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Chapter 9
Hilbert Proof Systems
Completeness of Classical Predicate Logic

CHAPTER 9 SLIDES

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Hilbert Proof Systems
Completeness of Classical Predicate Logic

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Chapter 9
Hilbert Proof Systems
Completeness of Classical Predicate Logic

Slides Set 1

PART 1: **Reduction** Predicate Logic to Propositional Logic

Proofs of Completeness Theorem

There are several quite distinct **approaches** to the proof of the **completeness** theorem
They correspond to the ways of thinking about **proofs**

Within each of these **approaches** there are endless variations in exact formulation, corresponding to the choice of **methods** we want to use to prove the **completeness** theorem

Different basic approaches are important, though, for they lead to different **applications**

Proofs of Completeness Theorem

We have already presented two of the **approaches**
for the propositional logic, namely
Hilbert style formalizations (proof systems) in chapter 5 and
Gentzen style **automated** proof systems in chapter 6

We have also presented, for each of these approaches
several **methods** of proving the **completeness** theorem:
two very different proofs for Hilbert style proof systems
in chapter 5 and **constructive** proofs for several **automated**
Gentzen style proof systems in chapter 6

Proofs of Completeness Theorem

There are many proofs of the **completeness** theorem for predicate (first order) logic

We present here in a great detail, a version of **Henkin's proof** as included in a classic

Handbook of Mathematical Logic, North Holland Publishing Company- Amsterdam - Newy York -Oxford (1977)

It contains a **method** for reducing certain problems of **first order** logic back to problems about **propositional** logic

Proofs of Completeness Theorem

We follow **Henkin method** and give independent **proof** of **compactness theorem** for propositional logic

As the next steps we prove the most important, classical logic theorems:

Reduction to Propositional Logic Theorem,
Compactness Theorem for first-order logic,
Löwenheim-Skolem Theorem and
Gödel Completeness Theorem

They all fall out of the **Henkin method**

Proofs of Completeness Theorem

We choose this particular proof of **completeness** not only for it being one of the oldest and most classical, but also for its **connection** with the propositional logic

Moreover, the proof of the **compactness** theorem is based on **semantical** version of **syntactical** notions and techniques crucial to the second proof of **completeness** theorem for propositional logic covered in chapter 5 and hence is familiar to the reader

Reduction Predicate Logic to Propositional Logic

Reduction Predicate Logic to Propositional Logic

Let $\mathcal{L} = \mathcal{L}(\mathbf{P}, \mathbf{F}, \mathbf{C})$ be a first order language with **equality**

We assume that the sets **P, F, C** are infinitely enumerable

We also assume that it has a full set of propositional connectives, i.e.

$$\mathcal{L} = \mathcal{L}_{\{\neg, \cap, \cup, \Rightarrow\}}(\mathbf{P}, \mathbf{F}, \mathbf{C})$$

Our **goal** now is to define a **propositional logic** within

$$\mathcal{L} = \mathcal{L}(\mathbf{P}, \mathbf{F}, \mathbf{C})$$

We do it in a sequence of **steps**

Reduction Predicate Logic to Propositional Logic

First we define a special subset PF of formulas of $\mathcal{L}$ called a set of all **propositional formulas** of $\mathcal{L}$

Intuitively, these are formulas of $\mathcal{L}$ which are **direct** propositional combination of **simpler formulas**, that are atomic formulas or formulas beginning with quantifiers

These simpler formulas are called **prime formulas** and are **formally** defined as follows.

Prime Formulas

Definition

Prime formula of $\mathcal{L}$ is any formula from the set

$$\mathcal{P} = A\mathcal{F} \ \cup \ \{\forall xB : \quad B \in \mathcal{F}\} \cup \{\exists xB : \quad B \in \mathcal{F}\}$$

where the set $A\mathcal{F}$ is the set of all **atomic** formulas of $\mathcal{L}$

The set

$$\mathcal{P} \subseteq \mathcal{F}$$

is called a set of all **prime formulas** of $\mathcal{L}$

Prime Formulas

Example

The following are **prime** formulas

$R(t_1, t_2),\ \ \forall x(A(x) \Rightarrow \neg A(x)),\ \ (c = c),\ \ \exists x(Q(x, y) \cap \forall y A(y))$

The following **are not** prime formulas.

$$(R(t_1, t_2) \Rightarrow (c = c)),\ \ (R(t_1, t_2) \cup \forall x(A(x) \Rightarrow \neg A(x))$$

Given a set $\mathcal{P}$ of **prime** formulas we define in a standard way the set $P\mathcal{F}$ of **propositional** formulas of $\mathcal{L}$ as follows

Propositional Formulas of $\mathcal{L}$

Definition (Propositional Formulas)

Let $\mathcal{F}$, $\mathcal{P}$ be sets of all formulas and prime formulas of $\mathcal{L}$, respectively

The **smallest** set $P\mathcal{F} \subseteq \mathcal{F}$, such that

(i) $\mathcal{P} \subseteq P\mathcal{F}$

(ii) If $A, B \in P\mathcal{F}$, then $(A \Rightarrow B)$, $(A \cup B)$, $(A \cap B)$ and $\neg A \in P\mathcal{F}$

is called a set of all **propositional formulas** of the predicate language $\mathcal{L}$

The set $\mathcal{P}$ is called the set of all **atomic propositional** formulas of $\mathcal{L}$

Propositional Semantics for $\mathcal{L}$

We define propositional semantics for propositional formulas in $\mathcal{PF}$ as follows

Definition (Truth assignment)
Let $\mathcal{P}$ be a set of **atomic propositional** formulas of $\mathcal{L}$ and $\{T, F\}$ be the set of logical values "true" and "false"
Any function

$$v : \mathcal{P} \longrightarrow \{T,\ F\}$$

is called a **truth assignment** in $\mathcal{L}$

Propositional Semantics for $\mathcal{L}$

We **extend** v to the set $P\mathcal{F}$ of all **propositional** formulas by defining the mapping

$$v^* : P\mathcal{F} \longrightarrow \{T, F\}$$

as follows

$v^*(A) = v(A)$ for $A \in \mathcal{P}$

and for any $A, B \in P\mathcal{F}$

$v^*(A \Rightarrow B) = v^*(A) \Rightarrow v^*(B)$

$v^*(A \cup B) = v^*(A) \cup v^*(B)$

$v^*(A \cap B) = v^*(A) \cap v^*(B)$

$v^*(\neg A) = \neg v^*(A)$

Propositional Model, Tautology

Definition

A truth assignment $v : \mathcal{P} \longrightarrow \{T, F\}$ is called a **propositional model** for a formula $A \in P\mathcal{F}$ if and only if $v^*(A) = T$

Definition

For any formula $A \in P\mathcal{F}$

$A \in P\mathcal{F}$ is a **propositional tautology** of $\mathcal{L}$ if and only if $v^*(A) = T$ for all $v : \mathcal{P} \longrightarrow \{T, F\}$

For the sake of simplicity we will often say model, tautology instead propositional model, propositional tautology when there is **no confusion**

Consistent Inconsistent Sets

Definition

Given a set S of propositional formulas

We say that v is a (propositional) **model** for the set S

if and only if

v is a model for **all** formulas $A \in S$

Definition (Consistent Set)

A set $S \subseteq \mathcal{PF}$ of propositional formulas of $\mathcal{L}$ is consistent if it has a (propositional) model

Definition (Inconsistent Set)

A set $S \subseteq \mathcal{PF}$ of propositional formulas of $\mathcal{L}$ is inconsistent if it **does not** have a (propositional) model

Compactness Theorem

Compactness Theorem for Propositional Logic of $\mathcal{L}$

A set $S \subseteq P\mathcal{F}$ of propositional formulas of $\mathcal{L}$ is **consistent** if and only if every finite subset of S is **consistent**

Proof

Assume that S is a **consistent** set. By definition, it has a **model**. Its model is also a model for **all** its subsets, including all finite subsets. Hence **all** its finite subsets are **consistent**

Compactness Theorem

To prove the converse implication, i.e. the **nontrivial** half of the **Compactness Theorem** we write it in a slightly modified form. To do so, we introduce the following definition

Definition

Any set S such that **all** its finite subsets are **consistent** is called **finitely consistent**

We re-write the compactness theorem as follows.

Compactness Theorem

A set S of propositional formulas of $\mathcal{L}$ is **consistent** if and only if S is **finitely consistent**

Compactness Theorem

The nontrivial half of the **Compactness Theorem** still to be proved is now stated now as follows

Every **finitely consistent** set of propositional formulas of $\mathcal{L}$ is **consistent**

The **proof** consists of the following **four steps**

S1 We introduce the notion of a maximal finitely **consistent** set

S2 We show that every maximal finitely **consistent** set is **consistent**

We do so by constructing its **model**

Compactness Theorem

S3 We show that every **finitely consistent** set S can be extended to a **maximal** finitely consistent set S^*

We show that

for every finitely **consistent** set S there is a set S^*, such that $S \subseteq S^*$ and S^* is **maximal** finitely consistent

S4 We use steps **S2** and **S3** to justify the following **reasoning**

Compactness Theorem

Given a finitely consistent set S

We **extend it**, via construction to be defined in the step **S3**

to a **maximal** finitely consistent set S^*

By the **S2**, the set S^* is **consistent** and so is the set S

This ends the proof of the **Compactness Theorem**

Proof of Step **S1**

Here are the details and **proofs** needed for completion of steps **S1** - **S4**

Step **S1** We introduce the following definition

Definition of Maximal **Finitely Consistent** Set (MFC)

Any set

$$S \subseteq P\mathcal{F}$$

is maximal **finitely consistent** if it is finitely **consistent** and for every formula A,

$$\text{either} \quad A \in S \quad \text{or} \quad \neg A \in S$$

We use notation MFC for maximal finitely consistent set, and FC for the finitely consistent set

Proof of Step **S2**

Step **S2** consists of proving the following Lemma

MFC Lemma

Any MFC set is consistent

Proof

Given a MFC set denoted by S^*

We prove consistency of S^* by constructing **model** for it

It means we are going to **construct** a truth assignment

$$v : \mathcal{P} \longrightarrow \{T, F\}$$

such that for **all** $A \in S^*$

$$v^*(A) = T$$

Proof of Step **S2**

Observe that directly from the definition we have the following property of the the MFC sets.

Property

For any MFC set S^* and for every $A \in P\mathcal{F}$, exactly one of the formulas $A, \neg A$ belongs to S^*

In particular, for any **atomic** formula $P \in \mathcal{P}$, we have that exactly **one** of formulas $P, \neg P$ belongs to S^*

This justifies the correctness of the following definition

Proof of Step **S2**

Definition

For any MFC set S^*, a mapping

$$v : \mathcal{P} \longrightarrow \{T, F\}$$

such that

$$v(P) = \begin{cases} T & \text{if } P \in S^* \\ F & \text{if } P \notin S^* \end{cases}$$

is called a truth assignment **defined** by S^*

Proof of Step **S2**

We extend v to

$$v^* : \mathcal{PF} \longrightarrow \{T, F\}$$

in a usual, standard way and we prove that the truth assignment v is a **model** for S^*

It means we show for any $A \in \mathcal{PF}$,

$$v^*(A) = \begin{cases} T & \text{if } A \in S^* \\ F & \text{if } A \notin S^* \end{cases}$$

We **prove** it by induction on the degree of the formula A as follows.

