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Chapter 5
Hilbert Proof Systems
Completeness of Classical Propositional Logic

CHAPTER 5 SLIDES

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PART 1: **Hilbert Proof Systems:** Proof System H_1

Hilbert Proof Systems

Hilbert proof systems are based on a language with implication and **contain** Modus Ponens as a rule of inference

Modus Ponens is probably the **oldest** of all known rules of inference as it was already known to the Stoics (3 B.C.)
It is also considered as the **most natural** to our intuitive thinking and the proof systems containing i Modus Ponens as the inference rule play a special role in logic.

Hilbert systems put major emphasis on logical axioms, keeping the **rules** of inference to minimum often **admitting** Modus Ponens as the **sole rule** of inference

Hilbert Proof Systems

There are many proof systems that describe classical propositional logic, i.e. that are **complete** with respect to the **classical semantics**

We present a **Hilbert** proof system for the classical propositional logic and discuss **two ways** of proving the **Completeness Theorem** for it

The **first proof** is based on the one included in Elliott Mendelson's book Introduction to Mathematical Logic

It is is a **constructive** proof that shows how one can use the assumption that a formula A is a **tautology** in order to construct its **formal proof**

The **second proof** is non-constructive

Its importance lies in a fact that the methods it uses can be applied to the proof of **completeness theorem** for classical predicate logic as we present it in (chapter 9)

It also **generalizes** to some non-classical logics

Hilbert Proof Systems

We prove completeness part of the **Completeness Theorem** by proving the converse implication to it

We show how one can **deduce** that a formula *A* **is not** a tautology **from** the fact that it **does not** have a proof

It is hence called a **counter-model** construction proof

Both proofs relay on the **Deduction Theorem** and so this is the theorem we are now going to prove

Hilbert Proof System H_1

We consider now a **Hilbert** proof system H_1 **based** on a language with implication as the **only** connective

The proof system H_1 has only **two** logical axioms and has the Modus Ponens as a **sole rule** of inference

Hilbert Proof System H_1

Definition

Hilbert system H_1 is defined as follows

$$H_1 = (\ \mathcal{L}_{\{\Rightarrow\}},\ \ \mathcal{F},\ \ \{A1, A2\},\ \ MP\ )$$

A1 (Law of simplification)

$(A \Rightarrow (B \Rightarrow A))$

A2 (Frege's Law)

$((A \Rightarrow (B \Rightarrow C)) \Rightarrow ((A \Rightarrow B) \Rightarrow (A \Rightarrow C)))$

MP is the Modus Ponens rule

$$MP\ \ \frac{A\ ;\ (A \Rightarrow B)}{B}$$

where A, B, C are any formulas from $\mathcal{F}$

Formal Proofs in H_1

The **formal proof** of

$$(A \Rightarrow A)$$

in H_1 is a sequence

$$B_1, \; B_2, \; B_3, \; B_4, B_5$$

as defined below

B_1 $((A \Rightarrow ((A \Rightarrow A) \Rightarrow A)) \Rightarrow ((A \Rightarrow (A \Rightarrow A)) \Rightarrow (A \Rightarrow A)))$
axiom A2 for $A = A$, $B = (A \Rightarrow A)$, and $C = A$

B_2 $(A \Rightarrow ((A \Rightarrow A) \Rightarrow A))$
axiom A1 for $A = A$, $B = (A \Rightarrow A)$

B_3 $A \Rightarrow (A \Rightarrow A)) \Rightarrow (A \Rightarrow A)))$
MP application to B_1 and B_2

B_4 $(A \Rightarrow (A \Rightarrow A))$,
axiom A1 for $A = A$, $B = A$

B_5 $(A \Rightarrow A)$
MP application to B_3 and B_4

Formal Proofs in H_1

We have hence proved the following

Fact

For any $A \in \mathcal{F}$, $\quad \vdash_{H_1} (A \Rightarrow A)$

It is easy to see that the **proof** of $(A \Rightarrow A)$ **wasn't** constructed automatically

The **main step** in its construction was the choice of a proper form (substitution) of logical axioms to **start with**, and to continue the proof with

This choice is **far from** obvious for un-experienced human and **impossible** for a machine, as the number of possible **substitutions** is infinite

Formal Proofs in H_1

In Chapter 4 we gave some **examples** of simple **proof systems** with inference rules such that it was **possible** to
"reverse" the usual way they were used
We could use them in a **reverse** manner in order to **search** for proofs.

Moreover and we were **able** to do so in an effective and fully automatic way

We called such proof systems **syntactically decidable** and we defined them formally as follows

Syntactically Decidable Proof Systems

Definition

A proof system $S = (\mathcal{L}, \mathcal{E}, LA, \mathcal{R})$ for which **there is** an **effective mechanical** procedure that finds (generates) a formal **proof** of any expression $E \in \mathcal{E}$, **if it exists**, is called a **syntactically semi- decidable** system

If additionally there is an **effective method** of deciding that **if** a proof of E is not found that it **does not exist**, the system S is called **syntactically decidable**

Otherwise S is **syntactically undecidable**

Searching for Proofs in a Proof Systems

We will argue now, that the presence of Modus Ponens inference rule in **Hilbert systems** makes them syntactically undecidable

A **general procedure** for automated search for proofs in a proof system S can be stated is as follows.

Let B be an expression of the system *S* that is not an axiom

If B has a **proof** in S, B must be the **conclusion** of one of the inference rules

Let's say it is a rule r

We find all its premisses, i.e. we evaluate $r^{-1}(B)$

If **all premisses** are axioms, the proof is **found**

Otherwise we **repeat** the procedure for any non-axiom **premiss**

Search for Proof by the Means of MP

Search for proofs in any **Hilbert System** S must involve, between other rules, if any, the Modus Ponens inference rule

Lets analyze a **search** for proofs by the means of Modus Ponens rule MP

The MP rule says: **given** two formulas A and $(A \Rightarrow B)$ we **conclude** a formula B

Assume now that we have a certain formula, we name it for convenience B

We want to **find** a proof of B

If B is an **axiom**, we have the proof; the formula itself

Search for Proof by the Means of MP

If B **is not** an axiom, it was obtained by the application of the Modus Ponens rule, to certain two formulas A and $(A \Rightarrow B)$

But there is infinitely many of formulas A, $(A \Rightarrow B)$, as A is any formula. It means that in for any B, $MP^{-1}(B)$ is countably infinite

Obviously, we have the following

Fact

Every **Hilbert System** S is not syntactically decidable

In particular, the system H_1 is not syntactically decidable

Semantic Links

Semantic Link 1

System H_1 is **sound** under classical, **Ł**, **H** semantics and **not sound** under **K** semantics

We leave the **proof** of the following theorem (by induction with respect of the length of the formal proof) as an easy **exercise**

Soundness Theorem for H_1

For any $A \in \mathcal{F}$, **if** $\vdash_{H_1} A$, **then** $\models A$

Semantic Links

Semantic Link 2

The system H_1 **is not complete** under classical semantics

It means that we have to show that **not all** classical **tautologies** have a proof in H_1

We have proved in Chapter 3 that one needs $\neg$ and one of the other connectives $\cup, \cap, \Rightarrow$ to express all classical connectives, and hence all classical **tautologies**

For **example** we can't express negation in term of implication alone and so a **tautology** $(\neg\neg A \Rightarrow A)$ is **not definable** in the language of H_1 , hence

$$\nvdash_{H_1} (\neg\neg A \Rightarrow A)$$

Proof from Hypothesis

We have constructed a **formal proof** of

$$(A \Rightarrow A)$$

in H_1 on a base of logical axioms, as an **example** of complexity of finding proofs in **Hilbert** systems

