Introduction to Predicate Logic

Part 2

CSE541

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Lecture Notes (2)

Predicate Logic Introduction

Part 2

- Predicate Logic Tautologies;
- Basic Laws of Quantifiers

- Intuitive Semantics for Predicate Logic

Basic Laws of Quantifiers
Predicate Logic Tautologies

De Morgan Laws

$\neg \forall x\, A(x) \equiv \exists x\, \neg A(x)$

$\neg \exists x\, A(x) \equiv \forall x\, \neg A(x)$

where A(x) is any formula with free variable x,

≡ means "logically equivalent"

Definability of Quantifiers

$\neg \forall x\, A(x) \equiv \exists x\, \neg A(x)$

$\neg \exists x\, A(x) \equiv \forall x\, \neg A(x)$

Example

De Morgan and other Laws Application in Mathematical Statements

$$\neg \forall x((x>0 \Rightarrow x+y > 0) \wedge \exists y\ (y>0))$$

≡ (by De Morgan's Law)

$$\exists x \neg ((x>0 \Rightarrow x+y > 0) \wedge \exists y\ (y > 0))$$

≡ (by De Morgan's Law and 1., 2., 3., 4.)

$$\exists x((x>0 \wedge x+y \leq 0) \vee \forall y(y \geq 0))$$

We used

1. $\neg (A \Rightarrow B) \equiv (A \wedge \neg B)$, 2. $\neg(A \wedge B) \equiv (\neg A \vee \neg B)$
3. $\neg(x + y) > 0) \equiv x+y \leq 0$
4. $\neg \exists y\ (y >0) \equiv \forall y \neg (y < 0)$

$\equiv \exists y\ (y \geq 0)$

Math Statement--- Logic Formula

Mathematical statement

$\neg \forall x((x>0 \Rightarrow x+y > 0) \wedge \exists y\,(y>0))$

Corresponding Logic Formula is

$\neg \forall x((P(x,c) \Rightarrow R(\, f(x,y),c)\,) \wedge \exists y\, P(y,c))$

More general; A(x), B(x) any formulas

$\neg \forall x((A(x) \Rightarrow B(\, x,y)) \wedge \exists y\, A(y))$

$\equiv \exists x\, \neg((A(x) \Rightarrow B(x,y)) \wedge \exists y\, A(y))$

$\equiv \exists x((A(x) \wedge \neg B(\, x,y)\,) \vee \neg \exists y\, A(y))$

$\equiv \exists x\, ((A(x) \wedge \neg B(\, x,y)) \vee \forall y\, \neg C(y))$

Distributivity Laws

1. **$\exists x(A(x) \vee B(x)) \equiv (\exists x A(x) \vee B(x))$**

   Existential quantifier is distributive over $\vee$, $(\exists x, \vee)$

2. **$\forall x (A(x) \wedge B(x)) \equiv (\forall x A(x) \wedge B(x))$**

Universal quantifier is distributive over $\wedge$, $(\forall x , \wedge)$

3. Existential quantifier is distributive over $\wedge$ only in one direction

$\exists x(A(x) \wedge B(x)) \Rightarrow (\exists x A(x) \wedge \exists x B(x))$

It is not true, that for any $X \neq \phi$ and any A(x), B(x)

$(\exists x A(x) \wedge \exists x B(x)) \Rightarrow \exists x(A(x) \wedge B(x))$

Example: for X = R, A(x) = $x > 0$, B(x) = x^2 we get

$\exists x (x>0) \wedge \exists x(x>0)$ is a **true** statement! in R(real numbers) and

$\exists x(x>0 \wedge x< 0)$ is a **false** statement in R!

Distributivity Laws

4. Universal quantifier is distributive over ∧ in only one direction:

$((\forall x\ A(x) \lor \forall x\ B(x)) \Rightarrow \forall x(A(x) \lor B(x)))$

Other direction counter example: take X=R (real numbers) and

$A(x) = x < 0$ $B(x) = x \geq 0$

$\forall x\ (x>0 \lor x \geq 0)$ is a **true** statement in R and

$\forall x(x<0) \lor \forall x(x \geq 0)$ is **false**

5. Universal quantifier is distributive over ⇒ in one direction:

$(\forall x(A(x) \Rightarrow B(x)) \Rightarrow (\forall x\ A(x) \Rightarrow \forall x\ B(x)))$

Other direction counter example:

