CSE 541 - Logic in Computer Science
Solutions for Selected problems on Predicate Logic

Exercises from Huth and Ryan, *Logic in Computer Science*, 2nd ed.

Exercise 2.1.3. We use the predicates,

$InBox(x)$:	x is in the box
$Red(x)$:	x is red
$Animal(x)$:	x is an animal
$Cat(x)$:	x is a cat
$Dog(x)$:	x is a dog
$Boy(x)$:	x is a boy
$Prize(x)$:	x is a prize
$Won(x, y)$:	x won y

to formalize the following sentences.

a. All red things are in the box.

$$\forall x(Red(x) \to InBox(x))$$

b. Only red things are in the box.

$$\forall x(InBox(x) \to Red(x))$$

c. No animal is both a cat and a dog.

$$\neg\exists x(Animal(x) \wedge (Cat(x) \wedge Dog(x)))$$

$$\text{or } \forall x(Animal(x) \to (\neg Cat(x) \vee \neg Dog(x)))$$

d. Every prize was won by a boy.

$$\forall x[Prize(x) \to \exists y\,(Boy(y) \wedge Won(y, x))]$$

e. A boy won every prize.

$$\exists y[Boy(y) \wedge \forall x\,(Prize(x) \to Won(y, x))]$$

Exercise 2.1.4. Let $F(x, y)$ mean that x is the father of y; $M(x, y)$, that x is the mother of y; $H(x, y)$, that x is the husband of y; $S(x, y)$, that x is the sister of y; and $B(x, y)$, that x is the brother of y. We use these predicate symbols to translate the following sentences into predicate logic.

a. Everybody has a mother.

$$\forall x \exists y M(y, x)$$

b. Everybody has a father and a mother.

$$\forall x[(\exists y F(y, x)) \wedge (\exists z M(z, x))]$$

or, equivalently,

$$\forall x \exists y \exists z (F(y, x) \wedge M(z, x))$$

c. Whoever has a mother has a father.

$$\forall x[(\exists y M(y, x)) \rightarrow (\exists z F(z, x))]$$

d. Ed is a grandfather.

$$\exists x \exists y (F(Ed, y) \wedge (F(y, x) \vee M(y, x)))$$

e. All fathers are parents.

$$\forall x[\exists y F(x, y) \rightarrow \exists z (F(x, z) \vee M(x, z))]$$

f. All husbands are spouses.

$$\forall x[\exists y H(x, y) \rightarrow \exists z (H(x, z) \vee H(z, x))]$$

Note that x is the wife of y means that y is the husband of x.

g. No uncle is an aunt.

According to Webster's Collegiate Dictionary, an uncle is (i) the brother of one's father or mother or (ii) the husband of one's aunt; where an aunt is either the sister of one's father or mother or else the wife of one's uncle.

Let $\alpha(x, y)$ be an abbreviation for the formula

$$\begin{aligned} &\exists z[B(x, z) \wedge (F(z, y) \vee M(z, y))] \vee \\ &\quad \exists z \exists w [H(x, z) \wedge S(z, w) \wedge (F(w, y) \vee M(w, y))] \end{aligned}$$

which expresses that x is an uncle of y; and let $\beta(x, y)$ be an abbreviation for the formula

$$\begin{aligned} &\exists z[S(x, z) \wedge (F(z, y) \vee M(z, y))] \vee \\ &\quad \exists z \exists w [H(z, x) \wedge B(z, w) \wedge (F(w, y) \vee M(w, y))] \end{aligned}$$

which expresses that x is an aunt of y.

The given sentence can then be formulated as:

$$\neg\exists x\exists y\exists z[\alpha(x,y)\wedge\beta(x,z)]$$

h. All brothers are siblings.

Siblings are individuals that have a common parent.

$\forall x\forall y(B(x,y)\wedge B(y,x)\rightarrow\exists z[(F(z,x)\wedge F(z,y))\vee(M(z,x)\wedge M(z,y))])$

i. Nobody's grandmother is anybody's father.

$$\neg\exists x\exists y[\exists z[M(y,z)\wedge(M(z,x)\vee F(z,x))]\wedge\exists wF(y,w)]$$

or, equivalently,

$$\forall x\forall y[(\exists z(M(y,z)\wedge(M(z,x)\vee F(z,x))))\rightarrow\neg\exists wF(y,w)]$$

j. Ed and Patsy are husband and wife.

$$H(Ed,Patsy)$$

k. Carl is Monique's brother-in-law.

A brother-in law is (i) the brother of one's spouse, (ii) the husband of one's sister, or (iii) the husband of one's spouse's sister.

$$\begin{aligned}\exists x\exists y[&(B(Carl,x)\wedge H(x,Monique))\\&\vee(H(Carl,x)\wedge S(x,Monique))\\&\vee(H(Carl,x)\wedge S(x,y)\wedge H(y,Monique))]\end{aligned}$$

Exercise 2.4.1. Let ϕ be the formula $\forall x\forall yQ(g(x,y),g(y,y),z)$.