Proof of Step **S2**

The **base case** of **atomic** formula $P \in \mathcal{P}$ follows immediately from the definition of v

Inductive Case: $A = \neg C$

1. Assume that $A \in S^*$

This means $\neg C \in S^*$ and by the MFC Property we have that $C \notin S^*$. So by the inductive assumption $v^*(C) = F$ and we get

$$v^*(A) = v^*(\neg C) = \neg v^*(C) = \neg F = T$$

2. Assume now that $A \notin S^*$.

By MFC Property we have that $C \in S^*$

By the inductive assumption $v^*(C) = T$ and

$$v^*(A) = v^*(\neg C) = \neg v^*(T) = \neg T = F$$

Proof of Step **S2**

We proved that for any formula $A \in P\mathcal{F}$,

$$v^*(\neg A) = \begin{cases} T & \text{if } \neg A \in S^* \\ F & \text{if } \neg A \notin S^* \end{cases}$$

Inductive Case: $A = (B \cup C)$

1. Assume that $A \in S^*$. i.e. $(B \cup C) \in S^*$

It is enough to prove that in this case $B \in S^*$ or $C \in S^*$, because then from the inductive assumption $v^*(B) = T$ and

$v^*(B \cup C) = v^*(B) \cup v^*(C) = T \cup v^*(C) = T$ for any C

The case $C \in S^*$ is similar

Proof of Step **S2**

Assume that $(B \cup C) \in S^*$, $B \notin S^*$ and $C \notin S^*$

Then by MFC Property we have that $\neg B \in S^*$, $\neg C \in S^*$ and consequently the set

$$\{(B \cup C), \neg B, \neg C\}$$

is a finite inconsistent subset of S^*, what **contradicts** the fact that S^* is finitely consistent

2. Assume now that $(B \cup C) \notin S^*$

By MFC Property, $\neg(B \cup C) \in S^*$ and by already proven **case** of $A = \neg C$ we have that $v^*(\neg(B \cup C)) = T$

But $v^*(\neg(B \cup C)) = \neg v^*((B \cup C)) = T$

This means that $v^*((B \cup C)) = F$, what **ends** the proof of this case

Step **S3**

The remaining cases of $A = (B \cap C)$ and $A = (B \Rightarrow C)$ are similar to the above and are left to the as an exercise

This **ends** the proof of MFC **Lemma** and completes the step **S2**

S3: Maximal finitely consistent (MFC) extension S^*

Given a finitely consistent set S

We **construct** the MFC **extension** S^* of the set S as follows

Proof of Step **S3**

The set of all formulas of $\mathcal{L}$ is infinitely countable and so is the set $P\mathcal{F}$. We assume that the set $P\mathcal{F}$ of all **propositional** formulas form a one-to-one sequence

$$(*) \quad A_1,\ A_2,\ \ldots,\ A_n,\ \ldots,$$

We **define** a chain

$$(**) \quad S_0 \subseteq S_1 \subseteq S_2,\ \ldots,\ \subseteq S_n \subseteq,\ \ldots$$

of **extensions** of the set S as follows

$$S_0 = S$$

$$S_{n+1} = \begin{cases} S_n \cup \{A_n\} & \text{if } S_n \cup \{A_n\} \text{ is } \textbf{finitely consistent} \\ S_n \cup \{\neg A_n\} & \text{otherwise.} \end{cases}$$

Proof of Step **S3**

We take

$$S^* = \bigcup_{n \in N} S_n$$

Obviously $S \subseteq S^*$ also is MFC as clearly and for every A, either $A \in S^*$ or $\neg A \in S^*$

To complete the **proof** that S^* is MFC set we have to show that it is **finitely** consistent

First, let observe that **if** all sets S_n are **finitely** consistent, **so is** the set $S^* = \bigcup_{n \in N} S_n$. Namely, let

$$S_F = \{B_1, ..., B_k\}$$

be a **finite** subset of S^*

Proof of Step **S3**

This means that there are sets $S_{i_1}, ...S_{i_k}$ in the chain $(**)$ such that

$$B_m \in S_{i_m} \quad \text{for} \quad m = 1, \dots k$$

Let $M = max(i_1, ...i_k)$. Obviously

$$S_F \subseteq S_M$$

and the set S_M is **finitely** consistent as an element of the chain $(**)$. This **proves** that **if** all sets S_n are **finitely consistent**, so is S^*

Now we have to **prove only** that **all** sets S_n **are** FC (finitely consistent)

We carry the proof by induction over the length of the chain

Proof of Step **S3**

Base Case

$S_0 = S$, so it is FC (finitely consistent) by assumption of the **Compactness Theorem**

Inductive Step

Assume now that S_n is FC (finitely consistent)

We prove that S_{n+1} is FC

We have two cases to consider

Case 1 $S_{n+1} = S_n \cup \{A_n\}$

Then S_{n+1} is FC by the definition of the chain

Case 2 $S_{n+1} = S_n \cup \{\neg A_n\}$

Observe that this can happen only if $S_n \cup \{A_n\}$ is **not** FC, i.e. there is a finite subset $S_n' \subseteq S_n$, such that $S_n' \cup \{A_n\}$ is **not** consistent

Proof of Step **S3**

Suppose now that S_{n+1} is **not** FC

This means that there is a finite subset $S_n'' \subseteq S_n$, such that $S_n'' \cup \{\neg A_n\}$ is **not** consistent

Take $S_n' \cup S_n''$. It is a finite subset of S_n so it is consistent by the inductive assumption

Let v be a **model** of $S_n' \cup S_n''$

Then **one of** $v^*(A), v^*(\neg A)$ **must** be T

This **contradicts** the inconsistency of both

$$S_n' \cup \{A_n\} \quad \text{and} \quad S_n' \cup \{\neg A_n\}$$

Thus, in ether case, S_{n+1} is FC

We hence proved that **all** sets S_n **are** FC (finitely consistent)

Compactness Theorem

This **completes** the proof of the step **S3**

We complete the proof of the **Compactness Theorem**
for propositional logic of $\mathcal{L}$ via the following argument
as presented in the step **S4**
Given a finitely consistent set S
We **extend** it, via construction defined in the step **S3** to
a **maximal** finitely consistent set S^*
By the **S2**, the set S^* is **consistent** and so is the set S

This ends the proof of the **Compactness Theorem**

Chapter 9
Hilbert Proof Systems
Completeness of Classical Predicate Logic

Slides Set 2

PART 2:

Henkin Method

Reduction to Propositional Logic Theorem,

Compactness Theorem,

Löwenheim-Skolem Theorem

Henkin Method

Propositional tautologies within $\mathcal{L}$ barely scratch the surface of the collection of **predicate** (first -order) tautologies

For **example** the following first-order formulas are **propositional** tautologies

$$(\exists x A(x) \cup \neg \exists x A(x)), \quad (\forall x A(x) \cup \neg \forall x A(x)$$

$$(\neg(\exists x A(x) \cup \forall x A(x)) \Rightarrow (\neg \exists x A(x) \cap \neg \forall x A(x)))$$

but the following are **predicate** (first order) tautologies that are not **propositional** tautologies

$$\forall x (A(x) \cup \neg A(x))$$

$$(\neg \forall x A(x) \Rightarrow \exists x \neg A(x))$$

Henkin Method

To stress the difference between the **propositional** tautologies of a propositional language and **predicate** tautologies the word **tautology** is used only for the propositional tautologies of a propositional language

The word a **valid formula** is used for the predicate tautologies in this case

We use here **both** notions, with preference to word **predicate tautology** or **tautology** for short when there is **no room** for misunderstanding

To make sure that there is no misunderstandings we **remind** the following basic definitions from chapter 8

Basic Definitions

Given a first order language $\mathcal{L}$ with the set of variables VAR and the set of formulas $\mathcal{F}$. Let

$$\mathcal{M} = [M, I]$$

be a **structure** for the language $\mathcal{L}$, with the **universe** M and the **interpretation** I and let

$$s : VAR \longrightarrow M$$

be an **assignment** of $\mathcal{L}$ in M

Here are some basic **definitions**

Basic Definitions

D1. A is **satisfied** in $\mathcal{M}$

Given a structure $\mathcal{M} = [M, I]$, we say that a formula A is **satisfied** in $\mathcal{M}$ if **there is** an assignment $s : VAR \longrightarrow M$ such that

$$(\mathcal{M}, s) \models A$$

D2. A is **true** in $\mathcal{M}$

Given a structure $\mathcal{M} = [M, I]$, we say that a formula A is **true** in $\mathcal{M}$ if

$$(\mathcal{M}, s) \models A$$

for **all** assignments $s : VAR \longrightarrow M$

Basic Definitions

D3. Model $\mathcal{M}$

If A is **true** in a structure $\mathcal{M} = [M, I]$, then $\mathcal{M}$ is called a **model** for A

We denote it as

$$\mathcal{M} \models A$$

D4. A is **predicate tautology (valid)**

A formula A is a **predicate** tautology (valid) if it is **true** in all structures $\mathcal{M} = [M, I]$, i.e. if all structures are **models** of A

We use use the term **predicate tautology** and and denote it, when there is no confusion with propositional case as

$$\models A$$

Basic Definitions

Case: A is a **sentence**

If the formula A is a sentence, then the truth or falsity of the statement $(\mathcal{M}, s) \models A$ is completely independent of s

Thus we write

$$\mathcal{M} \models A$$

and read $\mathcal{M}$ is a **model** of A, if for some (hence every) valuation s

$$(\mathcal{M}, s) \models A$$

D5. **Model** of a **set** S of formulas

$\mathcal{M}$ is a model of a set S (of sentences) if and only if $\mathcal{M} \models A$ for all $A \in S$. We write it

$$\mathcal{M} \models S$$

Predicate and Propositional Models

Relationship

Given a predicate language $\mathcal{L}$

The **predicate models** for $\mathcal{L}$ are defined in terms of

structures $\mathcal{M} = [M, I]$ and assignments $s : VAR \longrightarrow M$

The **propositional models** for $\mathcal{L}$ are defined in terms of of

truth assignments $v : \mathcal{P} \longrightarrow \{T, F\}$

The **relationship** between the predicate models and propositional models is established by the following **Lemma**

Relationship Lemma

Lemma

Let $\mathcal{M} = [M, I]$ be a structure for the language $\mathcal{L}$ and let $s : VAR \longrightarrow M$ an assignment in $\mathcal{M}$

There is a truth assignment

$$v : \mathcal{P} \longrightarrow \{T, F\}$$

such that for all formulas A of $\mathcal{L}$,

$$(\mathcal{M}, s) \models A \quad \text{if and only if} \quad v^*(A) = T$$

In particular, for any set S of sentences of $\mathcal{L}$,

if $\mathcal{M} \models S$ then S is **consistent** in the **propositional** sense

Relationship Lemma Proof

Proof

For any prime formula $A \in P$ we define

$$v(A) = \begin{cases} T & \text{if } (\mathcal{M}, s) \models A \\ F & \text{otherwise.} \end{cases}$$

Since every formula in $\mathcal{L}$ is either prime or is built up from prime formulas by means of propositional connectives, the conclusion is obvious

Relationship Lemma

Observe, that the converse of the **Lemma** implication:

if $\mathcal{M} \models S$ then S is **consistent** in the propositional sense

is **far** from **true**

Consider a set

$$S = \{\forall x(A(x) \Rightarrow B(x)), \forall x A(x), \exists x \neg B(x)\}$$

All formulas of S are different prime formulas

So S has and obvious **model** and hence is consistent in the propositional sense

Obviously S has **no** predicate **model**

Language with Equality

Definition (Language with Equality)

Let $\mathcal{L}$ be a **predicate** (first order) language with **equality**

$$\mathcal{L} = \mathcal{L}_{\{\neg,\cap,\cup,\Rightarrow\}}(\mathbf{P}, \mathbf{F}, \mathbf{C})$$

Equality Axioms

For any free variable or constant of $\mathcal{L}$, i.e for any $u, w, u_i, w_i \in (VAR \cup \mathbf{C})$,

E1 $\quad u = u$

E2 $\quad (u = w \Rightarrow w = u)$

E3 $\quad ((u_1 = u_2 \cap u_2 = u_3) \Rightarrow u_1 = u_3)$

E4

$((u_1 = w_1 \cap ... \cap u_n = w_n) \Rightarrow (R(u_1, ..., u_n) \Rightarrow R(w_1, ..., w_n)))$

E5

$((u_1 = w_1 \cap ... \cap u_n = w_n) \Rightarrow (t(u_1, ..., u_n) \Rightarrow t(w_1, ..., w_n)))$

where $R \in \mathbf{P}$ and $t \in \mathbf{T}$, i.e. R is an arbitrary n-ary relation symbol of $\mathcal{L}$ and $t \in \mathbf{T}$ is an arbitrary n-ary term of $\mathcal{L}$

Language with Equality

Observe that given any structure $\mathcal{M} = [M, I\,]$

We have by simple verification that

for all $s : VAR \longrightarrow M$, and

for all $A \in \{E1, E2, E3, E4, E5\}$,

$$(\mathcal{M}, s) \models A$$

This proves the following

Fact

All **equality axioms** are predicate tautologies of $\mathcal{L}$

This is why we call logic with **equality axioms** added to it, still just a **logic**

Henkin's Witnessing Expansion of $\mathcal{L}$

Henkin's Witnessing Expansion

Now we are going to **define** notions that are **fundamental** to the Henkin's technique for **reducing** predicate logic to propositional logic

The **first** one is that of witnessing expansion of $\mathcal{L}$

We construct an **expansion** of the language $\mathcal{L}$ by **adding** a set of new constants to it

It means the we **add** a specially constructed the set C to the set $\mathbf{C}$ of constants of $\mathcal{L}$ such that

$$C \cap \mathbf{C} = \emptyset$$

The language such **constructed** is called witnessing expansion of the language $\mathcal{L}$

The construction of the **expansion** is described as follows

Henkin's Witnessing Expansion

Definition
For any predicate language

$$\mathcal{L} = \mathcal{L}(\mathbf{P}, \mathbf{F}, \mathbf{C})$$

the language

$$\mathcal{L}(C) = \mathcal{L}(\mathbf{P}, \mathbf{F}, \mathbf{C} \cup C))$$

is called a **witnessing expansion** of $\mathcal{L}$

The set C of **new** constants and the language $\mathcal{L}(C)$ defined by the **construction** described below

We denote $\mathcal{L}(C)$ as

$$\mathcal{L}(C) = \mathcal{L} \cup C$$

Construction of the witnessing expansion of $\mathcal{L}$

We **define** the set C of **new** constants by constructing (by induction) an infinite sequence

$$C_0, C_1, ..., C_n, \dots$$

of **sets of constants** together with an infinite sequence

$$\mathcal{L}_0, \mathcal{L}_1, ..., \mathcal{L}_n, \dots$$

of **languages** as follows

$$C_0 = \emptyset \quad \text{and} \quad \mathcal{L}_0 = \mathcal{L} \cup C_0 = \mathcal{L}$$

We denote by

$$A[x]$$

the fact that the formula A has **exactly one** free variable

Henkin's Witnessing Expansion

For each such a formula $A[x]$ we introduce a distinct **new constant** denoted by

$$c_{A[x]}$$

We **define**

$$C_1 = \{c_{A[x]} : \ A[x] \in \mathcal{L}_0 \} \quad \text{and} \quad \mathcal{L}_1 = \mathcal{L} \cup C_1$$