Take $x \in R$, **$A(x) = x < 0$, $B(x) = x+1 >0$**

$(\forall x(x < 0) \Rightarrow \forall x(x+1 >0)$ is a False statement

Take x= -2, we get $(-2 < 0 \Rightarrow -2+1 > 0)$ False

Introduction and Elimination Laws

B- Formula without free variables

6. $\forall x(A(x) \Rightarrow B) \equiv (\exists x\, A(x) \Rightarrow B)$

7. $\exists x(A(x) \Rightarrow B) \equiv (\forall x\, A(x) \Rightarrow B)$

8. $\forall x(B \Rightarrow A(x)) \equiv (B \Rightarrow \forall x\, A(x))$

9. $\exists x(B \Rightarrow A(x)) \equiv (B \Rightarrow \exists x\, A(x))$

10. $\forall x(A(x) \vee B) \equiv (\forall x\, A(x) \vee B)$

11. $\forall x(A(x) \wedge B) \equiv (\forall x\, A(x) \wedge B)$

12. $\exists x(A(x) \vee B) \equiv (\exists x\, A(x) \vee B)$

13. $\exists x(A(x) \wedge B) \equiv (\exists x\, A(x) \wedge B)$

Remark: we prove 6 -9 from 10 – 13 + de Morgan + definability of implication

TRUTH SETS, Interpretations

We use truth sets for predicates in a set $X \neq \phi$ to define an intuitive semantics for predicate logic.

Given a set $X \neq \phi$ and a predicate P(x),

{x ∈ X: P(x)} is called a truth set for the predicate P(x) in the domain $X \neq \phi$

Example1:

Given P(x): $x+1 = 3$ **is called an interpretation of P(x)** in X.

X={1, 2, 3} then the truth set {x ∈ X: P(x)} = {x ∈ X: x+1 = 3 **} =** {2}, and we say that P(x) In TRUE in X under the interpretation P(x): x+1 = 3

Example2:

P(x): $x^2 \leq 0$ - **Interpretation of P(x)**

x = N x = N-{0}

{x: P(x) }= {0} {x:P(x)} = ϕ

TRUTH SETS

We use truth sets for predicates always for $X \neq \phi$

Conjunction:

$\{x \in X: (P(x) \wedge Q(x))\} = \{x: P(x)\} \wedge \{x: Q(x)\}$

Truth set for conjunction $(P(x) \wedge Q(x))$ is the set intersection of truth sets for its components.

Disjunction:

$\{x \in X: (P(x) \vee Q(x))\} = \{x: P(x) \vee \{x: Q(x)\}$

Truth set for disjunction $(P(x) \vee Q(x))$ is the set union of truth sets for its components.

Negation:

$\{x \in X: \neg P(x)\} = X - \{x \in X: P(x)\}$

$\neg$ is the negation and – is the set complement

Truth sets for Implication

Implication:

$\{x \in X: (P(x) \Rightarrow Q(x))\}$ $= X - \{x:P(x)\} \vee \{x : Q(x)\}$

$= -\{ x:P(x)\} \vee \{x : Q(x)\}$

$=\{x: \neg P(x)\} \vee \{x : Q(x)\}$

Example:

$\{x \in N: n>0 \Rightarrow n^2 <0\} = \{x \in N\ x \leq 0\} \vee \{\{x \in N : n^2<0\}$

$=\{0\} \vee \phi = \{0\}$

Truth Sets Semantics for Quantifiers

Definition:

$\forall x\, A(x) = T$ **iff** $\{x \in X: A(x)\} = X$

$X \neq \phi$ and $A(x)$ is any formula with x-free variable

Definition:

$\forall x\, A(x) = F$ **iff** $\{x \in X: A(x)\} \neq X$

where $X \neq \phi$ and $A(x)$ is any formula with x-free variable

Truth Sets for Quantifiers

Definition:

$\exists x\, A(x) = T$ (in $x \neq \phi$) iff $\{x \in X : A(x)\} \neq \phi$

Definition:

$\exists x\, A(x) = F$ (in $x \neq \phi$) iff $\{x \in X : A(x)\} = \phi$

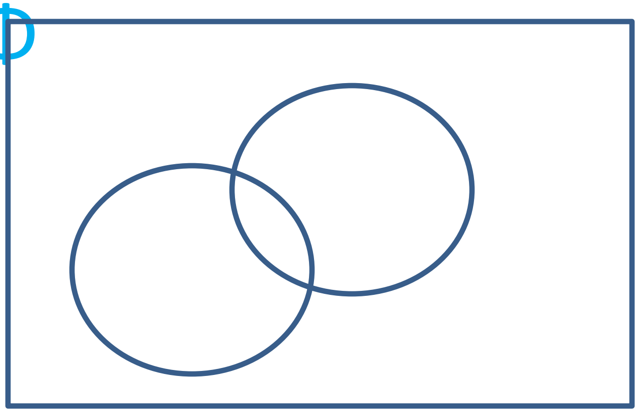
A(x) is a formula (complex) with free variable x.

