s \to p) \vee (t \to q) \vdash (s \to q) \vee (t \to p)$.

1	$\vdash p \vee \neg p$	LEM
2	$p, t \vdash p$	axiom
3	$p \vdash t \to p$	$\to$i 2
4	$p \vdash (s \to q) \vee (t \to p)$	$\vee$i 3
5	$\neg p \vdash \neg p$	axiom
6	$s \to p \vdash s \to p$	axiom
7	$s \to p, \neg p \vdash \neg s$	MT 6,5
8	$s \vdash s$	axiom
9	$s \to p, \neg p, s \vdash \bot$	$\neg$e 7,8
10	$s \to p, \neg p, s \vdash q$	$\bot$e 9
11	$s \to p, \neg p \vdash s \to q$	$\to$i 10
12	$s \to p, \neg p \vdash (s \to q) \vee (t \to p)$	$\vee$i 11
13	$s \to p \vdash (s \to q) \vee (t \to p)$	$\vee$e 1,4,12
14	$\vdash q \vee \neg q$	LEM
15	$q, s \vdash q$	axiom
16	$q \vdash s \to q$	$\to$i 15
17	$q \vdash (s \to q) \vee (t \to p)$	$\vee$i 16
18	$\neg q \vdash \neg q$	axiom
19	$t \to q \vdash t \to q$	axiom
20	$t \to q, \neg q \vdash \neg t$	MT 19,18
21	$t \vdash t$	axiom
22	$t \to q, \neg q, t \vdash \bot$	$\neg$e 20,21
23	$t \to q, \neg q, t \vdash q$	$\bot$e 22
24	$t \to q, \neg q \vdash s \to q$	$\to$i 23
25	$t \to q, \neg q \vdash (s \to q) \vee (t \to p)$	$\vee$i 24
26	$t \to q \vdash (s \to q) \vee (t \to p)$	$\vee$e 14,17,25
27	$(s \to p) \vee (t \to q) \vdash (s \to p) \vee (t \to q)$	axiom
28	$(s \to p) \vee (t \to q) \vdash (s \to q) \vee (t \to p)$	$\vee$e 27,13,26

Note that lines 1–13 and 14-26 represent similar subproofs.

Exercise 1.2.5(a). Proof of $\vdash ((p \to q) \to q) \to ((q \to p) \to p)$ in sequent style notation,

1	$p, \neg p \vdash q$	lemma, cf. 1.2.3(b)
2	$\neg p \vdash p \to q$	$\to$i 1
3	$(p \to q) \to q \vdash (p \to q) \to q$	axiom
4	$(p \to q) \to q, \neg p \vdash q$	$\to$e 3,2
5	$q \to p \vdash q \to p$	axiom
6	$\neg p \vdash \neg p$	axiom
7	$q \to p, \neg p \vdash \neg q$	MT 6,5
8	$(p \to q) \to q, q \to p, \neg p \vdash \bot$	$\neg$e 7,4
9	$(p \to q) \to q, q \to p \vdash p$	PBC 8
10	$(p \to q) \to q \vdash (q \to p) \to p$	$\to$i 9
11	$\vdash ((p \to q) \to q) \to ((q \to p) \to p)$	$\to$i 10

and a different proof of $(p \to q) \to q \vdash (q \to p) \to p$ using boxes,

1	$(p \to q) \to q$	premise
2	$q \to p$	assumption
3	$(p \to q) \vee \neg(p \to q)$	LEM
4	$p \to q$	assumption
5	q	$\to$e 1,4
6	p	$\to$e 2,5
7	$\neg(p \to q)$	assumption
8	$\neg p$	assumption
9	p	assumption
10	$\bot$	$\neg$e 8,9
11	q	$\bot$e 10
12	$p \to q$	$\to$i 9-11
13	$\bot$	$\neg$e 4,12
14	p	$\neg$i 5-13
15	p	$\vee$e 4-6,7-14
16	$(q \to p) \to p$	$\to$i 2-15

(courtesy of Shang Yang, Spring 2007).