Assume that we have already defined the set C_n of constants and the language $\mathcal{L}_n$

To each formula $A[x]$ of $\mathcal{L}_n$ which **is not** already a formula of $\mathcal{L}_{n-1}$ we assign distinct **new** constant symbol

$$c_{A[x]}$$

Henkin's Witnessing Expansion

We write it informally as $A[x] \in (\mathcal{L}_n - \mathcal{L}_{n-1})$ to denote that $A[x]$ of $\mathcal{L}_n$ which **is not** already a formula of $\mathcal{L}_{n-1}$

We define

$$C_{n+1} = C_n \cup \{c_{A[x]} : \; A[x] \in (\mathcal{L}_n - \mathcal{L}_{n-1})\}$$

$$\mathcal{L}_{n+1} = \mathcal{L} \cup C_{n+1}$$

We put

$$(*) \quad C = \bigcup C_n \quad \text{and} \quad \mathcal{L}(C) = \mathcal{L} \cup C$$

For any formula $A(x)$, a constant $c_{A[x]} \in C$ as defined by $(*)$ is called a **witnessing constant**

Reduction to Propositional Logic Theorem

Henkin Axioms

Definition(Henkin Axioms)

The following **sentences**

H1 $(\exists x A(x) \Rightarrow A(c_{A[x]}))$

H2 $(A(c_{\neg A[x]}) \Rightarrow \forall x A(x))$

are called **Henkin axioms**

The informal idea behind the Henkin axioms is the following

The axiom **H1** says:

If $\exists x A(x)$ is **true** *in a structure, choose an element a satisfying $A(x)$ and give it a* **new name** $c_{A[x]}$

The axiom **H2** says:

If $\forall x A(x)$ is **false**, *choose a counter example b and call it by a* **new name** $c_{\neg A[x]}$

Quantifiers Axioms

Definition (Quantifiers Axioms)

The following **sentences**

Q1 $(\forall x A(x) \Rightarrow A(t))$

where t is a closed term of $\mathcal{L}(C)$

Q2 $(A(t) \Rightarrow \exists x A(x))$

where t is a closed term of $\mathcal{L}(C)$

re called **quantifiers axioms**

Observe that the quantifiers axioms **Q1, Q2** obviously are predicate tautologies

Henkin Set

Any set of **sentences** of $\mathcal{L}(C)$ which are either Henkin axioms or quantifiers axioms is called the **Henkin set** and denoted by

$$S_{Henkin}$$

The sentences of S_{Henkin} are obviously **not true** in every $\mathcal{L}(C)$-structure. But we are going to show now the following

Every $\mathcal{L}$-structure can be **transformed** into an $\mathcal{L}(C)$-structure which is a **model** of S_{Henkin}

Before we do so we need to introduce two new notions

Reduct and Expansion

Given two languages $\mathcal{L}$ and $\mathcal{L}'$ such that

$$\mathcal{L} \subseteq \mathcal{L}'$$

Let $\mathcal{M}' = [M, I']$ be a structure for $\mathcal{L}'$. The structure

$$\mathcal{M} = [M, I' \mid \mathcal{L}]$$

is called the **reduct** of $\mathcal{M}'$ to the language $\mathcal{L}$ and $\mathcal{M}'$ is called the **expansion** of $\mathcal{M}$ to the language $\mathcal{L}'$

Thus the reduct of $\mathcal{M}'$ and the expansion of $\mathcal{M}$ are the same except that $\mathcal{M}'$ **assigns** meanings to the symbols in $\mathcal{L} - \mathcal{L}'$

Reduct and Expansion Lemma

Lemma

Let $\mathcal{M} = [M, I]$ be any structure for the language $\mathcal{L}$ and let $\mathcal{L}(C)$ be the **witnessing expansion** of $\mathcal{L}$

There is an **expansion** $\mathcal{M}' = [M, I']$ of $\mathcal{M} = [M, I]$ such that

$$\mathcal{M}' \models S_{Henkin}$$

Proof

In order to define the **expansion** of $\mathcal{M}$ to $\mathcal{M}'$ we have to **define** the interpretation I' for the symbols of the language $\mathcal{L}(C) = \mathcal{L} \cup C$, such that I' **restricted** to $\mathcal{L}$ is the interpretation I , i.e. such that

$$I' \mid \mathcal{L} = I$$

Lemma Proof

This means that we have to define $c_{I'}$ for all $c \in C$

By the definition, $c_{I'} \in M$, so this also means that we have to **assign** the elements of M to all constants $c \in C$ in such a way that the resulting expansion is a **model** for all sentences from S_{Henkin}

The **quantifier axioms** are predicate tautologies so they are going to be **true** regardless. So we have to worry only about the **Henkin axioms**

Lemma Proof

Observe now that if the **Lemma** holds for the Henkin axiom

H1 $(\exists x A(x) \Rightarrow A(c_{A[x]}))$

then it must hold for the axiom **H2**

Namely, let's consider the axiom **H2**:

$$(A(c_{\neg A[x]}) \Rightarrow \forall x A(x))$$

Assume that $A(c_{\neg A[x]})$ is **true** in the expansion $\mathcal{M}'$, i.e. that

$$\mathcal{M}' \models A(c_{\neg A[x]}) \quad \text{and that} \quad \mathcal{M}' \not\models \forall x A(x)$$

This means that

$$\mathcal{M}' \models \neg\forall x A(x)$$

and by the De Morgan Laws

$$\mathcal{M}' \models \exists x \neg A(x)$$

Lemma Proof

But we have assumed that $\mathcal{M}'$ is a **model** for **H1**

In particular

$$\mathcal{M}' \models (\exists x \neg A(x) \Rightarrow \neg A(c_{\neg A[x]}))$$

and hence as $\mathcal{M}' \models \exists x \neg A(x)$ we have that

$$\mathcal{M}' \models \neg A(c_{\neg A[x]})$$

This **contradicts** the assumption that

$$\mathcal{M}' \models A(c_{\neg A[x]})$$

Thus we **proved** that

if $\mathcal{M}'$ is a **model** for all axioms of the type **H1**, it is also a **model** for all axioms of the type **H2**

Lemma Proof

We **define** now $c_{I'}$ for all $c \in C$, where

$$C = \bigcup C_n$$

We do so by **induction** on n

Base case: $n = 1$ and $c_{A[x]} \in C_1$

By definition,

$$C_1 = \{c_{A[x]} : \; A[x] \in \mathcal{L}\}$$

In this case we have that $\exists x A(x) \in \mathcal{L}$ and hence the notion

$$\mathcal{M} \models \exists x A(x)$$

is well defined, as $\mathcal{M} = [M, I]$ is the structure for the language $\mathcal{L}$

Lemma Proof

As we consider arbitrary structure $\mathcal{M}$, there are two possibilities:

$$\mathcal{M} \models \exists x A(x) \quad \text{or} \quad \mathcal{M} \not\models \exists x A(x)$$

We **define** $c_{I'}$, for all $c \in C_1$ as follows

If $\mathcal{M} \models \exists x A(x)$, then $(\mathcal{M}, v') \models A(x)$ for certain $v'(x) = a \in M$. We set

$$(c_{A[x])})_{I'} = a$$

If $\mathcal{M} \not\models \exists x A(x)$, we set

$$(c_{A[x])})_{I'} \text{ arbitrarily}$$

Lemma Proof

This makes all the positive **H1** type Henkin axioms about the $c_{A[x]} \in C_1$ **true**, i.e.

$$\mathcal{M} = (M, I) \models (\exists x A(x) \Rightarrow A(c_{A[x]}))$$

But once $c_{A[x]} \in C_1$ are all interpreted in M, then the notion

$$\mathcal{M}' \models A$$

is defined for all formulas $A \in \mathcal{L} \cup C_1$

We carry the same argument and **define** $c_{I'}$, for all $c \in C_2$ and so on . . .

Lemma Proof

The **inductive step** is performed in the exactly the same way as the one above

Observe that we have aleady we **proved** that
if $\mathcal{M}'$ is a **model** for all axioms of the type **H1**, it is also
a **model** for all axioms of the type **H2**

Hence this ends the proof of the **Lemma**

Canonical Structure

Definition (Canonical Structure)

Given a structure $\mathcal{M} = [M, I]$ for the language $\mathcal{L}$
The **expansion**

$$\mathcal{M}' = [M, I']$$

of $\mathcal{M} = [M, I]$ is called a **canonical structure** for $\mathcal{L}(C)$
if all $a \in M$ are **denoted** by some $c \in C$. That is

$$M = \{c_{I'} : c \in C\}$$

Now we are ready to state and prove a theorem that provides the essential step in the proof of the **completeness theorem** for predicate logic

The Reduction to Propositional Logic

Theorem (The Reduction Theorem)

Let $\mathcal{L} = \mathcal{L}(\mathbf{P}, \mathbf{F}, \mathbf{C})$ be a predicate language and let $\mathcal{L}(C) = \mathcal{L}(\mathbf{P}, \mathbf{F}, \mathbf{C} \cup C)$ be a witnessing expansion of $\mathcal{L}$

For any set S of sentences of $\mathcal{L}$ the following conditions are equivalent

(i) S has a **model**, i.e. there is a structure $\mathcal{M} = [M, I]$ for the language $\mathcal{L}$ such that $\mathcal{M} \models A$ for all $A \in S$

(ii) There is a **canonical structure** $\mathcal{M} = [M, I]$ for $\mathcal{L}(C)$ which is a **model** for S, i.e. such that $\mathcal{M} \models A$ for all $A \in S$

(iii) The set $S \cup S_{Henkin} \cup EQ$ is **consistent** in sense of propositional logic, where EQ denotes the equality axioms $E1 - E5$

Reduction Theorem Proof

Proof

We have to prove that the conditions **(i), (ii), (iii)** of the theorem are equivalent

The implication **(ii)** $\rightarrow$ **(i)** is immediate

The implication $(\mathbf{i}) \rightarrow (\mathbf{iii})$ follows from the **Lemma**

We have to prove only the implication **(iii)** $\rightarrow$ **(ii)**

Assume **(iii)** , i.e. that the set $S \cup S_{Henkin} \cup EQ$ is consistent in sense of propositional logic and let v be a truth assignment to the prime sentences of $\mathcal{L}(C)$, such that

$$v^*(A) = T \quad \text{for all} \quad A \in S \cup S_{Henkin} \cup EQ$$

Reduction Theorem Proof

To prove the theorem, we construct a **canonical** $\mathcal{L}(C)$ structure $\mathcal{M} = [M, I]$ such that, for all sentences A of $\mathcal{L}(C)$,

$$\mathcal{M} \models A \quad \text{if and only if} \quad v^*(A) = T$$

By assumption, the truth assignment v is a propositional **model** for the set S_{Henkin}, so v^* satisfies the following conditions:

(i) $v^*(\exists x A(x)) = T$ if and only if $v^*(A(c_{A[x]})) = T$

(ii) $v^*(\forall x A(x)) = T$ if and only if $v^*(A(t)) = T$

for all **closed** terms t of $\mathcal{L}(C)$

Reduction Theorem Proof

The conditions (i) and (ii) allow us to construct the **canonical** $\mathcal{L}(C)$ **model** $\mathcal{M} = [M, I]$ out of the constants in C in the following way

To define $\mathcal{M} = [M, I]$ we must

(1.) specify the **universe** M of $\mathcal{M}$

(2.) define, for each n-ary predicate symbol $R \in \mathbf{P}$, the **interpretation** R_I as an n-argument relation in M

(3.) define, for each n-ary function symbol $f \in \mathbf{F}$, the **interpretation** $f_I : M^n \rightarrow M$, and

(4.) define, for each constant symbol c of $\mathcal{L}(C)$, i.e. $c \in \mathbf{C} \cup C$, its **interpretation** as element $c_I \in M$

Reduction Theorem Proof

The construction of the structure

$$\mathcal{M} = [M, I]$$

must be such that the condition

$$\text{(CM)} \qquad \mathcal{M} \models A \quad \text{if and only if} \quad v^*(A) = T$$

holds for for all sentences A of $\mathcal{L}(C)$

This condition (CM) tells us how to construct the definitions (1.) - (4.) above

Reduction Theorem Proof

Here are the definitions

(1.) **Definition** of the **universe** M of $\mathcal{M}$

In order to define the universe M we **first** define a relation $\approx$ on C as follows

$$c \approx d \quad \text{if and only if} \quad v(c = d)) = T$$

The **equality axioms** EQ guarantee that the relation $\approx$ is **equivalence** relation on C. Here is the proof

Reflexivity of $\approx$

All equality axioms EQ are predicate **tautologies**, so $v(c = d)) = T$ by axiom E1 and we have

$$c \approx c \quad \text{for all} \quad c \in C$$

Reduction Theorem Proof

Symmetry condition

$$\text{if} \quad c \approx d, \quad \text{then} \quad d \approx c$$

holds by axiom E2

Assume $c \approx d$, by definition $v(c = d)) = T$

By axiom E2

$$v^*((c = d \Rightarrow d = c)) = v(c = d) \Rightarrow v(d = c) = T$$

i.e. $T \Rightarrow v(d = c) = T$

This is possible **only if** $v(d = c) = T$

This proves that $d \approx c$

Reduction Theorem Proof

We prove **transitivity** in a similar way

Assume now that $c \approx d$ and $d \approx e$

By the **axiom** E3 we have that

$$v^{*}(((c = d \cap d = e) \Rightarrow c = e)) = T$$

Since $v(c = d)) = T$ and $v(d = e)) = T$ by the assumption $c \approx d$ and $d \approx e$, we evaluate