In order to make the construction of formal proofs easier by the use of **previously proved** formulas we use the notion of a formal proof from some **hypotheses** (and logical axioms) in any proof system

$$S = (\mathcal{L},\ \mathcal{E},\ LA, \mathcal{R})$$

as defined as follows in chapter 4

Proof from Hypothesis

Given a proof system $S = (\mathcal{L}, \mathcal{E}, LA, \mathcal{R})$

While proving expressions we often use some extra information available, besides the axioms of the proof system

This extra information is called **hypothesis** in the proof

Let $\Gamma \subseteq \mathcal{E}$ be a set expressions called hypothesis

Definition

A proof of $E \in \mathcal{E}$ from the set of hypothesis Γ in S is **a formal proof** in S, where the expressions from Γ are treated as additional hypothesis **added** to the set LA of the **logical axioms** of the system S

Notation: $\Gamma \vdash_S E$

We read it : E has a proof in S from the set Γ (and the logical axioms LA)

Formal Definition

Definition

We say that $E \in \mathcal{E}$ has a **formal proof** in S

from the set Γ and the logical axioms LA and denote it as $\Gamma \vdash_S E$

if and only if there is a sequence

$$A_1, \ldots, A_n$$

of expressions from $\mathcal{E}$, such that

$$A_1 \in LA \cup \Gamma, \quad A_n = E$$

and for each $1 < i \leq n$, either $A_i \in LA \cup \Gamma$ or A_i is

a **direct consequence** of some of the **preceding**

expressions by virtue of one of the rules of inference of S

Special Cases

Case 1: $\Gamma \subseteq \mathcal{E}$ is a **finite set** and $\Gamma = \{B_1, B_2, ..., B_n\}$

We write

$$B_1, B_2, ..., B_n \vdash_{\mathcal{S}} E$$

instead of $\{B_1, B_2, ..., B_n\} \vdash_{\mathcal{S}} E$

Case 2: $\Gamma = \emptyset$

By the **definition** of a proof of E from Γ, $\emptyset \vdash_S E$ means that in the proof of E we use **only** the logical axioms LA of S

We hence write

$$\vdash_S E$$

to denote that E has a proof from $\Gamma = \emptyset$

Proof from Hypothesis in H_1

Show that

$$(A \Rightarrow B),\ (B \Rightarrow C) \vdash_{H_1} (A \Rightarrow C)$$

We construct a formal proof

$$B_1, B_2, ...B_7$$

B_1 : $(B \Rightarrow C)$, hypothesis

B_2 : $(A \Rightarrow B)$, hypothesis

B_3 : $((A \Rightarrow (B \Rightarrow C)) \Rightarrow ((A \Rightarrow B) \Rightarrow (A \Rightarrow C)))$,
axiom A2

Proof from Hypothesis in H_1

B_4 : $((B \Rightarrow C) \Rightarrow (A \Rightarrow (B \Rightarrow C)))$,
axiom A1 for $A = (B \Rightarrow C)$, $B = A$

B_5 : $(A \Rightarrow (B \Rightarrow C))$,
B_1 and B_4 and MP

B_6 : $((A \Rightarrow B) \Rightarrow (A \Rightarrow C))$,
MP

B_7 : $(A \Rightarrow C)$

Deduction Theorem

In mathematical arguments, one often **proves** a statement B on the assumption of some other statement A and then concludes that we have **proved** the implication "if A, then B"

This reasoning is justified a theorem, called a **Deduction Theorem**

Reminder

We write $\Gamma, A \vdash B$ for $\Gamma \cup \{A\} \vdash B$

In general, we write $\Gamma, A_1, A_2, ..., A_n \vdash B$

for $\Gamma \cup \{A_1, A_2, ..., A_n\} \vdash B$

Deduction Theorem for H_1

Deduction Theorem for H_1

For any $A, B \in \mathcal{F}$ and $\Gamma \subseteq \mathcal{F}$

$$\Gamma,\ A \vdash_{H_1} B \quad \text{if and only if} \quad \Gamma \vdash_{H_1} (A \Rightarrow B)$$

In particular

$$A \vdash_{H_1} B \quad \text{if and only if} \quad \vdash_{H_1} (A \Rightarrow B)$$

H_1 Formal Proofs

The proof of the following **Lemma** provides a good example of multiple applications of the **Deduction Theorem**

Lemma

For any $A, B, C \in \mathcal{F}$,

(a) $(A \Rightarrow B), (B \Rightarrow C) \vdash_{H_1} (A \Rightarrow C)$,

(b) $(A \Rightarrow (B \Rightarrow C)) \vdash_{H_1} (B \Rightarrow (A \Rightarrow C))$

Observe that by **Deduction Theorem** we can re-write **(a)** as

(a') $(A \Rightarrow B),\ (B \Rightarrow C), A \vdash_{H_1} C$

Poof of **(a')**

We construct a formal proof

$$B_1, B_2, B_3, B_4, B_5$$

of $(A \Rightarrow B), (B \Rightarrow C), A \vdash_{H_1} C$ as follows.

B_1 : $(A \Rightarrow B)$
hypothesis

B_2 : $(B \Rightarrow C)$
hypothesis

B_3 : A
hypothesis

B_4 : B
B_1, B_3 and MP

B_5 : C
B_2, B_4 and MP

H_1 Formal Proofs

Thus we proved by **Deduction Theorem** that **(a)** holds, i.e.

$$(A \Rightarrow B), (B \Rightarrow C) \vdash_{H_1} (A \Rightarrow C)$$

Proof of **Lemma** part **(b)**

By **Deduction Theorem** we have that

$$(A \Rightarrow (B \Rightarrow C)) \vdash_{H_1} (B \Rightarrow (A \Rightarrow C))$$

if and only if

$$(A \Rightarrow (B \Rightarrow C)), B \vdash_{H_1} (A \Rightarrow C)$$

Formal Proofs

We construct a formal proof

$$B_1, B_2, B_3, B_4, B_5, B_6, B_7$$

of $(A \Rightarrow (B \Rightarrow C)), B \vdash_{H_1} (A \Rightarrow C)$ as follows.

B_1 : $(A \Rightarrow (B \Rightarrow C))$
hypothesis

B_2 : B
hypothesis

B_3 : $((B \Rightarrow (A \Rightarrow B))$
$A1$ for $A = B, B = A$

B_4 : $(A \Rightarrow B)$
B_2, B_3 and MP

H_1 Formal Proofs

B_5 : $((A \Rightarrow (B \Rightarrow C)) \Rightarrow ((A \Rightarrow B) \Rightarrow (A \Rightarrow C)))$
axiom A2

B_6 : $((A \Rightarrow B) \Rightarrow (A \Rightarrow C))$
B_1, B_5 and MP

B_7 : $(A \Rightarrow C)$

Thus we proved by **Deduction Theorem** that

$$(A \Rightarrow (B \Rightarrow C)) \vdash_{H_1} (B \Rightarrow (A \Rightarrow C))$$

Simpler Proof

Here i a simpler proof of **Lemma** part **(b)**

We apply the **Deduction Theorem** twice, i.e. we get

$$(A \Rightarrow (B \Rightarrow C)) \vdash_{H_1} (B \Rightarrow (A \Rightarrow C))$$

if and only if

$$(A \Rightarrow (B \Rightarrow C)), B \vdash_{H_1} (A \Rightarrow C)$$

if and only if

$$(A \Rightarrow (B \Rightarrow C)), B, A \vdash_{H_1} C$$

Simpler Proof

We now construct a proof of $(A \Rightarrow (B \Rightarrow C)), B, A \vdash_{H_1} C$

as follows

B_1 $(A \Rightarrow (B \Rightarrow C))$
hypothesis

B_2 B
hypothesis

B_3 A
hypothesis

B_4 $(B \Rightarrow C)$
B_1 , B_3 and MP

B_5 C
B_2 , B_4 and MP

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Slides Set 1

PART 2: **Proof** of **Deduction Theorem** for H_1

The Deduction Theorem for H_1

As we now **fix** the proof system to be H_1, we write $A \vdash B$ instead of $A \vdash_{H_1} B$