If $\mathcal{M}$ is a model with universe A, such that $Q^{\mathcal{M}}=A\times A\times A$, then $\mathcal{M}\models_l\phi$, for every environment l (regardless of what function $g^{\mathcal{M}}$ is).

On the other hand, if $\mathcal{M}'$ is a model with universe A, such that $Q^{\mathcal{M}}$ is the empty set, then $\mathcal{M}'\not\models_l\phi$, for any environment l.

Exercise 2.4.2. Let ϕ be the sentence

$$\forall x\exists y\exists z\ (P(x,y)\wedge P(z,y)\wedge(P(x,z)\rightarrow P(z,x))).$$

The following models all have the set $\mathbf{N}$ of natural numbers as universe.

a. The model $\mathcal{M}$ with $P^{\mathcal{M}} = \{(m,n) \mid m < n\}$ satisfies ϕ because for every environment l and every natural number k,

$$\mathcal{M} \models_{l[x \mapsto k][y \mapsto k+1][z \mapsto k]} P(x,y) \land P(z,y) \land (P(x,z) \to P(z,x)).$$

b. The model $\mathcal{M}'$ with $P^{\mathcal{M}'} = \{(m, 2*m) \mid m \in \mathbf{N}\}$ satisfies ϕ because for every environment l and every natural number k,

$$\mathcal{M}' \models_{l[x \mapsto k][y \mapsto 2*k][z \mapsto k]} P(x,y) \land P(z,y) \land (P(x,z) \to P(z,x)).$$

c. The model $\mathcal{M}''$ with $P^{\mathcal{M}''} = \{(m,n) \mid m < n+1\}$ satisfies ϕ because for every environment l and every natural number k,

$$\mathcal{M}'' \models_{l[x \mapsto k][y \mapsto k][z \mapsto k]} P(x,y) \land P(z,y) \land (P(x,z) \to P(z,x)).$$

Exercise 2.4.3. Let P be a predicate with two arguments.

If $\mathcal{M}$ is a model with universe A and for which $P^{\mathcal{M}}$ is the empty set, then $\mathcal{M} \models \forall x \neg P(x,x)$.

If $\mathcal{M}'$ is a model with universe A and for which $P^{\mathcal{M}'} = A \times A$, then $\mathcal{M}' \not\models \forall x \neg P(x,x)$.

Exercise 2.4.5. Let ϕ be the formula

$$\forall x \forall y \exists z\, (R(x,y) \to R(y,z)).$$

a. Let A be the set $\{a,b,c,d\}$ and $R^{\mathcal{M}}$ be $\{(b,c),(b,b),(b,a)\}$. We have $\mathcal{M} \not\models \phi$, because for every environment l,

$$\mathcal{M} \not\models_{l[x \mapsto b][y \mapsto c]} \exists z (R(x,y) \to R(y,z)).$$

(Note that there is no element $e \in A$, such that

$$\mathcal{M} \models_{l[x \mapsto b][y \mapsto c][z \mapsto e]} R(y,z),$$

whereas

$$\mathcal{M} \models_{l'[x \mapsto b][y \mapsto c]} R(x,y),$$

for every environment l'.)

b. Let A' be the set $\{a,b,c\}$ and $R^{\mathcal{M}'}$ be $\{(b,c),(a,b),(c,b)\}$. We have $\mathcal{M}' \models \phi$, because for every environment l and all elements e and f in A', there is an element $g \in A$, such that either

$$\mathcal{M}' \not\models_{l[x \mapsto e][y \mapsto f][z \mapsto g]} R(x,y)$$

or else

$$\mathcal{M}' \models_{l[x \mapsto e][y \mapsto f][z \mapsto g]} R(y,z).$$

Exercise 2.4.6.

Consider the three sentences

$$\begin{aligned}\phi_1 &= \forall x\, P(x,x)\\ \phi_2 &= \forall x \forall y\, (P(x,y) \to P(y,x))\\ \phi_3 &= \forall x \forall y \forall z\, (P(x,y) \wedge P(y,z) \to P(x,z))\end{aligned}$$

which express that the binary relation denoted by P is reflexive, symmetric, and transitive, respectively.

Let $\mathcal{M}$ be a model with universe A and for which $P^{\mathcal{M}}$ is the empty set. Then $\mathcal{M} \models \phi_2$ and $\mathcal{M} \models \phi_3$, but $\mathcal{M} \not\models \phi_1$.

Let $\mathcal{M}'$ be a model with the set of natural numbers as universe and for which $P^{\mathcal{M}'}$ is the less-than-or-equal-to relation. Then $\mathcal{M}' \models \phi_1$ and $\mathcal{M}' \models \phi_3$, but $\mathcal{M}' \not\models \phi_2$.