$v^{*}((c = d \cap d = e) \Rightarrow c = e) = (T \cap T \Rightarrow c = e) =$

$(T \Rightarrow c = e) = T$ and get that $(c = e) = T$ and hence

$$d \approx e$$

Reduction Theorem Proof

We denote by $[c]$ the equivalence class of c and we define the **universe** M of $\mathcal{M}$ as

$$M = \{[c] : \ c \in C\}$$

(2.) **Definition** of $R_I \subseteq M^n$

Let M be the the **universe** defined above

We define $R_I \subseteq M^n$ as follows

$([c_1], [c_2], \ \ldots, [c_n]) \in R_I$ if and only if $v(R(c_1, c_2, \ \ldots, c_n)) = T$

We have to prove now that R_I is *well defined* by the condition above

Reduction Theorem Proof

In order to prove that R_I is **well defined** we must verify:

$$\text{if } [c_1] = [d_1], \ldots, [c_n] = [d_n] \text{ and } ([c_1], [c_2], \ldots, [c_n]) \in R_I$$

$$\text{then } ([d_1], [d_2], \ldots, [d_n]) \in R_I$$

We have by the **axiom** E4 that

$$v^*(((c_1 = d_1 \cap ... c_n = d_n) \Rightarrow (R(c_1, .., c_n) \Rightarrow R(d_1, .., d_n)))) = T$$

By the assumption $[c_1] = [d_1], \ldots, [c_n] = [d_n]$ we have that

$$v(c_1 = d_1) = T, \ldots, v(c_n = d_n) = T$$

Reduction Theorem Proof

By the assumption $([c_1], [c_2], \ldots, [c_n]) \in R_I$, we have that

$$v(R(c_1, ..., c_n)) = T$$

Hence the **axiom** E4 condition becomes

$$(T \Rightarrow (T \Rightarrow v(R(d_1, ..., d_n)))) = T$$

It holds only when $v(R(d_1, ..., d_n)) = T$

and so we **proved** that

$$([d_1], [d_2], \ldots, [d_n]) \in R_I$$

Reduction Theorem Proof

(3.) **Definition** of $f_I : \ M^n \to M$

Let $c_1, \ c_2, \ \ldots, \ c_n \in C$ and $f \in \mathbf{F}$

We **claim** that **there is** $c \in C$ such that

$$f(c_1, \ c_2, \ \ldots, \ c_n) = c \quad \text{and} \quad v(f(c_1, \ c_2, \ \ldots, \ c_n) = c) = T$$

For consider the formula

$$A[x] \quad \text{given by} \quad f(c_1, \ c_2, \ \ldots, \ c_n) = x$$

$$\text{If} \quad v^*(\exists x A(x)) = v^*(\exists x \ f(c_1, \ c_2, \ \ldots, \ c_n) = x) = T$$

we want to **prove**

$$v^*(A(c_{A[x]})) = T \quad \text{i.e.} \quad v(f(c_1, \ c_2, \ \ldots, \ c_n) = c_A) = T$$

Reduction Theorem Proof

So suppose that $v(f(c_1, c_2, \ldots, c_n) = c_A) = F$

But one member of he Henkin set S_{Henkin} is the sentence

$$(A(f(c_1, c_2, \ldots, c_n)) \Rightarrow \exists x A(x))$$

so we must have that

$$v^*(A(f(c_1, c_2, \ldots, c_n))) = F$$

But this says that v assigns F to the atomic sentence

$$f(c_1, c_2, \ldots, c_n) = f(c_1, c_2, \ldots, c_n)$$

Reduction Theorem Proof

By the axiom E1, $v(c_i = c_i) = T$ for $i = 1, 2 \ldots n$

By axiom E5 we have that

$$(v^*(c_1 = c_1 \cap \ldots \ c_n = c_n) \Rightarrow v^*(f(c_1, \ \ldots, \ c_n) = f(c_1, \ldots, c_n))) = T$$

there is $c \in C$ such that

$$f(c_1, \ c_2, \ \ldots, \ c_n) = c \quad \text{and} \quad v(f(c_1, \ c_2, \ \ldots, \ c_n) = c) = T$$

We hence **define**

$f_I(([c_1], \ldots, [c_n]) = [c]$ for c such that $v(f(c_1, \ldots, \ c_n) = c) = T$

The argument similar to the one used in (2.) proves that f_I is **well defined**

Reduction Theorem Proof

(4.) **Definition** of $c_I \in M$

For any $c \in C$ we take

$$c_I = [c]$$

If $d \in \mathbf{C}$, then an argument similar to that used on (3.) shows that **there is** $c \in C$ such that $v(d = c) = T$, i.e. $d \approx c$, so we put

$$d_I = [c]$$

We hence completed the construction of the **canonical structure** $\mathcal{M} = [M, I]$

Reduction Theorem Proof

Observe that directly from the definition of the **canonical structure** $\mathcal{M} = [M, I]$ we have that the property

$$\text{(CM)} \qquad \mathcal{M} \models A \quad \text{if and only if} \quad v^*(A) = T$$

holds for **atomic** propositional sentences, i.e. we proved that

$$\mathcal{M} \models B \;\; \text{if and only if} \;\; v^*(B) = T \text{ for sentences } B \in \mathcal{P}$$

To complete the proof of the **Reduction Theorem** we prove now that the property (CM) holds for all other sentences

We carry the proof by induction on length of formulas

The **Base Case** is already proved

The **Inductive Case** is as follows

Reduction Theorem Proof

Case of propositional connectives is similar to the case of a formula $(A \cap B)$ below

$$\mathcal{M} \models (A \cap B) \quad \text{if and only if} \quad \mathcal{M} \models A \quad \text{and} \quad \mathcal{M} \models B$$

It follows directly from the **satisfaction** definition

$$\mathcal{M} \models A \text{ and } \mathcal{M} \models B \quad \text{if and only if} \quad v^*(A) = T \quad \text{and} \quad v^*(B) = T$$

$$\text{if and only if} \quad v^*(A \cap B) = T$$

It holds by the **induction** hypothesis

We proved

$$\mathcal{M} \models (A \cap B) \text{ if and only if } v^*(A \cap B) = T$$

for all sentences A, B of $\mathcal{L}(C)$

Reduction Theorem Proof

We prove now the case of a sentence B of the form

$$\exists x A(x)$$

We want to show that

$$\mathcal{M} \models \exists x A(x) \text{ if and only if } v^*(\exists x A(x)) = T$$

Let $v^*(\exists x A(x)) = T$

Then there is a c such that $v^*(A(c)) = T$, so by induction hypothesis, $\mathcal{M} \models A(c)$ so by definition

$$\mathcal{M} \models \exists x A(x)$$

Reduction Theorem Proof

On the other hand, if $v^*(\exists x A(x)) = F$, then by $S_{Henking}$ quantifier axiom **Q2** we have that

$$v^*(A(t)) = F$$

for all closed terms t of $\mathcal{L}(C)$. In particular, for every $c \in C$

$$v^*(A(c)) = F$$

By induction hypothesis,

$$\mathcal{M} \models \neg A(c) \quad \text{for all} \quad c \in C$$

Since every element of M is **denoted** by some $c \in C$ we have that

$$\mathcal{M} \models \neg \exists x A(x)$$

The **proof** of the case of a sentence B of the form $\forall x A(x)$ is similar and is left as and exercise

This ends the proof of the **Reduction Theorem**

Compactness Theorem
and
Löwenheim-Skolem Theorem

Compactness and Löwenheim-Skolem Theorems

The Reduction to Propositional Logic Theorem provides a powerful **method** of constructing **models** of theories out of **symbols** in a form of canonical models

It also gives us immediate **proofs** of the two important theorems: Compactness Theorem for the **predicate** logic and the Löwenheim-Skolem Theorem

Compactness theorem

Let S be any set of **predicate** formulas of $\mathcal{L}$

The set S has a **model** if and only if any finite subset S_0 of S has a **model**

Proof

Assume that S is a set of predicate formulas such that every finite subset S_0 of S has a **model**

We need to **show** that S has a **model**

The implication (**iii**) $\rightarrow$ (**i**) of the Reduction Theorem says:

" If The set $S \cup S_{Henkin} \cup EQ$ is **consistent** in sense of propositional logic, then S has a **model**"

So **showing** that S has a **model** this is equivalent to proving that $S \cup S_{Henkin} \cup EQ$ is **consistent** in the sense of propositional logic

Compactness Theorem

By already proved **Compactness Theorem** for propositional ogic of $\mathcal{L}$, it suffices to prove that for every finite subset $S_0 \subset S$, the set $S_0 \cup S_{Henkin} \cup EQ$ has a **model**

This follows from the assumption that S is a set such that every finite subset S_0 of S has a **model** and the implication $(i) \rightarrow (iii)$ of the **Reduction Theorem** that says:

" if S_0 has a **model**, then the set $S_0 \cup S_{Henkin} \cup EQ$ is consistent, " i.e. has a **model**

Löwenheim-Skolem Theorem

Let κ be an infinite cardinal

Let $\mathcal{L}$ be a **predicate** language with the **alphabet** $\mathcal{A}$ such that $card(\mathcal{A}) \leq \kappa$

Let Γ be a set of at most κ **formulas** of the $\mathcal{L}$

If the set S has a **model**, then there is a **model**

$$\mathcal{M} = [M, I]$$

of S such that

$$cardM \leq \kappa$$

Löwenheim-Skolem Theorem

Proof

Let $\mathcal{L}$ be a predicate language with the alphabet $\mathcal{A}$ such that $card(\mathcal{A}) \leq \kappa$

Obviously, $card(\mathcal{F}) \leq \kappa$

By the definition of the witnessing expansion $\mathcal{L}(C)$ of $\mathcal{L}$, $C = \bigcup_n C_n$ and for each n, $card(C_n) \leq \kappa$. So also $card C \leq \kappa$

Thus any canonical structure for $\mathcal{L}(C)$ has $\leq \kappa$ elements

By the implication $(i) \rightarrow (ii)$ of the **Reduction Theorem** that says: " if there is a model of S , then there is a canonical structure $\mathcal{M} = [M, I]$ for $\mathcal{L}(C)$ which is a **model** for S"

S has a model (canonical structure) with $\leq \kappa$ elements

This ends the proof

Chapter 9
Hilbert Proof Systems
Completeness of Classical Predicate Logic

Slides Set 3

PART 3: Proof of the**Completeness Theorem**

Completeness Theorem

The proof of Gödel's **completeness theorem** given by Kurt Gdel in his **doctoral dissertation** of 1929 and published as an article in 1930 is **not easy** to read today

It uses concepts and formalism that are no longer used and terminology that is often obscure

Gödel's proof was then **simplified** in 1947, when Leon Henkin observed in his **Ph.D. thesis** that the hard part of the Gödel's proof can be presented in the form of his **Model Existence Theorem** which published in 1949

Henkin's proof was **simplified** by Gisbert Hasenjaeger in 1953

Completeness Theorem

Other now classical **proofs** have been published by Rasiowa and Sikorski in 1951, 1952 using **Boolean algebraic** methods and by Beth in 1953, using **topological** methods

Still yet other **proofs** may be found in Hintikka (1955) and in Beth (1959)

We follow a **modern version** of of Henkin proof

Hilbert-style Proof System **H**

We define now a Hilbert style proof system **H** we are going to prove the **completeness theorem** for

Language $\mathcal{L}$

The language $\mathcal{L}$ of the proof system **H** is a predicate (first order) language with equality

We assume that the sets **P, F, C** are infinitely enumerable

We also assume that $\mathcal{L}$ has a full set of propositional connectives, i.e.

$$\mathcal{L} = \mathcal{L}_{\{\neg, \cap, \cup, \Rightarrow\}}(\mathbf{P}, \mathbf{F}, \mathbf{C})$$

Hilbert-style Proof System **H**

Logical Axioms LA

The set LA of **logical axioms** consists of three groups of axioms:

propositional axioms PA, equality axioms EA, and

quantifiers axioms QA

We write it symbolically as

$$LA = \{PA,\ EA,\ QA\}$$

For the set PA of **propositional axioms** we choose any **complete** set of axioms for propositional logic with a full set $\{\neg, \cap, \cap, \Rightarrow\}$ of propositional connectives

Hilbert-style Proof System **H**

In some formalizations, including the one in the *Handbook of Mathematical Logic, Barwise, ed.* (1977) we **base** our proof system **H** on, the authors just say for this group *PA* of **propositional axioms**: "all tautologies"

They of course mean all **predicate** formulas of $\mathcal{L}$ that are substitutions of propositional tautologies

This is done for the **need** of being able to use freely these **predicate** substitutions of propositional tautologies in the proof of **completeness theorem** for the proof system they formalize this way.

