CSE 541 - Logic in Computer Science
Sample Solutions for Selected Exercises on Modal Logic

Exercise 5.2.1 a

Answer key. (iv) false; (v) true; (vi) true; (vii) false; (viii) true; (ix) true; (x) true; (xi) false; (xii) true.

Exercises 5.2.1 b,c

i. Only worlds c, d and e satisfy the given formula.

ii. Only world b satisfies the given formula.

iii. Only worlds a, b and d satisfy the given formula.

iv. Only worlds a, b and d satisfy the given formula.

v. Only worlds c, d and e satisfy the given formula.

vi. All worlds satisfy the given formula.

Exercise 5.2.2

Let $\mathcal{M} = (W, R, L)$ be a model with $W = \{s_0\}$, $R = \emptyset$, and $L(s_0) = \emptyset$. The formula scheme $\phi \to \psi$ is not valid in $\mathcal{M}$ (e.g., $\neg p \to p$ is false), but has true instances, such as $p \to p$.

Exercise 5.2.3 b

i. World c satisfies the given formula.

ii. Worlds a and b satisfy the given formula.

iii. Worlds a, b and e satisfy the given formula.

iv. Worlds b, c, d and e satisfy the given formula.

v. All worlds satisfy the given formula.

Exercise 5.2.5

a. Let $\mathcal{M} = (W, R, L)$ be a model with $W = \{s_0, s_1, s_2\}$, $R = \{(s_0, s_1), (s_1, s_2)\}$, and $L(s_0) = L(s_2) = \emptyset$ and $L(s_1) = \{p\}$. Then $\mathcal{M}, s_0 \models \Box p$ but $\mathcal{M}, s_0 \not\models \Box\Box p$.

c. The formulas $\Box(p \wedge q)$ and $\Box p \wedge \Box q$ are equivalent; see the remarks on page 269.

d. Let $\mathcal{M} = (W, R, L)$ be a model with $W = \{s_0, s_1, s_2\}$, $R = \{(s_0, s_1), (s_0, s_2)\}$, and $L(s_0) = \emptyset$, $L(s_1) = \{p\}$, and $L(s_2) = \{q\}$. Then $\mathcal{M}, s_0 \models \Diamond p \wedge \Diamond q$ but $\mathcal{M}, s_0 \not\models \Diamond(p \wedge q)$.

e. Taking the same model $\mathcal{M}$, we also find that $\mathcal{M}, s_0 \models \Box(p \vee q)$ whereas $\mathcal{M}, s_0 \not\models \Box p \vee \Box q$.

f. The formulas $\Diamond(p \vee q)$ and $\Diamond p \vee \Diamond q$ are equivalent; see the next exercise.

g. Take again the same model $\mathcal{M}$ as before and note that $\mathcal{M}, s_0 \models (\Box p \rightarrow \Box q)$ but $\mathcal{M}, s_0 \not\models \Box(p \rightarrow q)$.

h. Let $\mathcal{M} = (W, R, L)$ be a model with $W = \{s_0\}$, R is the empty relation, and $L(s_0) = \emptyset$. Then $\mathcal{M}, s_0 \models \top$ but $\mathcal{M}, s_0 \not\models \Diamond\top$.

Exercise 5.2.6

a. We prove that $\Box(\phi \wedge \psi)$ and $\Box\phi \wedge \Box\psi$ are true in the same worlds.

   Consider first a model $\mathcal{M} = (W, R, L)$ and a world $x \in W$, such that $x \models \Box(\phi \wedge \psi)$. By the definition of the $\Box$ operator, we have $y \models (\phi \wedge \psi)$, for all $y \in W$ with xRy. By the semantics of conjunction, we obtain (i) $y \models \phi$, for all $y \in W$ with xRy, and (ii) $y \models \psi$, for all $y \in W$ with xRy. This implies both $x \models \Box\phi$ and $x \models \Box\psi$, and hence $x \models (\Box\phi \wedge \Box\psi)$, which completes the first part.