Venn Diagrams For Quantifiers

$\exists x(A(x) \wedge B(x))=T$ iff **$\{x:A(X)\} \wedge \{x:B(x)\} \neq \Phi$**

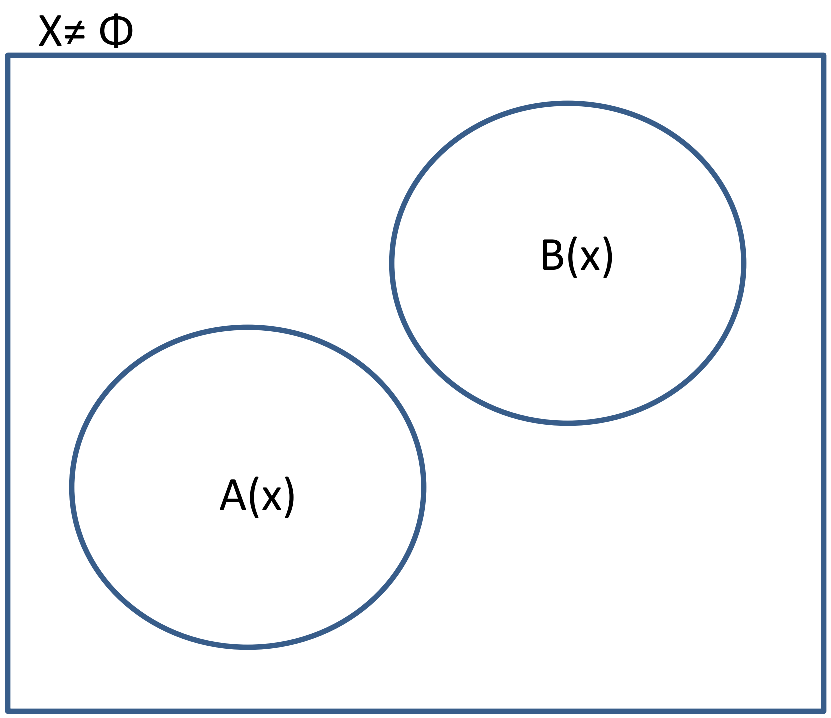
Picture

$X \neq \Phi$



$\exists x(A(x) \wedge B(x)) = F$ iff $\{x:A(x) \wedge \{x:B(x)\} = \Phi$

Picture



Remember {x:A(x)}, {x:B(x)}
Can be Φ!

X≠ Φ

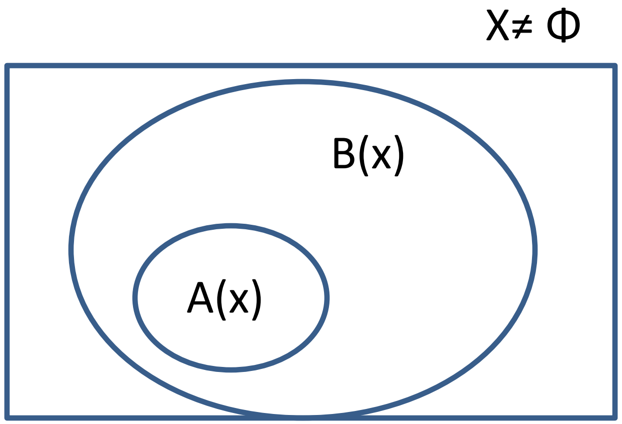
IMPLICATION

Observe that

$\forall x\ (A(x) \Rightarrow B(x)) = T$ iff $\{x \in X : A(x) \Rightarrow B(x)\} = X$

Iff **$\{x : A(x)\} \subseteq \{x : B(x)\}$**

Picture



Venn Diagrams For Implication

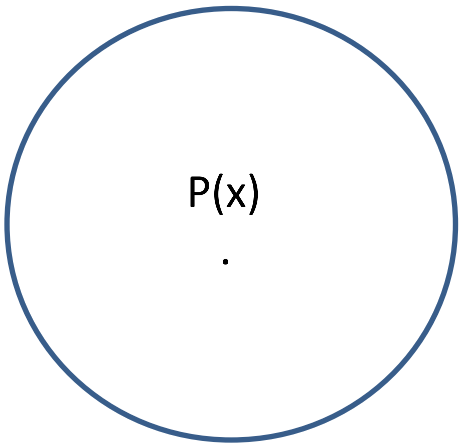
Example:

Draw a picture for a situation where (in $X \neq \Phi$)