Deduction Theorem (Herbrand, 1930) for H_1

For any formulas $A, B \in \mathcal{F}$,

$$\text{If } A \vdash B, \quad \text{then } \vdash (A \Rightarrow B)$$

Deduction Theorem (General case) for H_1

For any formulas $A, B \in \mathcal{F}$, $\Gamma \subseteq \mathcal{F}$

$$\Gamma, A \vdash B \quad \text{if and only if} \quad \Gamma \vdash (A \Rightarrow B)$$

Proof of The Deduction Theorem

Proof:

Part 1 We first prove the ”if” part:

$$\text{If} \quad \Gamma, \; A \vdash B \qquad \text{then} \quad \Gamma \vdash (A \Rightarrow B)$$

Assume that

$$\Gamma, \; A \; \vdash B$$

i.e. that we have a formal proof

$$B_1, B_2, ..., B_n$$

of B from the set of formulas $\Gamma \cup \{A\}$

We have to show that

$$\Gamma \vdash (A \Rightarrow B)$$

Proof of The Deduction Theorem

In order to prove that

$\Gamma \vdash (A \Rightarrow B)$ follows from $\Gamma,\ A \vdash B$

we prove a **stronger statement**, namely that

$$\Gamma \vdash (A \Rightarrow B_i)$$

for any B_i, $1 \leq i \leq n$ in the formal proof $B_1, B_2, ..., B_n$ of B also follows from $\Gamma,\ A \vdash B$

Hence in **particular case**, when $i = n$ we will obtain that $\Gamma \vdash (A \Rightarrow B)$ follows from $\Gamma,\ A \vdash B$

and that will end the proof of **Part 1**

Base Step

The proof of **Part 1** is conducted by **mathematical induction** on i, for $1 \leq i \leq n$

Step 1 $i = 1$ (base step)

Observe that when $i = 1$, it means that the **formal proof** $B_1, B_2, ..., B_n$ contains only **one element** B_1

By the **definition** of the formal proof from $\Gamma \cup \{A\}$, we have that

(1) B_1 is a logical axiom, or $B_1 \in \Gamma$, or

(2) $B_1 = A$

This means that $B_1 \in \{A1, A2\} \cup \Gamma \cup \{A\}$

Base Step

Now we have **two cases** to consider.

Case1: $B_1 \in \{A1, A2\} \cup \Gamma$

Observe that $(B_1 \Rightarrow (A \Rightarrow B_1))$ is the axiom $A1$

By assumption $B_1 \in \{A1, A2\} \cup \Gamma$

We get the **required proof** of $(A \Rightarrow B_1)$ from Γ

by the following application of the Modus Ponens rule

$$(MP)\ \frac{B_1 \ ;\ (B_1 \Rightarrow (A \Rightarrow B_1))}{(A \Rightarrow B_1)}$$

Base Step

Case 2: $B_1 = A$

When $B_1 = A$ then to prove $\Gamma \vdash (A \Rightarrow B_1)$

This means we have to prove

$$\Gamma \vdash (A \Rightarrow A)$$

This holds by **monotonicity** of the consequence and the fact that we have shown that

$$\vdash (A \Rightarrow A)$$

The above cases **conclude the proof** for $i = 1$ of

$$\Gamma \vdash (A \Rightarrow B_i)$$

Inductive Step

Inductive Step

Assume that

$$\Gamma \vdash (A \Rightarrow B_k)$$

for all $k < i$ (strong induction)

We will **show** that using this fact we can conclude that also

$$\Gamma \vdash (A \Rightarrow B_i)$$

Inductive Step

Consider a formula B_i in the formal proof

$$B_1, B_2, ..., B_n$$

By **definition** of the formal proof we have to show the following tow cases

Case 1 : $B_i \in \{A1, A2\} \cup \Gamma \cup \{A\}$ and

Case 2: B_i follows by MP from certain B_j, B_m such that $j < m < i$

Consider now the **Case 1:** $B_i \in \{A1, A2\} \cup \Gamma \cup \{A\}$

The proof of $(A \Rightarrow B_i)$

from Γ in this case is **obtained** from the proof of the **Step** $i = 1$ by replacement B_1 by B_i

and is omitted here as a **straightforward repetition**

Inductive Step

Case 2:

B_i is a **conclusion** of (MP)

If B_i is a conclusion of (MP), then we must have two formulas B_j, B_m in the formal proof

$$B_1, B_2, ..., B_n$$

such that $j < i$, $m < i$, $j \neq m$ and

$$(MP)\ \frac{B_j \ ; \ B_m}{B_i}$$

Inductive Step

By the **inductive assumption** the formulas B_j, B_m are such that $\Gamma \vdash (A \Rightarrow B_j)$ and $\Gamma \vdash (A \Rightarrow B_m)$

Moreover, by the definition of (MP) rule, the formula B_m has to have a form $(B_j \Rightarrow B_i)$

This means that

$$B_m = (B_j \Rightarrow B_i)$$

The inductive assumption can be re-written as follows

$$\Gamma \vdash (A \Rightarrow (B_j \Rightarrow B_i))$$

for $j < i$

Inductive Step

Observe now that the formula

$$((A \Rightarrow (B_j \Rightarrow B_i)) \Rightarrow ((A \Rightarrow B_j) \Rightarrow (A \Rightarrow B_i)))$$

is a **substitution of the axiom A2** and hence **has a proof** in our system

By the **monotonicity** of the consequence, it also has a proof from the set Γ, i.e.

$$\Gamma \vdash ((A \Rightarrow (B_j \Rightarrow B_i)) \Rightarrow ((A \Rightarrow B_j) \Rightarrow (A \Rightarrow B_i)))$$

Inductive Step

We know that

$$\Gamma \vdash ((A \Rightarrow (B_j \Rightarrow B_i)) \Rightarrow ((A \Rightarrow B_j) \Rightarrow (A \Rightarrow B_i)))$$

Applying the rule MP i.e. performing the following

$$\frac{(A \Rightarrow (B_j \Rightarrow B_i))\ ;\ ((A \Rightarrow (B_j \Rightarrow B_i)) \Rightarrow ((A \Rightarrow B_j) \Rightarrow (A \Rightarrow B_i)))}{((A \Rightarrow B_j) \Rightarrow (A \Rightarrow B_i))}$$

we get that also

$$\Gamma \vdash ((A \Rightarrow B_j) \Rightarrow (A \Rightarrow B_i))$$

Inductive Step

Applying again the rule MP i.e. performing the following

$$\frac{(A \Rightarrow B_j)\ ;\ ((A \Rightarrow B_j) \Rightarrow (A \Rightarrow B_i))}{(A \Rightarrow B_i)})$$

we get that

$$\Gamma \vdash (A \Rightarrow B_i)$$

what **ends the proof** of the inductive step

Proof of the Deduction Theorem

By the mathematical induction principle, we have **proved** that

$$\Gamma \vdash (A \Rightarrow B_i), \quad \text{for all} \quad 1 \leq i \leq n$$

In particular it is **true** for $i = n$, i.e. for $B_n = B$ and we proved that

$$\Gamma \vdash (A \Rightarrow B)$$

This ends the proof of the **first part** of the **Deduction Theorem**:

If $\Gamma, A \vdash B$, then $\Gamma \vdash (A \Rightarrow B)$

Proof of the Deduction Theorem

The **proof** of the second part, i.e. of the inverse implication:

$$\text{If} \quad \Gamma \vdash (A \Rightarrow B), \quad \text{then} \quad \Gamma, A \vdash B$$

is **straightforward** and goes as follows.