Let $\mathcal{M}''$ be a model with universe $A = \{a, b, c\}$ and for which

$$P^{\mathcal{M}''} = \{(a,a),(b,b),(c,c),(a,b),(b,a),(b,c),(c,b)\}.$$

Then $\mathcal{M}'' \models \phi_1$ and $\mathcal{M}'' \models \phi_2$, but $\mathcal{M}'' \not\models \phi_3$.

Exercise 2.4.8. We prove that

$$\forall x\, P(x) \vee \forall x\, Q(x) \models \forall x\, (P(x) \vee Q(x)).$$

Let $\mathcal{M}$ be a model with universe A such that

$$\mathcal{M} \models \forall x\, P(x) \vee \forall x\, Q(x).$$

We have to show that

$$\mathcal{M} \models \forall x\, (P(x) \vee Q(x)).$$

By the semantics of disjunction, we know that

$$\mathcal{M} \models \forall x\, P(x)$$

or

$$\mathcal{M} \models \forall x\, Q(x).$$

(i) We first consider the case that $\mathcal{M} \models \forall x\, P(x)$. By the semantics of universal quantification, we know that

$$\mathcal{M} \models_{l[x \mapsto a]} P(x)$$

for every environment l and all elements $a \in A$. But then we also have

$$\mathcal{M} \models_{l[x \mapsto a]} P(x) \vee Q(x)$$

for every environment l and all elements $a \in A$. This implies

$$\mathcal{M} \models \forall x\, (P(x) \vee Q(x)).$$

(ii) A similar argument can be applied if $\mathcal{M} \models \forall x\, Q(x)$.

Exercise 2.4.9. Let ϕ, ψ, and η be formulas of predicate logic that contain no free variables.

a. Let ψ be the tautology $P \vee \neg P$. Then $\phi \models \psi$ and $\neg\phi \models \psi$, for any formula ϕ. In other words, a formula ψ may be entailed both by ϕ and the negation $\neg\phi$.

b. Let ϕ be the atomic formula P, η be the atomic formula Q, and ψ be the conjunction $P \wedge Q$. Then $\phi \wedge \eta \models \psi$, but neither $\phi \models \psi$ nor $\eta \models \psi$.

c. Let ϕ be the atomic formula P, ψ also be P, and η be the formula $\neg P$. Then $\phi \models \psi$, but $\phi \vee \eta \not\models \psi$.

d. Suppose $\phi \to \psi$ is true in all models. By the semantics of implication (and since ϕ and ψ are sentences) this means that, for every model $\mathcal{M}$, $\mathcal{M} \models \psi$ holds whenever $\mathcal{M} \models \phi$ holds. Therefore, ϕ semantically entails ψ.

Exercise 2.4.11.

a. Let $\mathcal{M}$ be a model where the domain is the set of natural numbers, $P^{\mathcal{M}} = \{n \in \mathbf{N} \,|\, n > 0\}$, and $S^{\mathcal{M}} = \{(m, n) \in \mathbf{N} \times \mathbf{N} \,|\, m < n\}$. Informally, $P(x)$ means that x is positive, and $S(x, y)$ means that x is (strictly) less than y. The model $\mathcal{M}$ satisfies the formulas $\forall x \neg S(x, x)$, $\exists x P(x)$, $\forall x \exists y S(x, y)$, and $\forall x (P(x) \to \exists y S(y, x))$, which shows that the set of these four formulas is consistent.

b. The model from the preceding part also satisfies the formula $\forall x \forall y \forall z (S(x, y) \wedge S(y, z) \to S(x, z))$, which expresses that S denotes a transitive binary relation.

Exercise 2.4.12(h). The formula $\forall x \forall y\, ((P(x) \rightarrow P(y)) \wedge (P(y) \rightarrow P(x)))$ is not a theorem. For instance, it is false in the model $\mathcal{M}$ with the set of natural numbers as universe and for which $P^{\mathcal{M}}$ is the relation $\{n \mid n \text{ is even}\}$.
Exercise 2.4.12(i). The formula

$$(\forall x((P(x) \rightarrow Q(x)) \wedge (Q(x) \rightarrow P(x)))) \rightarrow ((\forall x\, P(x)) \rightarrow (\forall x\, Q(x)))$$

is a theorem. [Proof omitted]
Exercise 2.4.12(j). The formula

$$((\forall x\, P(x)) \rightarrow (\forall x\, Q(x))) \rightarrow (\forall x((P(x) \rightarrow Q(x)) \wedge (Q(x) \rightarrow P(x))))$$

is not a theorem. For example, it is false in the model $\mathcal{M}$ with the set of natural numbers as universe and for which $P^{\mathcal{M}}$ is the relation $\{n \mid n \text{ is even}\}$ and $Q^{\mathcal{M}}$ is the relation $\{n \mid n \text{ is odd}\}$.