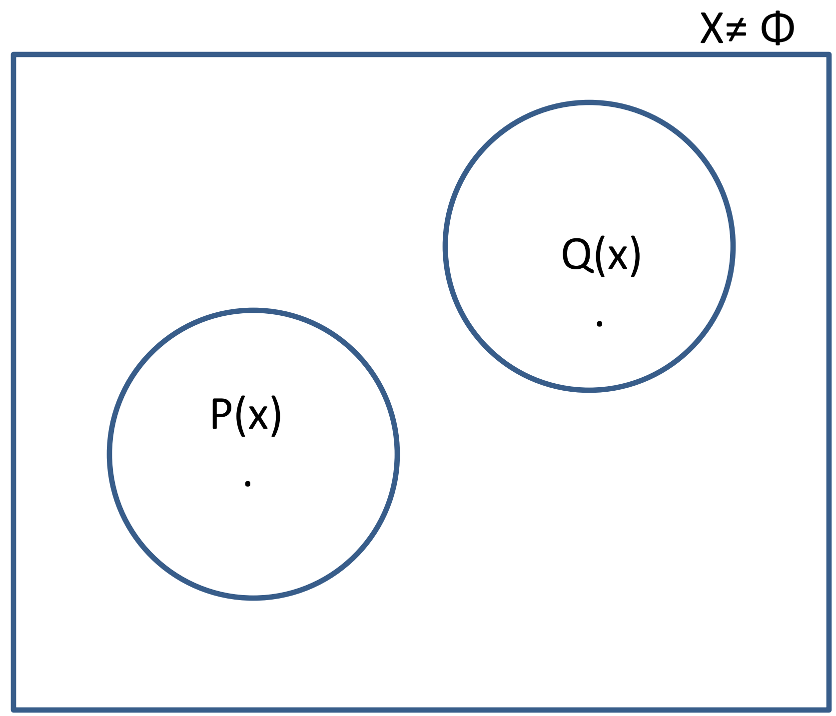
1. **$\exists x\, P(x) = T$,**
2. **$\exists x\, Q(x) = T$,**
3. **$\exists x (P(x) \wedge Q(x)) = F$ and**
4. **$\forall x\, (P(x) \vee Q(x) = F$**

1. $\exists x\, P(x) = T$ iff $\{\, x{:}P(x)\} \neq \Phi$
2. $\exists x\, Q(x) = T$ iff $\{x{:}Q\,(x)\} \neq \Phi$
3. $\{x{:}P(x)\} \wedge \{x{:}Q(x)\} \neq \Phi$
4. $\{x{:}P(x)\} \vee \{\, x{:}Q(x)\} \neq X$

Picture:



Denotes P(x) ≠ Φ



Proving Predicate Tautologies with TRUTH Sets

Prove that

$\models (\forall x\, A(x) \Rightarrow \exists x\, A(x))$

Proof:

Assume that not True

(Proof by contradiction) i.e. that there are $X \neq \Phi, A(x)$ such that.

$(\forall x\, A(x) \Rightarrow \exists x\, A(x)) = \mathbf{F}$

iff **$\forall x\, A(x)=T$** and **$\exists x\, A(x)=F$** $(A \Rightarrow B) = F$

iff (def) $x \neq \phi$

$\{x \in X : A(x)\} = X$ and **$\{x \in X : A(x)\} = \phi$**

iff $X= \phi$

Contradiction with **$x \neq \phi$,** hence proved.

Prove:

$\neg \forall x\, A(x) \equiv \exists x\, \neg A(x)$

$\exists x\, \neg A(x) = T$ in $X \neq \phi$ **iff** $\{x: \neg A(x)\} \neq \phi$ **iff**
$X - \{x: A(x)\} \neq \phi$ **iff** $\{x: A(x)\} \neq X$ **iff** $\forall x\, A(x) = F$
iff $\neg \forall x\, A(x) = T$

We assume that for any A(x),

the TRUTH set $\{x \in X: A(x)\}$ exists .

Russell Antinomy showed that that technique of TRUTH sets is not sufficient.

This is why we need a proper semantics!

Prove

$\exists x(A(x) \vee B(x)) \equiv \exists x\, A(x) \vee \exists x\, B(x)$

$\exists x(A(x)$ $\vee$ $B(x)\}$ = T iff

$\{x$: $(A(x)$ $\vee$ $B(x)\} \neq \phi$ (definition)

$= \{x$: $(A(x)\}$ $\vee$ $\{x$: $(B(x)\} \neq \phi$ iff

$\{x$: $A(x)\} \neq \phi$ or $\{x$: $B(x)\} \neq \phi$ iff

$= \exists x\, A(x)$=T or $\exists x\, B(x)$=T

We used: for any sets, A $\vee$ B $\neq \phi$ iff

A $\neq \phi$ and B $\neq \phi$