Assume that $\Gamma \vdash (A \Rightarrow B)$

By the monotonicity of the consequence we have also that $\Gamma, A \vdash (A \Rightarrow B)$

Obviously $\Gamma, A \vdash A$

Applying Modus Ponens to the above, we get the proof of B from $\{\Gamma, A\}$

We have hence proved that $\Gamma, A \vdash B$

This **ends** the proof

Proof of the Deduction Theorem

Deduction Theorem (General case) for H_1

For any formulas $A, B \in \mathcal{F}$ and any $\Gamma \subseteq \mathcal{F}$

$$\Gamma,\ A \vdash B \quad \text{if and only if} \quad \Gamma \vdash (A \Rightarrow B)$$

The particular case we get also the particular case

Deduction Theorem (Herbrand, 1930) for H_1

For any formulas $A, B \in \mathcal{F}$,

$$\text{If } A \vdash B, \quad \text{then } \vdash (A \Rightarrow B)$$

is obtained from the above by assuming that the set Γ is empty

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Slides Set 2

PART 3: Proof System H_2: Deduction Theorem, Exercises and Examples

Proof System H_2

The proof system H_1 is **sound** and strong enough to prove the Deduction Theorem, but, as we proved, is **not complete**

We extend now the language and the set of logical axioms of H_1 to form a new **Hilbert** system H_2 that is **complete** with respect to classical semantics

The proof of **Completeness Theorem** for H_2 is be presented in the next section (**Slides Set 3**)

Hilbert System H_2 Definition

Definition

$$H_2 = (\ \mathcal{L}_{\{\Rightarrow,\neg\}},\ \mathcal{F},\ \{A1, A2, A3\}\ \ (MP)\ )$$

A1 (Law of simplification)
$(A \Rightarrow (B \Rightarrow A))$

A2 (Frege's Law)
$((A \Rightarrow (B \Rightarrow C)) \Rightarrow ((A \Rightarrow B) \Rightarrow (A \Rightarrow C)))$

A3 $((\neg B \Rightarrow \neg A) \Rightarrow ((\neg B \Rightarrow A) \Rightarrow B)))$

MP (Rule of inference)

$$(MP)\ \frac{A\ ;\ (A \Rightarrow B)}{B}$$

where A, B, C are any formulas of the propositional language $\mathcal{L}_{\{\Rightarrow,\neg\}}$

Deduction Theorem for System H_2

Observation 1

The proof system H_2 is obtained by adding axiom A_3 to the system H_1

Observation 2

The language of H_2 is obtained by adding the connective $\neg$ to the language of H_1

Observation 3

The use of axioms $A1, A2$ in the proof of **Deduction Theorem** for the system H_1 is **independent** of the connective $\neg$ added to the language of H_1

Observation 4

Hence the proof of the **Deduction Theorem** for the system H_1 can be repeated **as it is** for the system H_2

Deduction Theorem for System H_2

Observations 1-4 prove that he Deduction Theorem holds for system H_2

Deduction Theorem for H_2

For any $\Gamma \subseteq \mathcal{F}$ and $A, B \in \mathcal{F}$

$$\Gamma,\ A \vdash_{H_2} B \text{ if and only if } \Gamma \vdash_{H_2} (A \Rightarrow B)$$

In particular

$$A \vdash_{H_2} B \text{ if and only if } \vdash_{H_2} (A \Rightarrow B)$$

Soundness and CompletenessTheorems

We get by easy verification that H_2 is a **sound** under classical semantics and hence we have the following

Soundness Theorem H_2
For every formula $A \in \mathcal{F}$

$$\text{if } \vdash_{H_2} A \quad \text{then} \quad \models A$$

We prove in the next section (**Slides Set 3**), that H_2 is also **complete** under classical semantics, i.e. we prove

Completeness Theorem for H_2

For every formula $A \in \mathcal{F}$,

$$\vdash_{H_2} A \ \text{ if and only if } \ \models A$$

CompletenessTheorems

The proof of completeness theorem (for a given semantics) is always a **main point** in creation of any new logic

There are many techniques to prove it, depending on the proof system, and on the semantics we define for it

We **present** in the next next section (Slides Set 2) two proofs of the **Completeness Theorem** for the system H_2

These proofs use very different techniques, hence the **reason** of presenting both of them

Proof System H_2: Exercises and Examples

Examples and Exercises

We present now some examples of **formal proofs** in H_2

There are **two reasons** for presenting them

First reason] is that all formulas we provide the **formal proofs** for play a crucial role in the proof of **Completeness Theorem** for H_2

The **second reason** is that they provide a "training ground" for a reader to **learn** how to develop **formal** proofs

For this **reason** we write some formal proofs in a **full detail** and we leave some for the reader to **complete** in a way explained in the following **example**

Important Lemma

We write $\vdash$ instead of $\vdash_{H_2}$ for the sake of simplicity

Reminder

In the construction of the formal proofs we often **use** the **Deduction Theorem** and the following **Lemma 1** that was proved in the previous section

Lemma 1

$$(\mathbf{a})\ \ (A \Rightarrow B), (B \Rightarrow C) \vdash_{H_2} (A \Rightarrow C)$$

$$(\mathbf{b})\ \ (A \Rightarrow (B \Rightarrow C)) \vdash_{H_2} ((B \Rightarrow (A \Rightarrow C))$$

Example 1

Here are consecutive steps

$$B_1, ..., B_5, \; B_6$$

of the proof in H_2 of $(\neg\neg B \Rightarrow B)$

B_1 : $((\neg B \Rightarrow \neg\neg B) \Rightarrow ((\neg B \Rightarrow \neg B) \Rightarrow B))$

B_2 : $((\neg B \Rightarrow \neg B) \Rightarrow ((\neg B \Rightarrow \neg\neg B) \Rightarrow B))$

B_3 : $(\neg B \Rightarrow \neg B)$

B_4 : $((\neg B \Rightarrow \neg\neg B) \Rightarrow B)$

B_5 : $(\neg\neg B \Rightarrow (\neg B \Rightarrow \neg\neg B))$

B_6 : $(\neg\neg B \Rightarrow B)$

Exercise 1

Complete the proof presented in **Example 1** by providing comments how each step of the proof was **obtained**

Remark

The **solution** presented on the next slide shows how to write details of solutions

Solutions of other **problems** presented later are less detailed

Exercise 1 Solution

Solution

The comments that **complete** the proof are as follows.

B_1 : $((\neg B \Rightarrow \neg\neg B) \Rightarrow ((\neg B \Rightarrow \neg B) \Rightarrow B))$

Axiom $A3$ for $A = \neg B$, $B = B$

B_2 : $((\neg B \Rightarrow \neg B) \Rightarrow ((\neg B \Rightarrow \neg\neg B) \Rightarrow B))$

B_1 and **Lemma 1 (b)** for
$A = (\neg B \Rightarrow \neg\neg B)$, $B = (\neg B \Rightarrow \neg B)$, $C = B$,

i.e. we have

$((\neg B \Rightarrow \neg\neg B) \Rightarrow ((\neg B \Rightarrow \neg B) \Rightarrow B)) \vdash ((\neg B \Rightarrow \neg B) \Rightarrow ((\neg B \Rightarrow \neg\neg B) \Rightarrow B))$

Exercise 1 Solution

$B_3: \quad (\neg B \Rightarrow \neg B)$

We proved for H_1 and hence for H_2 that $\vdash (A \Rightarrow A)$ and we substitute $A = \neg B$