Hilbert-style Proof System **H**

In this case these **tautologies** are listed as **axioms** of the system and hence are provable in it

This is a convenient approach, but also the one that makes such a proof system **not** to be finitely axiomatizable

We **avoid** the infinite axiomatization by choosing a proper finite set of predicate language version of propositional **axioms** that is known (proved already for propositional case) to be **complete**, i.e. the one in which all propositional tautologies are **provable**

We choose, for name of the proof system **H** for Hilbert

Moreover, historical sake, we adopt Hilbert (1928) set of **axioms** from chapter 5

Hilbert-style Proof System **H**

For the set *EA* of **equational axioms** we choose the same set as in before because they were used in the proof of **Reduction to Propositional Logic Theorem**

We want to be able to carry this proof **within** the system **H**

For the set *QA* of **quantifiers axioms** we choose the **axioms**

such that the Henkin set S_{Henkin} axioms **Q1, Q2** are their particular cases

This again is needed, so the proof of the **Reduction Theorem** can be carried within **H**

Hilbert-style Proof System **H**

Rules of inference $\mathcal{R}$

There are four inference rules:

Modus Ponens (MP) and three quantifiers rules $(G), (G1), (G2)$, called **Generalization Rules**

We **define** the proof system **H** as follows

$$\mathbf{H} = (\mathcal{L}_{\{\neg,\cap,\cup,\Rightarrow\}}(\mathbf{P},\mathbf{F},\mathbf{C}),\ \mathcal{F},\ LA,\ \mathcal{R} = \{(MP),(G),(G1),(G2)\})$$

where $\mathcal{L} = \mathcal{L}_{\{\neg,\cap,\cup,\Rightarrow\}}(\mathbf{P},\mathbf{F},\mathbf{C})$ is **predicate** language with **equality**

We assume that the sets **P, F, C** are infinitely enumerable

$\mathcal{F}$ is the set of all well formed **formulas** of $\mathcal{L}$

Hilbert-style Proof System **H**

LA is the set of **logical axioms**

$$LA = \{PA, EA, QA\}$$

for PA, EA, QA defined as follows

PA is the set of **propositional axioms** (Hilbert, 1928)

A1 $(A \Rightarrow A)$

A2 $(A \Rightarrow (B \Rightarrow A))$

A3 $((A \Rightarrow B) \Rightarrow ((B \Rightarrow C) \Rightarrow (A \Rightarrow C)))$

A4 $((A \Rightarrow (A \Rightarrow B)) \Rightarrow (A \Rightarrow B))$

A5 $((A \Rightarrow (B \Rightarrow C)) \Rightarrow (B \Rightarrow (A \Rightarrow C)))$

A6 $((A \Rightarrow B) \Rightarrow ((B \Rightarrow C) \Rightarrow (A \Rightarrow C)))$

Hilbert-style Proof System **H**

A7 $((A \cap B) \Rightarrow A)$

A8 $((A \cap B) \Rightarrow B)$

A9 $((A \Rightarrow B) \Rightarrow ((A \Rightarrow C) \Rightarrow (A \Rightarrow (B \cap C))))$

A10 $(A \Rightarrow (A \cup B))$

A11 $(B \Rightarrow (A \cup B))$

A12 $((A \Rightarrow C) \Rightarrow ((B \Rightarrow C) \Rightarrow ((A \cup B) \Rightarrow C)))$

A13 $((A \Rightarrow B) \Rightarrow ((A \Rightarrow \neg B) \Rightarrow \neg A))$

A14 $(\neg A \Rightarrow (A \Rightarrow B))$

A15 $(A \cup \neg A)$

for any $A, B, C \in \mathcal{F}$

Hilbert-style Proof System **H**

EA is the set of **equality axioms**

E1 $u = u$

E2 $(u = w \Rightarrow w = u)$

E3 $((u_1 = u_2 \cap u_2 = u_3) \Rightarrow u_1 = u_3)$

E4

$((u_1 = w_1 \cap ... \cap u_n = w_n) \Rightarrow (R(u_1, ..., u_n) \Rightarrow R(w_1, ..., w_n)))$

E5

$((u_1 = w_1 \cap ... \cap u_n = w_n) \Rightarrow (t(u_1, ..., u_n) \Rightarrow t(w_1, ..., w_n)))$

for any free variable or **constant** of $\mathcal{L}$, $R \in \mathbf{P}$, and $t \in \mathbf{T}$

where R is an arbitrary n-ary **relation** symbol of $\mathcal{L}$ and $t \in \mathbf{T}$ is an arbitrary n-ary **term** of $\mathcal{L}$

Hilbert-style Proof System **H**

QA is the set of **quantifiers axioms**.

Q1 $(\forall x A(x) \Rightarrow A(t))$

Q2 $(A(t) \Rightarrow \exists x A(x))$

where where t is a **term**

A(t) is a result of **substitution** of t for all free occurrences of x in $A(x)$ and

t is **free for** x in A(x), i.e. **no** occurrence of a variable in t becomes a **bound** occurrence in A(t)

Hilbert-style Proof System **H**

$\mathcal{R}$ is the set of **rules of inference**

$$\mathcal{R} = \{(MP),\ (G),\ (G1),\ (G2)\}$$

(MP) is **Modus Ponens** rule

$$(MP)\ \ \frac{A\ ;\ (A \Rightarrow B)}{B}$$

for any formulas $A, B \in \mathcal{F}$

(G) is a quantifier **generalization** rule

$$(G)\ \ \frac{A}{\forall x A}$$

where $A \in \mathcal{F}$ and in particular we write

$$(G)\ \ \frac{A(x)}{\forall x A(x)}$$

for $A(x) \in \mathcal{F}$ and $x \in VAR$

Hilbert-style Proof System **H**

(G1) is a quantifier **generalization** rule

$$(G1)\ \frac{(B \Rightarrow A(x))}{(B \Rightarrow \forall x A(x))}$$

where for $A(x), B \in \mathcal{F}$, $x \in VAR$, and B is such that x is **not free** in B

(G2) is a quantifier **generalization** rule

$$(G2)\ \frac{(A(x) \Rightarrow B)}{(\exists x A(x) \Rightarrow B)}$$

where for $A(x), B \in \mathcal{F}$, $x \in VAR$, and B is such that x is **not free** in B

Hilbert-style Proof System **H**

We define now, as we do for any proof system, a notion of a **formal proof** of a formula A **from** a set S of formulas in **H** as a finite **sequence**

$$B_1, B_2, \ \ldots \ B_n$$

of formulas with each of which is **either** a logical axiom of **H**, a member of S , **or** else follows from earlier formulas in the sequence by one of the inference rules from $\mathcal{R}$

and is such that

$$B_n = A$$

We write it formally as follows.

Formal Proof in **H**

Definition

Let $\Gamma \subseteq \mathcal{F}$ be any set of formulas of $\mathcal{L}$

A **proof** in **H** of a formula $A \in \mathcal{F}$ **from** a set Γ of formulas is a sequence

$$B_1, B_2, \ \ldots \ B_n$$

of formulas, such that

$$B_1 \in LA \cup \Gamma, \qquad B_n = A$$

and for each $1 < i \leq n$, **either** $B_i \in LA \cup \Gamma$ **or** B_i is a **conclusion** of some of the preceding expressions in the sequence $B_1, B_2, \ \ldots \ B_n$ by virtue of one of the rules of inference from $\mathcal{R}$

Formal Proof in **H**

We write

$$\Gamma \vdash_{\mathbf{H}} A$$

to denote that the formula A has a **proof** from Γ in **H**

The case when $\Gamma = \emptyset$ is a special one

By the definition, $\emptyset \vdash_{\mathbf{H}} A$ means that in the proof of A **only** logical axioms LA are used. We hence write

$$\vdash_{\mathbf{H}} A$$

to denote that a formula A has a proof in **H**

Formal Proof in **H**

As we work now with a **fixed** (and only one) proof system **H**, we use the notation

$$\Gamma \vdash A \quad \text{and} \quad \vdash A$$

to denote the **proof** of a formula A **from** a set Γ in **H** and the proof of a formula A in **H**, respectively

Completeness Theorem

Any proof of the **completeness theorem** for a given proof system consists always of **two parts**

First we have show that

all formulas that have a proof in the system are tautologies

This is called a **soundness theorem** or soundness part of the completeness theorem

Completeness Theorem

The **second** implication says:

if a formula is a tautology then it has a proof in the proof system

This alone is sometimes called a **completeness theorem** (on assumption that the proof system is**sound**)

Traditionally it is called a completeness part of the **completeness theorem**

Soundness Theorem

We know that all **axioms** of **H** are predicate tautologies (proved in chapter 8)

All **rules** of inference from $\mathcal{R}$ are **sound** as the corresponding formulas were also proved in chapter 8 to be predicate tautologies and so the system **H** is **sound** i.e. the following holds for **H**

Soundness Theorem

For every formula $A \in \mathcal{F}$ of the language $\mathcal{L}$ of the proof system **H**,

$$\text{if} \quad \vdash A \quad \text{then} \quad \models A$$

Completeness Theorem

The **soundness theorem** proves that the proofs in the system **H** ”produce” only tautologies

We show here, as the next step that our proof system **H** ”produces” not only tautologies, but that all tautologies are **provable** in it

This is called a **completeness theorem** for classical predicate (first order logic, as it all is proven with respect to **classical** semantics

This is why it is called a **completeness** of classical predicate logic

Completeness Theorem

The goal is now to prove the **completeness part** of the following original theorem Gödel' s theorem

Theorem (completeness of predicate logic)

For any formula A of the language $\mathcal{L}$ of the proof system **H**,

A is **provable** in **H** if and only if

A is a predicate **tautology (valid)**

We write it symbolically as

$$\vdash A \quad \text{if and only if} \quad \models A$$

Completeness Theorem

We are going to prove the above **Theorem** (completeness of predicate logic) as a **particular case** of the **Gödel Completeness Theorem** that follows

This theorem is its more general, and more modern version

Its formulation, as well as the method of proving it, was first introduced by Henkin in 1947

It uses a notion of a **logical implication**, and some other notions that we introduce now below

Completeness Theorem

Sentence, Closure

Any formula of $\mathcal{L}$ **without** free variables is called a **sentence**

For any formula $A(x_1, \ldots x_n)$, a sentence

$$\forall x_1 \forall x_2 \ldots \forall x_n\, A(x_1, \ldots x_n)$$

is called a **closure** of $A(x_1, \ldots x_n)$

Directly from the above definition have that the following hold

Closure Fact

For any formula $A(x_1, \ldots x_n)$,

$$\models A(x_1, \ldots x_n) \quad \text{if and only if} \quad \models \forall x_1 \forall x_2 \ldots \forall x_n\, A(x_1, \ldots x_n)$$

Completeness Theorem

Logical Implication

For any set $\Gamma \subseteq \mathcal{F}$ of formulas of $\mathcal{L}$ and any $A \in \mathcal{F}$, we say that the set Γ **logically implies** the formula A and write it as

$$\Gamma \models A$$

if and only if all **models** of Γ are **models** of A

Observe, that in order to **prove** that $\Gamma \models B$ we have to show that the implication

$$\text{if } \mathcal{M} \models \Gamma \text{ then } \mathcal{M} \models B$$

holds for all structures $\mathcal{M} = [U, I]$ for $\mathcal{L}$

Completeness Theorem

Directly from the **Closure Lemma** we get the following

Lemma

Let Γ be a set of sentences of $\mathcal{L}$

For any formula $A(x_1, \ldots x_n)$ that **is not** a sentence,

$\Gamma \vdash A(x_1, \ldots x_n)$ if and only if $\Gamma \models \forall x_1 \forall x_2 \ldots \forall x_n \, A(x_1, \ldots x_n)$

Completeness Theorem

The above **Lemma** and **Closure Lemma** show that we need to consider only **sentences** (closed formulas) of $\mathcal{L}$ since they prove two properties:

(1) a formula of $\mathcal{L}$ is a **tautology** if and only if its closure is a **tautology**

(2) a formula of $\mathcal{L}$ is **provable** from Γ if and only if its closure is **provable** from Γ

This justifies the following **generalization** of the original Gödel' s completeness of predicate logic theorem

Completeness Theorem

Gödel Completeness Theorem

Let Γ be any set of sentences and A any sentence of a language $\mathcal{L}$ of Hilbert proof system $\mathbf{H}$

A sentence A is **provable** from Γ in $\mathbf{H}$ if and only if the set Γ **logically implies** A

We write it in symbols,

$$\Gamma \vdash A \quad \text{if and only if} \quad \Gamma \models A.$$

Remark

We want to remind that the Section: Reduction Predicate Logic to Propositional Logic is an integral and the first part of the proof the **Gödel Completeness Theorem**

We presented it separately for two reasons

R1. The reduction method and theorems and their proofs are purely **semantical** in their nature and hence are independent of the proof system **H**

R2. Because of the reason **R1.** the reduction method can be **used/adapted** to a proof of **completeness theorem** of any other proof system one needs to prove the classical **completeness theorem** for

Consistency

There are two definitions of consistency: semantical and syntactical

The **semantical** definition uses the notion of a model and says, in plain English:

a set of formulas is consistent if it has a model

The **syntactical** one uses the notion of provability and says:

a set of formulas is consistent if one can't prove a contradiction from it

We have used, in the Proof Two of the **Completeness Theorem** for propositional logic (chapter 5) the syntactical definition of consistency

We use now the following **semantical** definition

Consistency

Definition (Consistent/Inconsistent)