   Next consider a model $\mathcal{M} = (W, R, L)$ and a world $x \in W$, such that $x \models (\Box\phi \wedge \Box\psi)$. By the semantics of conjunction, we have $x \models \Box\phi$ and $x \models \Box\psi$. By the definition of the $\Box$ operator, we obtain $y \models \phi$, for all $y \in W$ with xRy, as well as $y \models \psi$, for all $y \in W$ with xRy. Thus we also have $y \models (\phi \wedge \psi)$, for all $y \in W$ with xRy. This implies $x \models \Box(\phi \wedge \psi)$, which completes the second part.

b. This part can be proved in a similar way.

c. We prove that $\Box\top$ is true in all worlds of all models. Let $\mathcal{M} = (W, R, L)$ be a model and $x \in W$. To show $x \models \Box\top$ it suffices to prove $y \models \top$, for all $y \in W$ with xRy. The latter assertion can easily be seen to be true, as $z \models \top$ holds for all $z \in W$.

d. Note that $\Diamond\bot$ is equivalent to $\neg\Box\neg\bot$ and to $\neg\Box\top$. By the preceding part, the last formula is equivalent to $\neg\top$. We thus obtain that $\Diamond\bot$ is equivalent to $\bot$, which implies that $\Diamond\bot \leftrightarrow \bot$ is valid.

e. We prove that $\Diamond\top \to (\Box\phi \to \Diamond\phi)$ is a valid formula of basic modal logic.

   Consider an arbitrary model $\mathcal{M} = (W, R, L)$ and a world $x \in W$. We have to show that $x \models \Diamond\top \to (\Box\phi \to \Diamond\phi)$. For that purpose, let us assume $x \models \Diamond\top$. We need to show $x \models (\Box\phi \to \Diamond\phi)$. Let us therefore assume $x \models \Box\phi$, and then show $x \models \Diamond\phi$.

   To prove $x \models \Diamond\phi$ we have to show that there exists a world $y \in W$ such that xRy and $y \models \phi$. Since $x \models \Diamond\top$ we know that there exists a world, say z, with xRz. Furthermore, $x \models \Box\phi$ implies $x' \models \phi$, for all worlds x' with xRx'. Consequently we have $z \models \phi$, which completes the proof.

Exercise 5.3.14

The frame (W, R) with $W = \{x, y\}$ and $R = \{(x, x), (x, y), (y, y)\}$ is reflexive and transitive, but not symmetric. If we choose a labelling function L with $L(x) = \{p\}$ and $L(y) = \emptyset$, then x does not satisfy $p \to \Box\Diamond p$. But x does satisfy $p \to \Box\Diamond p$ if we choose a labelling function L with $L(x) = L(y) = \emptyset$.

Exercise 5.3.15

Any frame (W, R) where $R = \emptyset$ is Euclidean. Also the frame (W_1, R_1), with $W_1 = \{x, y\}$ and $R_1 = \{(x, y), (y, y)\}$, and the frame (W_2, R_2), with $W_2 = \{x, y, z\}$ and $R_2 = \{(x, y), (x, z), (y, y), (y, z), (z, y), (z, z)\}$, are Euclidean. The frame (W_3, R_3), with $W_3 = \{x, y\}$ and $R_3 = \{(x, y)\}$, and the frame (W_4, R_4), with $W_4 = \{x, y, z\}$ and $R_4 = \{(x, y), (x, z)\}$, are not Euclidean.

Exercise 5.3.16

1. The relation R corresponding to $\phi \to \Box\phi$ is characterized by the condition that it be a subset of the identity relation (i.e., for all x and y, if xRy then $x = y$).

2. The relation corresponding to $\Box\bot$ is the empty relation.

3. The relation R corresponding to $\Diamond\Box\phi \to \Box\Diamond\phi$ is characterized by the following condition (also known as "diamond property" or "confluence"): for all x, y and z such that xRy and xRz, there exists v with yRv and zRv.