$B_4: \quad ((\neg B \Rightarrow \neg\neg B) \Rightarrow B)$

B_2, B_3 and MP

$B_5: \quad (\neg\neg B \Rightarrow (\neg B \Rightarrow \neg\neg B))$

Axiom $A1$ for $A = \neg\neg B, \quad B = \neg B$

$B_6: \quad (\neg\neg B \Rightarrow B)$

B_4, B_5 and **Lemma 1 (a)** for
$A = \neg\neg B, \quad B = (\neg B \Rightarrow \neg\neg B), C = B$

i.e. we have

$$(\neg\neg B \Rightarrow (\neg B \Rightarrow \neg\neg B)), \ ((\neg B \Rightarrow \neg\neg B) \Rightarrow B) \vdash (\neg\neg B \Rightarrow B)$$

Proofs from Axioms Only

General remark

Observe that in steps

$$B_2, B_3, B_5, B_6$$

of the proof we **called on** previously proved **facts** and used them as a part of the proof

We can always **obtain** a formal proof that uses **only** axioms of the system by **inserting** previously constructed **formal proofs** of these facts into the places occupying by the respective **steps** B_2, B_3, B_5, B_6 where these **facts** were used

Proofs from Axioms

Example

Consider the step

$B_3 : \quad (\neg B \Rightarrow \neg B)$

The formula $(\neg B \Rightarrow \neg B)$ is a previously proved **fact**

We **replace** the formula $(\neg B \Rightarrow \neg B)$ (in step step B_3 by its **formal proof** that uses uses **only** axioms

We obtain this proof from the the previously constructed proof of $(A \Rightarrow A)$ by **replacing** A by $\neg B$

The last step of the **inserted proof** becomes now "old" step B_3 and we re-numerate all other steps accordingly

Proofs from Axioms Only

Here are consecutive first THREE steps of the proof of $(\neg\neg B \Rightarrow B)$

B_1 : $((\neg B \Rightarrow \neg\neg B) \Rightarrow ((\neg B \Rightarrow \neg B) \Rightarrow B))$

B_2 : $((\neg B \Rightarrow \neg B) \Rightarrow ((\neg B \Rightarrow \neg\neg B) \Rightarrow B))$

B_3 : $(\neg B \Rightarrow \neg B)$

We insert now the proof of $(\neg B \Rightarrow \neg B)$ after step B_2 and **erase** the B_3

The **last step** of the inserted proof becomes the **erased** B_3

Proofs from Axioms Only

A part of new **transformed** proof is

B_1 : $((\neg B \Rightarrow \neg\neg B) \Rightarrow ((\neg B \Rightarrow \neg B) \Rightarrow B))$ (Old B_1)

B_2 : $((\neg B \Rightarrow \neg B) \Rightarrow ((\neg B \Rightarrow \neg\neg B) \Rightarrow B))$ (Old B_2)

We insert here the proof from axioms only of Old B_3

B_3 : $((\neg B \Rightarrow ((\neg B \Rightarrow \neg B) \Rightarrow \neg B)) \Rightarrow ((\neg B \Rightarrow (\neg B \Rightarrow \neg B)) \Rightarrow (\neg B \Rightarrow \neg B)))$, (New B_3)

B_4 : $(\neg B \Rightarrow ((\neg B \Rightarrow \neg B) \Rightarrow \neg B))$

B_5: $((\neg B \Rightarrow (\neg B \Rightarrow \neg B)) \Rightarrow (\neg B \Rightarrow \neg B))$

B_6: $(\neg B \Rightarrow (\neg B \Rightarrow \neg B))$

B_7: $(\neg B \Rightarrow \neg B)$ (Old B_3)

Proofs from Axioms Only

We repeat our procedure by **replacing** the step B_2 by its formal proof as defined in the proof of the **Lemma 1 (b)**

We continue the process for all other steps which involved application of the **Lemma 1** until we get a full **formal proof** from the axioms of H_2 only

Usually we don’t do it and we don’t need to do it, but it is important to remember that **it always can be done**

Example 2

Here are consecutive steps

$$B_1, \; B_2, \; ....., \; B_5$$

in a proof of $(B \Rightarrow \neg\neg B)$

B_1 $((\neg\neg\neg B \Rightarrow \neg B) \Rightarrow ((\neg\neg\neg B \Rightarrow B) \Rightarrow \neg\neg B))$

B_2 $(\neg\neg\neg B \Rightarrow \neg B)$

B_3 $((\neg\neg\neg B \Rightarrow B) \Rightarrow \neg\neg B)$

B_4 $(B \Rightarrow (\neg\neg\neg B \Rightarrow B))$

B_5 $(B \Rightarrow \neg\neg B)$

Exercise 2

Exercise 2

Complete the proof presented in **Example 2** by providing detailed comments how each step of the proof was obtained.

Solution

The comments that complete the proof are as follows.

B_1 $((\neg\neg\neg B \Rightarrow \neg B) \Rightarrow ((\neg\neg\neg B \Rightarrow B) \Rightarrow \neg\neg B))$

Axiom $A3$ for $A = B$, $B = \neg\neg B$

B_2 $(\neg\neg\neg B \Rightarrow \neg B)$

Example 1 for $B = \neg B$

Exercise 2

B_3 $((\neg\neg\neg B \Rightarrow B) \Rightarrow \neg\neg B)$

B_1, B_2 and MP

i.e. we have that

$$\frac{(\neg\neg\neg B \Rightarrow \neg B);((\neg\neg\neg B \Rightarrow \neg B) \Rightarrow ((\neg\neg\neg B \Rightarrow B) \Rightarrow \neg\neg B))}{((\neg\neg\neg B \Rightarrow B) \Rightarrow \neg\neg B)}$$

B_4 $(B \Rightarrow (\neg\neg\neg B \Rightarrow B))$

Axiom $A1$ for $A = B, \quad B = \neg\neg\neg B$

B_5 $(B \Rightarrow \neg\neg B)$

B_3, B_4 and Lemma 1 **(a)** for
$A = B, \quad B = (\neg\neg\neg B \Rightarrow B), C = \neg\neg B,$

i.e. we have that

$(B \Rightarrow (\neg\neg\neg B \Rightarrow B)), ((\neg\neg\neg B \Rightarrow B) \Rightarrow \neg\neg B) \vdash (B \Rightarrow \neg\neg B)$

Example 3

Here are consecutive steps

$B_1, \; B_2, \; ..., \; B_{12}$ in a proof of $\quad (\neg A \Rightarrow (A \Rightarrow B))$

$B_1 \quad \neg A$

$B_2 \quad A$

$B_3 \quad (A \Rightarrow (\neg B \Rightarrow A))$

$B_4 \quad (\neg A \Rightarrow (\neg B \Rightarrow \neg A))$

$B_5 \quad (\neg B \Rightarrow A)$

$B_6 \quad (\neg B \Rightarrow \neg A)$

$B_7 \quad ((\neg B \Rightarrow \neg A) \Rightarrow ((\neg B \Rightarrow A) \Rightarrow B))$

Example 3

B_8 $((\neg B \Rightarrow A) \Rightarrow B)$

B_9 B

B_{10} $\neg A, A \vdash B$

B_{11} $\neg A \vdash (A \Rightarrow B)$

B_{12} $(\neg A \Rightarrow (A \Rightarrow B))$

Exercise 3

1. Complete the proof from the **Example 3** by providing comments how each step of the proof was obtained.

2. Prove that

$$\neg A,\ A \vdash B$$

Exercise 4

Example 4

Here are consecutive steps $B_1, ..., B_7$

in a proof of $((\neg B \Rightarrow \neg A) \Rightarrow (A \Rightarrow B))$

B_1 $(\neg B \Rightarrow \neg A)$

B_2 $((\neg B \Rightarrow \neg A) \Rightarrow ((\neg B \Rightarrow A) \Rightarrow B))$

B_3 $(A \Rightarrow (\neg B \Rightarrow A))$

B_4 $((\neg B \Rightarrow A) \Rightarrow B)$

B_5 $(A \Rightarrow B)$

B_6 $(\neg B \Rightarrow \neg A) \vdash (A \Rightarrow B)$

B_7 $((\neg B \Rightarrow \neg A) \Rightarrow (A \Rightarrow B))$

Exercise 4

Complete the proof from **Example 4** by providing comments how each step of the proof was obtained

Example 5

Here are consecutive steps $B_1, ..., B_9$

in a proof of $((A \Rightarrow B) \Rightarrow (\neg B \Rightarrow \neg A))$

B_1 $(A \Rightarrow B)$

B_2 $(\neg\neg A \Rightarrow A)$

B_3 $(\neg\neg A \Rightarrow B)$

B_4 $(B \Rightarrow \neg\neg B)$

B_5 $(\neg\neg A \Rightarrow \neg\neg B)$

B_6 $((\neg\neg A \Rightarrow \neg\neg B) \Rightarrow (\neg B \Rightarrow \neg A))$

B_7 $(\neg B \Rightarrow \neg A)$

B_8 $(A \Rightarrow B) \vdash (\neg B \Rightarrow \neg A)$

B_9 $((A \Rightarrow B) \Rightarrow (\neg B \Rightarrow \neg A))$

Exercise 5

Exercise 5

Complete the proof of **Example 5** by providing comments how each step of the proof was obtained.