A set $\Gamma \subseteq \mathcal{F}$ of formulas of $\mathcal{L}$ is **consistent**

if and only if it has a **model**, otherwise, is **inconsistent**

Directly from the above definition we have the following

Inconsistency Lemma

For any set $\Gamma \subseteq \mathcal{F}$ of formulas of $\mathcal{L}$ and any $A \in \mathcal{F}$,

if $\Gamma \models A$, then the set $\Gamma \cup \{\neg A\}$ is **inconsistent**

Proof

Assume $\Gamma \models A$ and $\Gamma \cup \{\neg A\}$ is **consistent**

It means there is a structure $\mathcal{M} = [U, I]$, such that

$\mathcal{M} \models \Gamma$ and $\mathcal{M} \models \neg A$, i.e. $\mathcal{M} \not\models A$

This is a **contradiction** with $\Gamma \models A$

Crucial Lemma

Now we are going to prove the following **Lemma** that is crucial, to the proof of the Completeness Theorem

Crucial Lemma

Let Γ be any set of **sentences** of a language $\mathcal{L}$ of **H**

The following conditions hold for any formulas $A, B \in \mathcal{F}$ of $\mathcal{L}$

(i) If $\Gamma \vdash (A \Rightarrow B)$ and $\Gamma \vdash (\neg A \Rightarrow B)$, then $\Gamma \vdash B$

(ii) If $\Gamma \vdash ((A \Rightarrow C) \Rightarrow B)$, then $\Gamma \vdash (\neg A \Rightarrow B)$ and $\Gamma \vdash (C \Rightarrow B)$

(iii) If x does not appear in B and if $\Gamma \vdash ((\exists y A(y) \Rightarrow A(x)) \Rightarrow B)$, then $\Gamma \vdash B$

(iv) If x does not appear in B and if $\Gamma \vdash ((A(x) \Rightarrow \forall y A(y)) \Rightarrow B)$, then $\Gamma \vdash B$

Crucial Lemma Proof

Proof

(i) Notice that the formula $((A \Rightarrow B) \Rightarrow ((\neg A \Rightarrow B) \Rightarrow B))$ is a substitution of a propositional tautology, hence by definition of **H**, is **provable** in it

By monotonicity, $\Gamma \vdash ((A \Rightarrow B) \Rightarrow ((\neg A \Rightarrow B) \Rightarrow B))$

By assuption $\Gamma \vdash (A \Rightarrow B)$ and by Modus Ponens we get

$$\Gamma \vdash ((\neg A \Rightarrow B) \Rightarrow B)$$

By assuption $\Gamma \vdash (\neg A \Rightarrow B)$ and Modus Ponens we get

$$\Gamma \vdash B$$

Crucial Lemma Proof

(ii) The formulas

$$(1)\ (((A \Rightarrow B) \Rightarrow (\neg A \Rightarrow B)))$$

$$(2)\ (((A \Rightarrow B) \Rightarrow B) \Rightarrow (C \Rightarrow B))$$

are substitution of a propositional tautologies, hence are **provable** in **H**

Assume $\Gamma \vdash ((A \Rightarrow C) \Rightarrow B)$

By monotonicity and (1) we get

$$\Gamma \vdash (\neg A \Rightarrow B)$$

and by (2) we get

$$\vdash (C \Rightarrow B)$$

Crucial Lemma Proof

(iii) Assume

$$\Gamma \vdash ((\exists y A(y) \Rightarrow A(x)) \Rightarrow B)$$

Observe that it is a particular case of assumption

$$\Gamma \vdash ((A \Rightarrow C) \Rightarrow B)$$

in (ii), for $A = \exists y A(y)$, $C = A(x)$ and B= B

Hence by (ii) we have that

$$\Gamma \vdash (\neg \exists y A(y) \Rightarrow B) \text{ and } \Gamma \vdash (A(x) \Rightarrow B)$$

Apply Generalization Rule G2 to

$$\Gamma \vdash (A(x) \Rightarrow B)$$

and we have

$$\Gamma \vdash (\exists y A(y) \Rightarrow B)$$

Crucial Lemma Proof

Then by (i) applied to

$$\Gamma \vdash (\exists y A(y) \Rightarrow B) \quad \text{and} \quad \Gamma \vdash (\neg\exists y A(y) \Rightarrow B)$$

we get

$$\Gamma \vdash B$$

The proof of (iv) is similar to (iii) but uses the Generalization Rule G1

This **ends** the proof of the **Lemma**

Completeness Theorem for **H**

Now we are ready to conduct the proof of the Completeness Theorem for **H** stated as follows

H Completeness Theorem

Let Γ be any set of sentences and A any sentence of a language $\mathcal{L}$ of Hilbert proof system **H**

$$\Gamma \vdash A \quad \text{if and only if} \quad \Gamma \models A$$

In particular, for any formula A of $\mathcal{L}$,

$$\vdash A \quad \text{if and only if} \quad \models A$$

Proof of Completeness Theorem for **H**

Proof

We prove the **completeness part** , i.e. we prove the implication

$$\text{if } \Gamma \models A, \text{ then } \Gamma \vdash A$$

Suppose that $\Gamma \models A$

This means that we assume that all $\mathcal{L}$ models of Γ are models of A

By the **Inconsistency Lemma** the set $\Gamma \cup \{\neg A\}$ is inconsistent

Let $\mathcal{M} \models \Gamma$

We **construct**, as a next step, a witnessing expansion language $\mathcal{L}(C)$ of $\mathcal{L}$

Proof of Completeness Theorem for **H**

By the **Reduction Theorem** the set

$$\Gamma \cup S_{Henkin} \cup EQ$$

is **consistent** in a sense of propositional logic in $\mathcal{L}$

The set S_{Henkin} is a Henkin Set and EQ are equality axioms that are also the equality axioms EQ of **H**

By the **Compactness Theorem** for propositional logic of $\mathcal{L}$ there is a finite set

$$S_0 \subseteq \Gamma \cup S_{Henkin} \cup EQ$$

such that $S_0 \cup \{\neg A\}$ is **inconsistent** in the sense of propositional logic in $\mathcal{L}$

Proof of Completeness Theorem for **H**

We list all elements of S_0 in a sequence

$$A_1,\ A_2, \ldots, A_n,\ B_1,\ B_2, \ldots, B_m$$

where the sequence

$$A_1,\ A_2, \ldots, A_n$$

consists of those elements of S_0 which are **either** in $\Gamma \cup EQ$ **or else** are quantifiers axioms that are particular cases of the quantifiers axioms QA of **H**. We list them in any order

The sequence

$$B_1,\ B_2, \ldots, B_m$$

consists of elements of S_0 which are Henkin Axioms but listed **carefully** as to be described as follows

Proof of Completeness Theorem for **H**

Observe that by definition,

$$\mathcal{L}(C) = \bigcup_{n \in N} \mathcal{L}_n \text{ for } \mathcal{L} = \mathcal{L}_0 \subseteq \mathcal{L}_1 \subseteq \dots$$

We **define** the **rank** of $A \in \mathcal{L}(C)$ to be the **least** n, such that $A \in \mathcal{L}_n$

Now we choose for B_1 a Henkin Axiom in S_0 of the maximum **rank**

We choose for B_2 a Henkin Axiom in $S_0 - \{B_1\}$ of the maximum **rank**

We choose for B_3 a Henkin Axiom in $S_0 - \{B_1, B_2\}$ of the maximum **rank**, etc. ...

Proof of Completeness Theorem for **H**

The point of choosing the formulas B_i in this way is to make sure that the **witnessing constant** about which B_i speaks, **does not** appear in

$$B_{i+1},\ B_{i+2},\ \ldots,\ B_m$$

For **example** , if B_1 is

$$(\exists x A(x) \Rightarrow A(c_{A[x]}))$$

then $A[x]$ **does not** appear in any of the other $B_2,\ \ldots,\ B_m$, by the **maximality condition** on B_1

Proof of Completeness Theorem for **H**

We know that that $S_0 \cup \{\neg A\}$ is **inconsistent** in the sense of propositional logic, i.e.

$S_0 \cup \{\neg A\}$ **does not** have a (propositional) model

This means that

$$v^*(\neg A) \neq T \text{ for all } v \quad \text{and so} \quad v^*(A) = T \text{ for all } v$$

Hence a sentence

$$(S) \quad (A_1 \Rightarrow (A_2 \Rightarrow \ldots (A_n \Rightarrow (B_1 \Rightarrow \ldots (B_m \Rightarrow A))..)$$

is a **propositional tautology**

Proof of Completeness Theorem for **H**

We now replace in the sentence (S) each witnessing constant by a distinct new variable and write the result as

$$(S') \quad (A_1{}' \Rightarrow (A_2{}' \Rightarrow \ldots (A_n{}' \Rightarrow (B_1{}' \Rightarrow \ldots (B_m{}' \Rightarrow A))..)$$

We have $A' = A$ since A has **no** witnessing constant in it

The result is still a **tautology** and hence is **provable** in **H** from propositional axioms PA and Modus Ponens

By monotonicity

$$S_0 \vdash (A_1{}' \Rightarrow (A_2{}' \Rightarrow \ldots (A_n{}' \Rightarrow (B_1{}' \Rightarrow \ldots (B_m{}' \Rightarrow A))..)$$

Proof of Completeness Theorem for **H**

Each of $A_1', A_2', \ldots, A_n'$ is **either** a quantifiers axiom from QA of **H** **or else** in S_0, so

$$S_0 \vdash A_i' \quad \text{for all} \quad 1 \leq i \leq n$$

We apply Modus Ponens to the above and (S') n times and get

$$S_0 \vdash (B_1' \Rightarrow (B_2' \Rightarrow \ldots (B_m' \Rightarrow A))..)$$

Proof of Completeness Theorem for **H**

For **example**, if $B_1{}'$ is

$$(\exists x C(x) \Rightarrow C(x))$$

we have

$$S_0 \vdash ((\exists x C(x) \Rightarrow C(x)) \Rightarrow B)$$

for $B = (B_2{}' \Rightarrow \ldots (B_m{}' \Rightarrow A))..)$

By the **Crucial Lemma** part (iii) that says:

(iii) If x does not appear in B and if

$\Gamma \vdash ((\exists y A(y) \Rightarrow A(x)) \Rightarrow B)$, then $\Gamma \vdash B$

we get $S_0 \vdash B$, i.e.

$$S_0 \vdash (B_2{}' \Rightarrow \ldots (B_m{}' \Rightarrow A))..)$$

Proof of Completeness Theorem for **H**

If, for **example**, $B_2{}'$ is

$$(D(x) \Rightarrow \forall x D(x))$$

we have

$$S_0 \vdash ((\exists x C(x) \Rightarrow C(x)) \Rightarrow D)$$

for $D = (B_3{}' \Rightarrow \ldots (B_m{}' \Rightarrow A))..)$

By the **Crucial Lemma** part (iv) that says:

(iv) If x does not appear in B and if
$\Gamma \vdash ((A(x) \Rightarrow \forall y A(y)) \Rightarrow B)$, then $\Gamma \vdash B$
we get $S_0 \vdash D$, i.e.

$$S_0 \vdash (B_3{}' \Rightarrow \ldots (B_m{}' \Rightarrow A))..)$$

Proof of Completeness Theorem for **H**

We hence apply parts (iii) and (iv) of the **Crucial Lemma** to successively remove all

$$B_1{}', \ldots, B_m{}'$$

and obtain

$$S_0 \vdash A$$

This ends the proof that

$$\Gamma \vdash A$$

We hence we **completed the proof** of the completeness part of the first part

$$\Gamma \vdash A \quad \text{if and only if} \quad \Gamma \models A$$

of the **H Completeness Theorem**

Gödel' s Completeness Theorem

The soundness part of the **H Completeness Theorem** i.e. the implication

$$\text{if } \Gamma \vdash A, \text{ then } \Gamma \models A$$

holds for any sentence A of $\mathcal{L}$ directly by **Closure Lemma** and **Soundness Theorem**

The original Gödel' s **Theorem**, is expressed by the second part of the **H** Completeness Theorem:

$$\vdash A \quad \text{if and only if} \quad \models A$$

It follows from **Closure Lemma** and the first part for $\Gamma = \emptyset$

Chapter 9
Hilbert Proof Systems
Completeness of Classical Predicate Logic

Slides Set 4

PART 4: Deduction Theorem

PART 5: Some other Axiomatizations

Chapter 9
Hilbert Proof Systems
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Slides Set 4

PART 4: Deduction Theorem

Deduction Theorem

In mathematical arguments, one often assumes a statement A on the assumption (hypothesis) of some other statement B and then concludes that we have proved the implication

"if A, then B"

This reasoning is justified by the following theorem, called a **Deduction Theorem**

It was first formulated and **proved** for a certain Hilbert proof system S for the classical **propositional** logic by Herbrand in 1930 in a form stated as follows

Deduction Theorem

Deduction Theorem (Herbrand,1930)

For any formulas A, B of the language of a **propositional** proof system S,

$$\text{if} \quad A \vdash_S B \ \text{ then } \ \vdash_S (A \Rightarrow B)$$

In chapter 5 we formulated and proved the following, more genera l version of the Herbrand Theorem for a very simple (two logical axioms and Modus Ponens) propositional proof system $H1$

Deduction Theorem

For any subset Γ of the set of formulas $\mathcal{F}$ of H_1 and for any formulas $A, B \in \mathcal{F}$,

$$\Gamma,\ A \vdash_{H_1} B \quad \text{if and only if} \quad \Gamma \vdash_{H_1} (A \Rightarrow B)$$

In particular,

$$A \vdash_{H_1} B \quad \text{if and only if} \quad \vdash_{H_1} (A \Rightarrow B)$$

A natural question arises:

does **deduction theorem** hold for the **predicate** logic in general and for its proof system **H** we defined here?.