   Proof. (i) Suppose R satisfies the diamond property. Let $\mathcal{M}$ be an arbitrary Kripke model (W, R, L). We have to show that $\Diamond\Box\phi \to \Box\Diamond\phi$ is true in all worlds of W. Let $x \in W$ be such that $x \models \Diamond\Box\phi$. We need to show that $x \models \Box\Diamond\phi$; or that $y \models \Diamond\phi$, for all y with xRy. Let $y \in W$ be such that xRy. To prove $y \models \Diamond\phi$ we need to find a $z \in W$ such that $z \models \phi$.

   We know $x \models \Diamond\Box\phi$, from which we may infer that there exists $y' \in W$ such that xRy' and $y' \models \Box\phi$. Since we also have xRy we may use the diamond property to infer that there exists a $z \in W$ such that yRz and $y'Rz$. From $y' \models \Box\phi$ and $y'Rz$ we obtain $z \models \phi$, which completes the first part of the proof.

   (ii) Suppose $\mathcal{F}$ is a frame (W, R) that satisfies $\Diamond\Box\phi \to \Box\Diamond\phi$. In other words, the latter formula is true in each world of each Kripke model (W, R, L) based on $\mathcal{F}$.

   We have to prove that R satisfies the diamond property. For that purpose, suppose x, y and z are elements of W such that xRy and xRz. We need to show that there exists a $v \in W$ such that yRv and zRv.

   Now consider a specific Kripke model (W, R, L_y), where the labelling function L_y is defined (in terms of y) by: $L_y(u) = \{p\}$, if yRu, and $L_y(u) = \emptyset$, otherwise. In this Kripke model we obviously have $y \models \Box p$, and since xRy, also $x \models \Diamond\Box p$. Since $\mathcal{F}$ satisfies the general schema $\Diamond\Box\phi \to \Box\Diamond\phi$, we get in particular, $x \models \Diamond\Box p \to \Box\Diamond p$. By modus ponens we obtain $x \models \Box\Diamond p$ and, since xRz, also $z \models \Diamond p$. Thus there exists a $v \in W$ such that zRv and $v \models p$. By the definition of L_y we must also have yRv. This completes the second part of the proof.

Exercise 5.3.17

A binary relation R on a set W is said to be dense if for all x and y in W with xRy, there exists $z \in W$, such that xRz and zRy.

We prove that a frame $\mathcal{F} = (W, R)$ satisfies the formula (scheme) $\Box\Box\phi \to \Box\phi$ iff the accessibility relation R is dense.

(i) Suppose the relation R is dense. Let $\mathcal{M}$ be any model (W, R, L) and $x \in W$. We need to show that x satisfies $\Box\Box\phi \to \Box\phi$. For that purpose, suppose $x \models \Box\Box\phi$. We need to show $x \models \Box\phi$. In other words, we have to prove that $y \models \phi$, for all $y \in W$ with xRy. Let y be an arbitrary world in W such that xRy. Since R is dense, there exists a world $z \in W$, such that xRz and zRy. We also know $x \models \Box\Box\phi$, and hence may infer that $z \models \Box\phi$. But since zRy, this implies $y \models \phi$, which completes this part of the proof.

(ii) Suppose $\mathcal{F}$ is a frame (W, R) that satisfies $\Box\Box\phi \to \Box\phi$. We have to show that R is dense.

Let x and y be arbitrary elements of W with xRy. We need to show that there is a world $z \in W$ such that xRz and zRy.

Let now L be a labelling function such that $L(u) = \{p\}$ if there exists $w \in W$ such that xRw and wRu; and $L(u) = \emptyset$ otherwise. Let $\mathcal{M}$ be the model (W, R, L). Then $\mathcal{M}, x \models \Box\Box p$. Since the frame (W, R) satisfies $\Box\Box\phi \to \Box\phi$, we may infer that $\mathcal{M}, x \models \Box p$. In other words, we must have $\mathcal{M}, x' \models p$, for all worlds x' with xRx'. In particular, we obtain $\mathcal{M}, y \models p$, or $p \in L(y)$. By the definition of L, this implies that there exists $z \in W$ such that xRz and zRy, which completes the proof.