Solution

$B_1 \quad (A \Rightarrow B)$
Hypothesis

$B_2 \quad (\neg\neg A \Rightarrow A)$
Example 1 for $B = A$

$B_3 \quad (\neg\neg A \Rightarrow B)$
Lemma 1 **(a)** for $A = \neg\neg A, \quad B = A, \quad C = B$

$B_4 \quad (B \Rightarrow \neg\neg B)$
Example 2

Exercise 5

B_5 $(\neg\neg A \Rightarrow \neg\neg B)$

Lemma 1 **(a)** for $A = \neg\neg A, \; B = B, \; C = \neg\neg B$

B_6 $((\neg\neg A \Rightarrow \neg\neg B) \Rightarrow (\neg B \Rightarrow \neg A))$

Example 4 for $B = \neg A, \; A = \neg B$

B_7 $(\neg B \Rightarrow \neg A)$

B_5, B_6 and MP

B_8 $(A \Rightarrow B) \vdash (\neg B \Rightarrow \neg A)$

$B_1 - B_7$

B_9 $((A \Rightarrow B) \Rightarrow (\neg B \Rightarrow \neg A))$

Deduction Theorem

Example 6

Prove that

$$\vdash \ (A \Rightarrow (\neg B \Rightarrow (\neg(A \Rightarrow B))))$$

Solution

Here are consecutive steps (with comments) of building the formal proof

$B_1 \quad A, (A \Rightarrow B) \vdash B$

This is MP

Example 6

$B_2 \quad A \vdash ((A \Rightarrow B) \Rightarrow B)$

Deduction Theorem

$B_3 \quad \vdash (A \Rightarrow ((A \Rightarrow B) \Rightarrow B))$

Deduction Theorem

$B_4 \quad \vdash (((A \Rightarrow B) \Rightarrow B) \Rightarrow (\neg B \Rightarrow \neg(A \Rightarrow B)))$

Example 5 for $A = (A \Rightarrow B), \quad B = B$

$B_5 \quad \vdash (A \Rightarrow (\neg B \Rightarrow (\neg(A \Rightarrow B)))$

$B_3, \; B_4$ and Lemma 2 **(a)** for

$A = A \quad B = ((A \Rightarrow B) \Rightarrow B), \quad C = (\neg B \Rightarrow (\neg(A \Rightarrow B))$

Observe that the proof presented is not the only proof

Example 7

Here are consecutive steps $B_1, \ldots, B_{12}$

in a proof of $((A \Rightarrow B) \Rightarrow ((\neg A \Rightarrow B) \Rightarrow B))$

B_1 $(A \Rightarrow B)$

B_2 $(\neg A \Rightarrow B)$

B_3 $((A \Rightarrow B) \Rightarrow (\neg B \Rightarrow \neg A))$

B_4 $(\neg B \Rightarrow \neg A)$

B_5 $((\neg A \Rightarrow B) \Rightarrow (\neg B \Rightarrow \neg\neg A))$

B_6 $(\neg B \Rightarrow \neg\neg A)$

B_7 $((\neg B \Rightarrow \neg\neg A) \Rightarrow ((\neg B \Rightarrow \neg A) \Rightarrow B)))$

Example 7

B_8 $((\neg B \Rightarrow \neg A) \Rightarrow B)$

B_9 B

B_{10} $(A \Rightarrow B), (\neg A \Rightarrow B) \vdash B$

B_{11} $(A \Rightarrow B) \vdash ((\neg A \Rightarrow B) \Rightarrow B)$

B_{12} $((A \Rightarrow B) \Rightarrow ((\neg A \Rightarrow B) \Rightarrow B))$

Exercise 7

Complete the proof in **Example 7** by providing comments how each step of the proof was obtained

Exercise 7

Solution

B_1 $(A \Rightarrow B)$
Hypothesis

B_2 $(\neg A \Rightarrow B)$
Hypothesis

B_3 $((A \Rightarrow B) \Rightarrow (\neg B \Rightarrow \neg A))$
Example 5

B_4 $(\neg B \Rightarrow \neg A)$
B_1, B_3 and MP

B_5 $((\neg A \Rightarrow B) \Rightarrow (\neg B \Rightarrow \neg\neg A))$
Example 5 for $A = \neg A, \quad B = B$

B_6 $(\neg B \Rightarrow \neg\neg A)$
$B2, B_5$ and MP

Exercise 7

B_7 $((\neg B \Rightarrow \neg\neg A) \Rightarrow ((\neg B \Rightarrow \neg A) \Rightarrow B)))$
Axiom A3 for $B = B, \quad A = \neg A$

B_8 $((\neg B \Rightarrow \neg A) \Rightarrow B)$
B_6, B_7 and MP

B_9 B
B_4, B_8 and MP

B_{10} $(A \Rightarrow B), (\neg A \Rightarrow B) \vdash B$
$B1 - B9$

B_{11} $(A \Rightarrow B) \vdash ((\neg A \Rightarrow B) \Rightarrow B)$
Deduction Theorem

B_{12} $((A \Rightarrow B) \Rightarrow ((\neg A \Rightarrow B) \Rightarrow B))$
Deduction Theorem

Example 8

Example 8

Here are consecutive steps

$$B_1, ..., B_3$$

in a proof of

$$((\neg A \Rightarrow A) \Rightarrow A)$$

B_1 $((\neg A \Rightarrow \neg A) \Rightarrow ((\neg A \Rightarrow A) \Rightarrow A)))$

B_2 $(\neg A \Rightarrow \neg A)$

B_3 $((\neg A \Rightarrow A) \Rightarrow A))$

Exercise 8

Complete the proof of **Example 8** by providing comments how each step of the proof was obtained

Solution

B_1 $((\neg A \Rightarrow \neg A) \Rightarrow ((\neg A \Rightarrow A) \Rightarrow A)))$

Axiom A3 for $B = A$

B_1 $(\neg A \Rightarrow \neg A)$

Already proved $(A \Rightarrow A)$ for $A = \neg A$

B_1 $((\neg A \Rightarrow A) \Rightarrow A))$

B_1, B_2 and MP

LEMMA

We **summarize** all the formal proofs in H_2 provided in our **Examples** and **Exercises** in a form of a following lemma