Deduction Theorem

The **Deduction Theorem** can not be carried directly to the **predicate** logic, but it nevertheless **holds** with some modifications. Here is where the problem lays.

Fact

Given the proof system

$$\mathbf{H} = (\mathcal{L}(\mathbf{P}, \mathbf{F}, \mathbf{C}),\ \mathcal{F},\ LA,\ \mathcal{R} = \{(MP),\ (G), (G1),\ (G2)\})$$

For any formula $A(x) \in \mathcal{F}$,

$$A(x) \vdash \forall x A(x)$$

but it is **not always** the case that

$$\vdash (A(x) \Rightarrow \forall x A(x))$$

Deduction Theorem

Proof

Obviously, $A(x) \vdash \forall x A(x)$ by Generalization rule (G)

Let now $A(x)$ be an atomic formula $P(x)$

By the **H Completeness Theorem**

$$\vdash (P(x) \Rightarrow \forall x P(x)) \quad \text{if and only if} \quad \models (P(x) \Rightarrow \forall x P(x))$$

Consider a structure

$$\mathcal{M} = [M, I]$$

where M contains at least two elements c and d

We define $P_I \subseteq M$ as a property that holds only for c, i.e.

$$P_I = \{c\}$$

Deduction Theorem

Take any assignment $s : VAR \longrightarrow M$

Then $(\mathcal{M}, s) \models P(x)$ only when $s(x) = c$ for all $x \in VAR$

$$\mathcal{M} = [M, I] \text{ is a } \textbf{counter model} \text{ for } (P(x) \Rightarrow \forall x P(x))$$

as we found s such $(\mathcal{M}, s) \models P(x)$ and obviously $(\mathcal{M}, s) \not\models \forall x P(x)$

We proved that $\not\models (P(x) \Rightarrow \forall x P(x))$

By the **H Completeness Theorem** this is equivalent to

$$\nvdash \ (P(x) \Rightarrow \forall x P(x))$$

and the **Deduction Theorem** fails as

$$Px \ \vdash \ \forall x P(x)$$

Deduction Theorem

The **Fact** shows that the problem is with application of the generalization rule (G) to the formula $A \in \Gamma$

To handle this we introduce, after Mendelson(1987) the following notion

Definition

Let A be one of formulas in Γ and let

$$(P) \quad B_1, B_2, ..., B_n$$

be a proof (deduction) of B_n from Γ, together with justification at each step. We say that the formula

B_i **depends upon** A in the proof $B_1, B_2, ..., B_n$

if and only if the following holds

(1) B_i is A and the justification for B_i is $B_i \in \Gamma$

or

(2) B_i is justified as direct consequence by MP

or

(G) of some preceding formulas in the proof sequence (P), where at least one of these preceding formulas **depends upon** A

Example

Here is a proof (deduction)

$$B_1,\ B_2,\ \ldots,\ B_5$$

showing that

$$A,\ (\forall xA \Rightarrow C)\ \vdash\ \forall xC$$

B_1 A

Hyp

B_1 depends upon A

B_2 $\forall xA$

B_1, (G)

B_2 depends upon A

B_3 $(\forall xA \Rightarrow C)$

Hyp

B_3 depends upon $(\forall xA \Rightarrow C)$

Deduction Theorem

B_3 $(\forall x A \Rightarrow C)$

Hyp

B_3 depends upon $(\forall x A \Rightarrow C)$

B_4 C

MP on B_2, B_3

B_4 depends upon A and $(\forall x A \Rightarrow C)$

B_5 $\forall x C$

(G)

B_4 depends upon A and $(\forall x A \Rightarrow C)$

Observe that the formulas A, C may, or may not have x as a free variable

Deduction Theorem

DT Lemma

If B **does not** depend upon A in a proof (deduction) showing that $\Gamma, A \vdash B$, then $\Gamma \vdash B$

Proof

Let

$$B_1, B_2, \ldots, B_n = B$$

be a proof (deduction) of B from Γ, A,

in which B **does not** depend upon A

We prove by induction over the length of the proof that

$$\Gamma \vdash B$$

Deduction Theorem

Assume that **DT Lemma** holds for all proofs of the length less than n

If $B \in \Gamma$ or $B \in LA$, by definition then $\Gamma \vdash B$

If B is a direct **consequence** of two preceding formulas, then, since B **does not** depend upon A, **neither do** theses preceding formulas

By inductive hypothesis, theses preceding formulas have a proof from Γ alone

Hence **so does** B, i.e.

$$\Gamma \vdash B$$

Now we are ready to formulate and prove the **Deduction Theorem** for predicate logic

Deduction Theorem

For any formulas A, B of the language of proof system **H** the following holds

(1) Assume that **in some** proof (deduction) showing that

$$\Gamma, A \vdash B$$

no application of the generalization rule (G) **to** a formula that **depends** upon A has as its quantified variable a free variable of the formula A

Then we have that

$$\Gamma \vdash (A \Rightarrow B)$$

(2) If $\Gamma \vdash (A \Rightarrow B)$, then $\Gamma, A \vdash B$

Deduction Theorem

Proof

The proof we present extends the proof of the **Deduction Theorem** for propositional logic from chapter 5

We **adopt** the propositional proof to the system **H** and add the relevant predicate cases

For the sake of clarity and **independence** we write now the whole proof in all **details**

Deduction Theorem

(1) Assume that

$$\Gamma,\ A \ \vdash B$$

i.e. that we have a formal proof

$$B_1,\ B_2,\ \ldots, B_n$$

of B from the set of formulas $\Gamma \cup \{A\}$

In order to prove that

$$\Gamma \vdash (A \Rightarrow B)$$

we will prove the following a **stronger** statement

(**S**) $\Gamma \vdash (A \Rightarrow B_i)$ for all B_i $(1 \leq i \leq n)$ in the proof of B

Deduction Theorem

Hence, in particular case, when $i = n$, we will obtain that also

$$\Gamma \vdash (A \Rightarrow B)$$

The proof of the statement (**S**) is conducted by induction on $1 \leq i \leq n$

Base Step $i = 1$

When $i = 1$, it means that the formal proof contains only one element B_1

By the definition of the formal proof from $\Gamma \cup \{A\}$, we have that $B_1 \in LA$, or $B_1 \in \Gamma$, or $B_1 = A$, i.e.

$$B_1 \in LA \cup \Gamma \cup \{A\}$$

Here we have two cases

Deduction Theorem

Case 1 $B_1 \in LA \cup \Gamma$

Observe that the formula

$$(B_1 \Rightarrow (A \Rightarrow B_1))$$

is a particular case of the axiom A2 of **H**

By assumption $B_1 \in LA \cup \Gamma$, hence we get the required proof of $(A \Rightarrow B_1)$ from Γ by the following application of the MP rule

$$(MP)\ \frac{B_1 \ ; \ (B_1 \Rightarrow (A \Rightarrow B_1))}{(A \Rightarrow B_1)}$$

Deduction Theorem

Case 2 $B_1 = A$

When $B_1 = A$, then to prove

$$\Gamma \vdash (A \Rightarrow B)$$

means to prove $\Gamma \vdash (A \Rightarrow A)$

But $(A \Rightarrow A) \in LA$ (axiom A1) of **H**, i.e. $\vdash (A \Rightarrow A)$. By the monotonicity of the consequence we have that

$$\Gamma \vdash (A \Rightarrow A)$$

The above cases conclude the proof of the Base Case $i = 1$

Deduction Theorem

Inductive Step

Assume that

$$\Gamma \vdash (A \Rightarrow B_k)$$

for all $k < i$, we will show that using this fact we can conclude that also

$$\Gamma \vdash (A \Rightarrow B_i)$$

Consider a formula B_i in the proof sequence

By the definition, $B_i \in LA \cup \Gamma \cup \{A\}$

or B_i follows by MP from certain B_j, B_m such that $j < m < i$

We have to consider again two cases

Case 1

$B_i \in LA \cup \Gamma \cup \{A\}$

The proof of $(A \Rightarrow B_i)$ from Γ in this case is obtained from the proof of the Base Step for $i = 1$ by replacement B_1 by B_i and will be omitted here as a straightforward repetition

Case 2

B_i is a conclusion of MP

If B_i is a conclusion of MP , then we must have two formulas B_j, B_m in the proof sequence, such that $j < i, m < i, \ j \neq m$ and

$$(MP)\ \frac{B_j \ ; \ B_m}{B_i}$$

item[[] By the inductive assumption, the formulas B_j, B_m are such that

$$\Gamma \vdash (A \Rightarrow B_j) \quad \text{and} \quad \Gamma \vdash (A \Rightarrow B_m)$$

Deduction Theorem

Moreover, by the definition of the Modus Ponens rule, the formula B_m has to have a form $(B_j \Rightarrow B_i)$, i.e.

$$B_m = (B_j \Rightarrow B_i)$$

and the the inductive assumption can be re-written as

$$(*) \quad \Gamma \vdash (A \Rightarrow B_j) \quad \text{and} \quad \Gamma \vdash (A \Rightarrow (B_j \Rightarrow B_i)) \quad \text{for } j < i$$

Observe now that the formula

$$((A \Rightarrow (B_j \Rightarrow B_i)) \Rightarrow ((A \Rightarrow B_j) \Rightarrow (A \Rightarrow B_i)))$$

is a substitution of the axiom A3 of **H** and hence

$$\vdash \; ((A \Rightarrow (B_j \Rightarrow B_i)) \Rightarrow ((A \Rightarrow B_j) \Rightarrow (A \Rightarrow B_i)))$$

Deduction Theorem

By the monotonicity,

$$(**)\quad \Gamma \vdash ((A \Rightarrow (B_j \Rightarrow B_i)) \Rightarrow ((A \Rightarrow B_j) \Rightarrow (A \Rightarrow B_i)))$$

Applying the rule MP to formulas $(*)$ and $(**)$ i.e. performing the following

$$(MP)\ \frac{(A \Rightarrow (B_j \Rightarrow B_i));\ ((A \Rightarrow (B_j \Rightarrow B_i)) \Rightarrow ((A \Rightarrow B_j) \Rightarrow (A \Rightarrow B_i)))}{((A \Rightarrow B_j) \Rightarrow (A \Rightarrow B_i))}$$

we get that also

$$\Gamma \vdash ((A \Rightarrow B_j) \Rightarrow (A \Rightarrow B_i))$$

Deduction Theorem

Applying again the rule MP to formulas $(*)$ and the above

$$\Gamma \vdash ((A \Rightarrow B_j) \Rightarrow (A \Rightarrow B_i))$$

i.e. performing the following

$$(MP)\ \frac{(A \Rightarrow B_j)\ ;\ ((A \Rightarrow B_j) \Rightarrow (A \Rightarrow B_i))}{(A \Rightarrow B_i)}$$

we get that

$$\Gamma \vdash (A \Rightarrow B_i)$$

Deduction Theorem

Finally, suppose that there is some $j < i$ such that

$$B_i \text{ is } \forall x B_j$$

By inductive assumption

$$\Gamma \vdash (A \Rightarrow B_j)$$

and either

(i) B_j **does not** depend upon A or

(ii) x is **not free** variable in A

We want to prove

$$\Gamma \vdash B_i$$

We have theses two cases **(i)** and **(ii)** to consider.