Exercise 5.5.3

a. $K_1 p$

b. $K_1 (p \vee q)$

c. $K_1 p \ \vee \ K_1 q$

d. $K_1 p \ \vee \ K_1 \neg p$

e. $\neg K_1 (p \vee q) \ \wedge \ \neg K_1 \neg (p \vee q)$

f. $K_1 K_2 p \ \vee \ K_1 \neg K_2 p$

g. $K_1 (K_2 p \ \vee K_2 \neg p) \vee K_1 (\neg K_2 p \wedge \neg K_2 \neg p)$

h. $\neg K_1 p \ \wedge \cdots \wedge \neg K_n p$

i. $\neg (E p \vee E \neg p)$

j. $(K_1 p \to K_1 q) \wedge \cdots \wedge (K_n p \to K_n q)$

k. $(K_1 p \wedge \neg K_1 q) \vee \cdots \vee (K_n p \wedge \neg K_n q)$

l. $E(K_1 p \vee \cdots \vee K_n p)$

Exercise 5.5.4
Consider the $KT45^3$ model in Figure 5.13.

a. We have $x_1 \not\models K_1 p$, as $x_1 R_1 x_1$ but $x_1 \not\models p$.

b. We have $x_3 \models K_1(p \vee q)$.

c. We have $x_1 \models K_2 q$.

d. We have $x_3 \models E(p \vee q)$.

e. We have $x_1 \not\models Cq$, as $x_1 \not\models Eq$.

f. We have $x_1 \not\models D_{\{1,3\}} p$, because $x_1 R_1 x_1$ and $x_1 R_3 x_1$, yet $x_1 \not\models p$.

g. For similar reasons we also have $x_1 \not\models D_{\{1,2\}} p$.

h. We have $x_6 \models E\neg q$.

i. We have $x_6 \not\models K_3 K_1 \neg q$ and hence $x_6 \not\models EE\neg q$. Therefore, we obtain $x_6 \not\models C\neg q$.

j. We have $x_6 \models C_{\{3\}} \neg q$, as $x_6 \models K_3 \neg q$.

Exercise 5.5.5

Let $\mathcal{M} = (W, R, L)$ be a $KT45^2$ model with $W = \{s_0, s_1, s_2\}$,

$$\begin{aligned} R_1 &= \{(s_0, s_0), (s_0, s_1), (s_1, s_0), (s_1, s_1), (s_2, s_2)\} \\ R_2 &= \{(s_0, s_0), (s_1, s_1), (s_1, s_2), (s_2, s_1), (s_2, s_2)\} \end{aligned}$$

and such that ϕ is true in s_0 and s_1, but not in s_2.

Then $E\phi$ is true in s_0, but not in s_1. Consequently, s_0 does not satisfy $E\phi \to EE\phi$, while s_1 does not satisfy $\neg E\phi \to E\neg E\phi$.

Aces and Eights
You play a card game, *aces and eights*, with Alice and Bob. The game is played with eight cards, four aces and four eights. Each player is dealt two cards and raises them up so that the other players can see them but he or

she cannot. The remaining two cards are left face down. The players take turns trying to determine which cards they are holding: (i) two aces, (ii) two eights, or (iii) and ace and an eight. (Assume all players are intelligent, perceptive, and truthful.)

- In the first game Alice, who holds two aces, goes first and Bob, who holds two eights, goes second. They both cannot determine what cards they are holding. This implies that you are holding neither two eights (for then Alice would know that she is holding a pair of aces) nor two aces (for then Bob would know he is holding a pair of eights), but an ace and an eight.

- In the second game you go first. Alice, who goes second, holds two eights, and Bob, who goes third, holds an ace and an eight. No one is able to give a definite answer in the first turn. Since Alice and Bob hold three of the four eights, you hold either a pair of aces or else an ace and an eight. If you had a pair of aces, then Bob, culd have used similar arguments as in the first game to deduce that he is holding an ace and an eight. The fact that he does not know what cards he is holding allows you to conclude that you are holding an ace and an eight.