Lemma

The following formulas **are provable** in H_2

1. $(A \Rightarrow A)$

2. $(\neg\neg B \Rightarrow B)$

3. $(B \Rightarrow \neg\neg B)$

4. $(\neg A \Rightarrow (A \Rightarrow B))$

5. $((\neg B \Rightarrow \neg A) \Rightarrow (A \Rightarrow B))$

6. $((A \Rightarrow B) \Rightarrow (\neg B \Rightarrow \neg A))$

7. $(A \Rightarrow (\neg B \Rightarrow (\neg(A \Rightarrow B)))$

8. $((A \Rightarrow B) \Rightarrow ((\neg A \Rightarrow B) \Rightarrow B))$

9. $((\neg A \Rightarrow A) \Rightarrow A)$

Completeness Theorem for H_2

Formulas **1, 3, 4**, and **7-9** from the set of provable formulas from the **Lemma** are all formulas needed together with the logical axioms of H_2 to **execute** the two proofs of the **Completeness Theorem** for H_2

We present these proofs in the **Slides Set 3**

The two proofs **represent** two different methods of proving the **Completeness Theorem**

Chapter 5
Hilbert Proof Systems
Completeness of Classical Propositional Logic

Slides Set 3

PART 4: Completeness Theorem Proof One : Constructive Proof

Completeness Theorem: Proof One

The **Proof One** of the **Completeness Theorem** for H_2 presented here is similar in its structure to the proof of the **Deduction Theorem**

The **Proof One** is due to Kalmar, 1935 and is a detailed version of the one published in Elliott Mendelson's book Introduction to Mathematical Logic, 1987

The **Proof One** is, as **Deduction Theorem** was, constructive

It means it **defines** a method how one can **use** the assumption that a formula A is a **tautology** in order to **construct** its formal proof

Completeness Theorem: Proof One

The **Proof One** relies heavily on the **Deduction Theorem** and is very elegant and simple but its methods are **applicable only** to the classical propositional logic

The **Proof One** is specific to a propositional language

$$\mathcal{L}_{\{\neg,\ \Rightarrow\}}$$

and to the proof system H_2

Nevertheless, the H_2 based **Proof One** can be **adopted** and **extended** to other classical propositional languages containing implication and negation

Completeness Theorem: Proof One

For example we can **adopt** the **Proof One** to languages

$$\mathcal{L}_{\{\neg,\, \cup,\, \Rightarrow\}}, \quad \mathcal{L}_{\{\neg,\, \cap,\, \cup, \Rightarrow\}}, \quad \mathcal{L}_{\{\neg,\, \cap,\, \cup, \Rightarrow, \Leftrightarrow\}}$$

and **appropriate** proof systems based for them

We do so by **adding** new special logical axioms to the logical axioms of the proof system H_2

Such obtained proof systems are called **extensions** of the system H_2

Completeness Theorem: Proof One

One can think about the system H_2 with its axiomatization given by set

$$\{A1, A2, A3\}$$

of logical **axioms**, and its **language**

$$\mathcal{L}_{\{\neg, \Rightarrow\}}$$

as in a sense, a "minimal" **Hilbert System** for classical propositional logic

The **Proof One** can not be extended to the classical predicate logic, **neither** to the variety of non-classical logics

Proof System H_2

Reminder: H_2 is the following proof system:

$$H_2 = (\ \mathcal{L}_{\{\Rightarrow,\neg\}},\ \ \mathcal{F},\ \ \{A1, A2, A3\},\ \ MP\ )$$

The axioms $A1 - A3$ are defined as follows.

A1 $(A \Rightarrow (B \Rightarrow A))$,

A2 $((A \Rightarrow (B \Rightarrow C)) \Rightarrow ((A \Rightarrow B) \Rightarrow (A \Rightarrow C)))$,

A3 $((\neg B \Rightarrow \neg A) \Rightarrow ((\neg B \Rightarrow A) \Rightarrow B)))$

$$(MP)\ \frac{A\ ;\ (A \Rightarrow B)}{B}$$

Proof System H_2

Obviously, the selected axioms $A1, A2, A3$ are **tautologies**, and the MP rule leads from tautologies to tautologies.

Hence our proof system H_2 is **sound** and the following theorem holds

Soundness Theorem

For every formula $A \in \mathcal{F}$,

If $\vdash_{H_2} A$, then $\models A$

System H_2 Lemma

We have proved and presented in **Slides Set 2** the following

Lemma

The following formulas a **are provable** in H_2

1. $(A \Rightarrow A)$

2. $(\neg\neg B \Rightarrow B)$

3. $(B \Rightarrow \neg\neg B)$

4. $(\neg A \Rightarrow (A \Rightarrow B))$

5. $((\neg B \Rightarrow \neg A) \Rightarrow (A \Rightarrow B))$

6. $((A \Rightarrow B) \Rightarrow (\neg B \Rightarrow \neg A))$

7. $(A \Rightarrow (\neg B \Rightarrow (\neg(A \Rightarrow B)))$

8. $((A \Rightarrow B) \Rightarrow ((\neg A \Rightarrow B) \Rightarrow B))$

9. $((\neg A \Rightarrow A) \Rightarrow A)$

Proof One

The Proof One of **Completeness Theorem** presented here is very elegant and simple, but is applicable only to the classical propositional logic

This proof **is,** as was the proof of Deduction Theorem, a fully **constructive**

The technique it uses , because of its specifics can't be used even in a case of classical predicate logic, not to mention variaty of non-classical logics

Completeness Theorem

The **Proof One** is similar in its structure to the proof of the **Deduction Theorem** and is due to Kalmar, 1935

It is a constructive proof and relies heavily on the **Deduction Theorem**

It is possible to prove the **Completeness Theorem** independently of the Deduction Theorem and we will discus such a proofs in later chapters

Main Lemma

Some Notations

We write $\vdash A$ instead of $\vdash_S A$ as the system S is fixed.

Let A be a formula and $b_1, b_2, ..., b_n$ be all propositional variables that occur in A, we write it as $A = A(b_1, b_2, ..., b_n)$

Lemma Definition

Let v be a truth assignment $v : VAR \longrightarrow \{T, F\}$

We define, for $A, b_1, b_2, ..., b_n$ and truth assignment v corresponding formulas A', $B_1, B_2, ..., B_n$ as follows:

$$A' = \begin{cases} A & \text{if } v^*(A) = T \\ \neg A & \text{if } v^*(A) = F \end{cases}$$

$$B_i = \begin{cases} b_i & \text{if } v(b_i) = T \\ \neg b_i & \text{if } v(b_i) = F \end{cases}$$

for $i = 1, 2, ..., n$

Examples

Example

Let A be a formula $(a \Rightarrow \neg b)$

Let v be such that $v(a) = T, \;\; v(b) = F$

In this case we have that $b_1 = a, \;\; b_2 = b$, and

$v^*(A) = v^*(a \Rightarrow \neg b) = v(a) \Rightarrow \neg v(b) = T \Rightarrow \neg F = T$

The corresponding A', B_1, B_2 are:

$A' = A$ as $v^*(A) = T$

$B_1 = a$ as $v(a) = T$

$B_2 = \neg b$ as $v(b) = F$

Examples

Example 2

Let A be a formula $((\neg a \Rightarrow \neg b) \Rightarrow c)$

and let v be such that v(a)= T, v(b) =F, v(c)=F

Evaluate $A', B_1, ...B_n$ as defined by the **definition 1**

In this case $n = 3$ and $b_1 = a,\ b_2 = b, b_3 = c$

and we evaluate

$v^*(A) = v^*((\neg a \Rightarrow \neg b) \Rightarrow c) = ((\neg v(a) \Rightarrow \neg v(b)) \Rightarrow v(c)) = ((\neg T \Rightarrow \neg F) \Rightarrow F) = (T \Rightarrow F) = F$

The corresponding A', B_1, B_2, B_2 are:

$A' = \neg((\neg a \Rightarrow \neg b) \Rightarrow c)$ as $v^*(A) = F$

$B_1 = a$ as $v(a) = T$, $B_2 = \neg b$ as $v(b) = F$, and

$B_3 = \neg c$ as $v(c) = F$

Main Lemma

The Main Lemma stated below **describes** a method of transforming a **semantic** notion of a tautology into a **syntactic** notion of provability