Deduction Theorem

Case **(i)**

$$\Gamma \vdash (A \Rightarrow B_j)$$

and B_j **does not** depend upon A

Then by **DT Lemma** we have that $\Gamma \vdash B_j$

and, consequently, by the generalization rule (G)

$$\Gamma \vdash \forall x B_j$$

Thus we proved

$$\Gamma \vdash B_i$$

Deduction Theorem

Now, from just proved

$$\Gamma \vdash B_i$$

and axiom A2 of **H**

$$\vdash (B_i \Rightarrow (A \Rightarrow B_i))$$

and monotonicity

$$\Gamma \vdash (B_i \Rightarrow (A \Rightarrow B_i))$$

and MP applied to them we get

$$\Gamma \vdash (A \Rightarrow B_i)$$

Deduction Theorem

Case **(ii)**

$\Gamma \vdash (A \Rightarrow B_j)$ and x **is not** free variable in A

We know that $\models (\forall x(A \Rightarrow B_j) \Rightarrow (A \Rightarrow \forall x B_j))$

hence the **Completeness Theorem** we get

$\vdash (\forall x(A \Rightarrow B_j) \Rightarrow (A \Rightarrow \forall x B_j))$

Since $\Gamma \vdash (A \Rightarrow B_j)$ by inductive assumption, we get by the generalization rule (G) and nmonotonicity

$$\Gamma \vdash \forall x(A \Rightarrow B_j)$$

By MP applied to the above

$$\Gamma \vdash (A \Rightarrow \forall x B_j)$$

That is we got

$$\Gamma \vdash A \Rightarrow B_i)$$

Deduction Theorem

Since $\Gamma \vdash (A \Rightarrow B_j)$ by inductive assumption, we get by the generalization rule (G),

$$\Gamma \vdash \forall x(A \Rightarrow B_j)$$

and so, by MP

$$\Gamma \vdash A \Rightarrow \forall x B_j)$$

That is we proved

$$\Gamma \vdash (A \Rightarrow B_i)$$

This completes the induction and the **proves** part (1) of the **Deduction Theorem**

Deduction Theorem part (2)

The **proof** of the implication

$$\text{if } \Gamma \vdash (A \Rightarrow B) \text{ then } \Gamma, A \vdash B$$

is straightforward

Assume $\Gamma \vdash (A \Rightarrow B)$. By monotonicity we have also that

$$\Gamma, A \vdash (A \Rightarrow B)$$

Obviously, $\Gamma, A \vdash A$. Applying MP to the above, we get the proof of B from $\{\Gamma, A\}$ i.e. we have proved that

$$\Gamma, A \vdash B$$

This **ends** the proof of the **Deduction Theorem** for **H**

PART 5: Some other Axiomatizations

We present here some of most known, and historically important axiomatizations of classical **predicate** logic, i.e. the following Hilbert style proof systems

1. Hilbert and Ackermann (1928)

This formalization is based on D. Hilbert and W. Ackermann book *Grundzügen der Theoretischen Logik* (Principles of Theoretical Logic), Springer - Verlag, 1928

The book grew from the **courses** on logic and foundations of mathematics Hilbert gave in years 1917-1922

He received **help** in writeup from Barnays and the material was **put into** the book by Ackermann and Hilbert

Hilbert and Ackermann

The Hilbert and Ackermann book was conceived as an introduction to mathematical logic and was followed by another two volumes book written by D. Hilbert and P. Bernays, *Grundzügen der Mathematik I, II* , Springer -Verlag, 1934, 1939

Hilbert and Ackermann **formulated** and **asked** a question of the **completeness** for their deductive (proof) system

It was **answered** affirmatively by Kurt Gödel in 1929 with proof of his **Completeness Theorem**

Hilbert and Ackermann

We define the Hilbert and Ackermann proof system **HA** following a pattern established for the **H** system

The original **language** used by Hilbert and Ackermann contained **only** negation $\neg$ and disjunction $\cup$ and so do we

We **define**

$$\mathbf{HA} = (\mathcal{L}_{\{\neg,\cup\}}(\mathbf{P},\mathbf{F},\mathbf{C}),\ \mathcal{F},\ LA,\ \mathcal{R})$$

where

$$\mathcal{R} = \{(MP),\ (SB),\ (G1),\ (G2)\}$$

The set LA of logical axioms is as follows

Hilbert and Ackermann (1928)

Propositional Axioms

A1 $(\neg(A \cup A) \cup A)$

A2 $(\neg A \cup (A \cup B))$

A3 $(\neg(A \cup B) \cup (B \cup A))$

A4 $(\neg(\neg B \cup C) \cup (\neg(A \cup B) \cup (A \cup C)))$

for any $A, B, C, \in \mathcal{F}$

Quantifiers Axioms

Q1 $(\neg \forall x A(x) \cup A(x))$

Q2 $(\neg A(x) \cup \exists x A(x))$

Q3 $(\neg A(x) \cup \exists x A(x))$,

for any $A(x) \in \mathcal{F}$

Rules of Inference $\mathcal{R}$

(MP) is the Modus Ponens rule. It has, in the language $\mathcal{L}_{\{\neg,\cup\}}$, a form

$$(MP)\ \frac{A\ ;\ \ (\neg A\cup B)}{B}$$

(SB) is a **substitution rule**

$$(SB)\ \frac{A(x_1,x_2,\ \dots\ x_n)}{A(t_1,t_2,\ \dots\ t_n)}$$

where $A(x_1,x_2,\ \dots\ x_n)\in\mathcal{F}$ and $t_1,t_2,\ \dots\ t_n\in\mathbf{T}$

.

Hilbert and Ackermann

$(G1), (G2)$ are **quantifiers generalization rules**

$$(G1) \quad \frac{(\neg B \cup A(x))}{(\neg B \cup \forall x A(x))}$$

$$(G2) \quad \frac{(\neg A(x) \cup B)}{(\neg \exists x A(x) \cup B)}$$

where $A(x), B \in \mathcal{F}$ and B is such that x is **not free** in B

Hilbert and Ackermann

The **HA** system is usually written now with the use of implication, i.e. is based on a language

$$\mathcal{L} = \mathcal{L}_{\{\neg, \Rightarrow\}}(\mathbf{P}, \mathbf{F}, \mathbf{C})$$

We define

$$\mathbf{HAI} = (\mathcal{L}_{\{\neg, \Rightarrow\}}(\mathbf{P}, \mathbf{F}, \mathbf{C}), \mathcal{F}, LA, \mathcal{R})$$

for

$$\mathcal{R} = \{(MP), (SB), (G1), (G2)\}$$

and the set LA of logical axioms as follows

Hilbert and Ackermann

Propositional Axioms

A1 $((A \cup A) \Rightarrow A)$

A2 $(A \Rightarrow (A \cup B))$

A3 $((A \cup B) \Rightarrow (B \cup A))$

A4 $((\neg B \cup C) \Rightarrow ((A \cup B) \Rightarrow (A \cup C)))$

for any

$A, B, C, \in \mathcal{F}$

Quantifiers Axioms

Q1 $(\forall x A(x) \Rightarrow A(x))$

Q2 $(A(x) \Rightarrow \exists x A(x))$

for any $A(x) \in \mathcal{F}$

Hilbert and Ackermann

Rules of Inference $\mathcal{R}$

(MP) is Modus Ponens rule

$$(MP)\quad \frac{A\ ;\ (A \Rightarrow B)}{B}$$

for any formulas $A, B \in \mathcal{F}$

(SB) is a **substitution rule**

$$(SB)\quad \frac{A(x_1, x_2, \dots x_n)}{A(t_1, t_2, \dots t_n)}$$

where $A(x_1, x_2, \dots x_n) \in \mathcal{F}$ and $t_1, t_2, \dots t_n \in \mathbf{T}$

Hilbert and Ackermann

$(G1), (G2)$ are **quantifiers generalization rules**.

$$(G1) \quad \frac{(B \Rightarrow A(x))}{(B \Rightarrow \forall x A(x))}$$

$$(G2) \quad \frac{(A(x) \Rightarrow B)}{(\exists x A(x) \Rightarrow B)}$$

where $A(x), B \in \mathcal{F}$ and B is such that x is **not free** in B

The form of the **quantifiers** axioms Q1, Q2, and **quantifiers generalization** rule (G2) is due to Bernays

Mendelson (1987)

Here is the **first order** logic proof system as introduced in Elliott Mendelson's book *Introduction to Mathematical Logic* (1987). Hence the name **HM**

HM is a generalization to the **predicate** language of the proof system H_2 for **propositional** logic defined after Mendelson's book and studied in Chapter 5

$$\mathbf{HM} = (\mathcal{L}_{\{\neg,\cup\}}(\mathbf{P}, \mathbf{F}, \mathbf{C}),\ \mathcal{F},\ LA,\ \mathcal{R} = \{(MP),\ (G)\})$$

The **HM** components are as follows

Mendelson (1987)

Propositional Axioms

A1 $(A \Rightarrow (B \Rightarrow A))$

A2 $((A \Rightarrow (B \Rightarrow C)) \Rightarrow ((A \Rightarrow B) \Rightarrow (A \Rightarrow C)))$

A3 $((\neg B \Rightarrow \neg A) \Rightarrow ((\neg B \Rightarrow A) \Rightarrow B)))$

for any $A, B, C, \in \mathcal{F}$

Mendelson

Quantifiers Axioms

Q1 $(\forall x A(x) \Rightarrow A(t))$

where t is a term, A(t) is a result of **substitution** of t for all **free** occurrences of x in $A(x)$ and t is **free for** x in A(x), i.e. **no** occurrence of a variable in t becomes a **bound** occurrence in A(t)

Q2 $(\forall x(B \Rightarrow A(x)) \Rightarrow (B \Rightarrow \forall x A(x)))$

where $A(x), B \in \mathcal{F}$ and B is such that x is **not free** in B

Rules of Inference $\mathcal{R}$

(MP) is the Modus Ponens rule

$$(MP) \quad \frac{A \; ; \; (A \Rightarrow B)}{B}$$

for any formulas $A, B \in \mathcal{F}$

(G) is the **generalization rule**

$$(G) \quad \frac{A(x)}{\forall x A(x)}$$

where $A(x) \in \mathcal{F}$ and $x \in VAR$

Rasiowa, Sikorski (1950)

Helena Rasiowa and Roman Sikorski are the authors of the first **algebraic proof** of the Gödel **completeness theorem** ever given in 1950

Other **algebraic** proofs were later given by Rieger, Beth, Łos in 1951 , and Scott in 1954

Rasiowa and Sikorski (1950)

Here is Rasiowa- Sikorski original formalization

$$RS = (\mathcal{L}_{\{\neg,\cap,\cup,\Rightarrow\}}(\mathbf{P},\mathbf{F},\mathbf{C}),\ \mathcal{F},\ LA,\ \mathcal{R})$$

for

$$\mathcal{R} = \{(MP),(SB),(Q1),(Q2),(Q3),(Q4)\}$$

The logical axioms LA are as follows

Propositional Axioms

A1 $((A \Rightarrow B) \Rightarrow ((B \Rightarrow C) \Rightarrow (A \Rightarrow C)))$

A2 $(A \Rightarrow (A \cup B))$

A3 $(B \Rightarrow (A \cup B))$

Rasiowa and Sikorski

A4 $((A \Rightarrow C) \Rightarrow ((B \Rightarrow C) \Rightarrow ((A \cup B) \Rightarrow C)))$

A5 $((A \cap B) \Rightarrow A)$

A6 $((A \cap B) \Rightarrow B)$

A7 $((C \Rightarrow A) \Rightarrow ((C \Rightarrow B) \Rightarrow (C \Rightarrow (A \cap B)))$

A8 $((A \Rightarrow (B \Rightarrow C)) \Rightarrow ((A \cap B) \Rightarrow C))$

A9 $(((A \cap B) \Rightarrow C) \Rightarrow (A \Rightarrow (B \Rightarrow C))$

A10 $(A \cap \neg A) \Rightarrow B)$

A11 $((A \Rightarrow (A \cap \neg A)) \Rightarrow \neg A)$

A12 $(A \cup \neg A)$

for any $A, B, C \in \mathcal{F}$

Rasiowa and Sikorski

Rules of Inference $\mathcal{R}$

(MP) is **Modus Ponens** rule

$$(MP) \quad \frac{A \; ; \; (A \Rightarrow B)}{B}$$

for any formulas $A, B \in \mathcal{F}$

(SB) is a **substitution rule**

$$(SB) \quad \frac{A(x_1, x_2, \ \dots \ x_n)}{A(t_1, t_2, \ \dots \ t_n)}$$

where $A(x_1, x_2, \ \dots \ x_n) \in \mathcal{F}$ and $t_1, t_2, \ \dots \ t_n \in \mathbf{T}$

Rasiowa and Sikorski

$(G1), (G2)$ are the following **quantifiers introduction rules**

$$(G1) \quad \frac{(B \Rightarrow A(x))}{(B \Rightarrow \forall x A(x))}$$

$$(G2) \quad \frac{(A(x) \Rightarrow B)}{(\exists x A(x) \Rightarrow B)}$$

where $A(x), B \in \mathcal{F}$ and B is such that x is **not free** in B

Rasiowa and Sikorski

$(G3), (G3)$ are the following **quantifiers elimination rules**.

$$(G3) \quad \frac{(B \Rightarrow \forall x A(x))}{(B \Rightarrow A(x))}$$

$$(G4) \quad \frac{\exists x(A(x) \Rightarrow B)}{(A(x) \Rightarrow B)}$$

where $A(x), B \in \mathcal{F}$ and B is such that x is **not free** in B

Rasiowa and Sikorski

The **algebraic logic** starts from purely logical considerations, abstracts from them, places them into a general algebraic context, and makes use of **other branches** of mathematics such as topology, set theory, and functional analysis

For **example**, Rasiowa and Sikorski **algebraic generalization** of the **completeness theorem** for classical predicate logic is the following

Rasiowa and Sikorski

Algebraic Completeness Theorem (Rasiowa, Sikorski 1950)

For every formula A of the classical predicate calculus RS the following conditions are equivalent

i A is derivable in RS;

ii A is valid in every realization of $\mathcal{L}$;

iii A is valid in every realization of $\mathcal{L}$ in any complete Boolean algebra;

iv A is valid in every realization of $\mathcal{L}$ in the field $B(X)$ of all subsets of any set $X \neq \emptyset$;

Rasiowa and Sikorski

v A is valid in every semantic realization of $\mathcal{L}$ in any enumerable set;

vi there exists a non-degenerate Boolean algebra $\mathcal{A}$ and an infinite set J such that A is valid in every realization of $\mathcal{L}$ in J and $\mathcal{A}$;

vii $A_R(\mathbf{I}) = V$ for the canonical realization R of $\mathcal{L}$ in the Lindenbaum-Tarski algebra $\mathcal{LT}$ of RS and the identity valuation $\mathbf{I}$;

viii A is a predicate tautology.