It **defines**, for any formula A and a truth assignment v a corresponding **deducibility relation**

Main Lemma

For any formula $A = A(b_1, b_2, ..., b_n)$ and any truth assignment v

If $A^{'}, B_1, B_2, ..., B_n$ are corresponding formulas defined by **Lemma Definition**, then

$$B_1, B_2, ..., B_n \vdash A'$$

Examples

Example

Let A be a formula $(a \Rightarrow \neg b)$

Let v be such that $v(a) = T, \;\; v(b) = F$

We have that $A' = A, \;\; B_1 = a, \;\; B_2 = \neg b$

Main Lemma asserts that

$$a, \neg b \vdash (a \Rightarrow \neg b)$$

Example

Let A be a formula $((\neg a \Rightarrow \neg b) \Rightarrow c)$ and let v be such that v(a)= T, v(b) =F, v(c)=F

Main Lemma asserts that

$$a, \neg b, \neg c \vdash \neg((\neg a \Rightarrow \neg b) \Rightarrow c)$$

Proof of the Main Lemma

The proof is by induction on the **degree of the formula** A

Base Case $n = 0$

In this case A is atomic and so consists of a single propositional variable, say a

If $v^*(A) = T$ then we have by **Lemma Definition**

$A' = A = a, \ B_1 = a$

We obtain, by **definition of provability** from a set Γ of hypothesis for $\Gamma = \{a\}$ that

$$a \vdash a$$

Proof of the Main Lemma

If $v^*(A) = F$ we have by **Lemma Definition** that

$$A' = \neg A = \neg a \quad \text{and} \quad B_1 = \neg a$$

We obtain, by **definition of provability** from a set Γ of hypothesis for $\Gamma = \{\neg a\}$ that

$$\neg a \vdash \neg a$$

This **proves** that **Main Lemma** holds for n=0

Proof of the Main Lemma

Inductive Step

Assume that the **Main Lemma** holds for any formula with $j < n$ connectives

Need to prove: the **Main Lemma** holds for A with n connectives

There are several sub-cases to deal with

Case: A is $\neg A_1$

By the **inductive assumption** we have the formulas

$$A_1^{'}, \quad B_1, B_2, ..., B_n$$

corresponding to the A_1 and the propositional variables $b_1, b_2, ..., b_n$ in A_1, such that

$$B_1, B_2, ..., B_n \vdash A_1^{'}$$

Proof of the Main Lemma

Observe that the formulas A and $\neg A_1$ have the same propositional variables

So the corresponding formulas

$$B_1, \quad B_2, \ \ldots, \ B_n$$

are the **same** for both of them

We are going to show that the **inductive assumption** allows us to prove that

$$B_1, B_2, ..., B_n \ \vdash \ A^{'}$$

There are two cases to consider.

Proof of the Main Lemma

Case: $v^*(A_1) = T$

If $v^*(A_1) = T$ then by **Lemma Definition** $A_1^{'} = A_1$ and by the inductive assumption

$$B_1, B_2, ..., B_n \vdash A_1$$

In this case: $v^*(A) = v^*(\neg A_1) = \neg v^*(T) = F$

So we have that

$$A^{'} = \neg A = \neg\neg A_1$$

Proof of the Main Lemma

By **Lemma** formula **3.** we have that that

$$\vdash (A_1 \Rightarrow \neg\neg A_1)$$

we obtain by the monotonicity that also

$$B_1, B_2, ..., B_n \vdash (A_1 \Rightarrow \neg\neg A_1)$$

By **inductive assumption**

$$B_1, B_2, ..., B_n \vdash A_1$$

and by MP we have

$$B_1, B_2, ..., B_n \vdash \neg\neg A_1$$

and as $A' = \neg A = \neg\neg A_1$ we get $B_1, B_2, ..., B_n \vdash \neg A$ and so we proved that

$$B_1, B_2, ..., B_n \vdash A'$$

Proof of the Main Lemma

Case: $v^*(A_1) = F$

If $v^*(A_1) = F$ then $A_1' = \neg A_1$ and $v^*(A) = T$ so

$$A' = A$$

Therefore by the **inductive assumption** we have that

$$B_1, B_2, ..., B_n \vdash \neg A_1$$

as $A' = \neg A_1$ we get

$$B_1, B_2, ..., B_n \vdash A'$$

Proof of the Main Lemma

Case: A is $(A_1 \Rightarrow A_2)$

If A is $(A_1 \Rightarrow A_2)$ then A_1 and A_2 have less than n connectives

$A = A(b_1, ... b_n)$ so there are some **subsequences** $c_1, ..., c_k$ and $d_1, ...d_m$ for $k, m \leq n$ of the sequence $b_1, ..., b_n$ such that

$$A_1 = A_1(c_1, ..., c_k) \quad \text{and} \quad A_2 = A(d_1, ...d_m)$$

Proof of the Main Lemma

A_1 and A_2 have less than n connectives and so by the **inductive assumption** we have appropriate formulas $C_1, ..., C_k$ and $D_1, ...D_m$ such that

$$C_1, C_2, \ldots, C_k \vdash A_1{}' \quad \text{and} \quad D_1, D_2, \ldots, D_m \vdash A_2{}'$$

and $C_1, C_2, ..., C_k$, $D_1, D_2, ..., D_m$ are **subsequences** of formulas $B_1, B_2, ..., B_n$ corresponding to the propositional variables in A

By **monotonicity** we have the also

$$B_1, B_2, ..., B_n \vdash A_1{}' \quad \text{and} \quad B_1, B_2, ..., B_n \vdash A_2{}'$$

Now we have the following **sub-case** to consider

Proof of the Main Lemma

Case: $v^*(A_1) = v^*(A_2) = T$

If $v^*(A_1) = T$ then $A_1{}' = A_1$ and

if $v^*(A_2) = T$ then $A_2{}' = A_2$

We also have $v^*(A_1 \Rightarrow A_2) = T$ and so $A' = (A_1 \Rightarrow A_2)$

By the above and the **inductive assumption**

$$B_1, B_2, ..., B_n \vdash A_2$$

and By **Axiom 1** and by **monotonicity** we have

$$B_1, B_2, ..., B_n \vdash (A_2 \Rightarrow (A_1 \Rightarrow A_2))$$

By above and MP we have $B_1, B_2, ..., B_n \vdash (A_1 \Rightarrow A_2)$
that is

$$B_1, B_2, ..., B_n \vdash A'$$

Proof of the Main Lemma

Case: $v^*(A_1) = T, \; v^*(A_2) = F$

If $v^*(A_1) = T$ then $A_1{}' = A_1$ and

if $v^*(A_2) = F$ then $A_2{}' = \neg A_2$

Also we have in this case $v^*(A_1 \Rightarrow A_2) = F$ and so

$A' = \neg(A_1 \Rightarrow A_2)$

By the **above**, the **inductive assumption** and **monotonicity**

$B_1, B_2, ..., B_n \vdash \neg A_2$

By Lemma **7.** and by **monotonicity** we have

$$B_1, B_2, ..., B_n \vdash (A_1 \Rightarrow (\neg A_2 \Rightarrow \neg(A_1 \Rightarrow A_2)))$$

By above and MP **twice** we have

$B_1, B_2, ..., B_n \vdash \neg(A_1 \Rightarrow A_2)$ that is

$$B_1, B_2, ..., B_n \vdash A'$$

Proof of the Main Lemma

Case: $v^*(A_1) = F$

Observe that if $v^*(A_1) = F$ then $A_1{}'$ is $\neg A_1$ and, whatever value v gives A_2, we have

$$v^*(A_1 \Rightarrow A_2) = T$$

So A' is $(A_1 \Rightarrow A_2)$

Therefore

$$B_1, B_2, \ldots, B_n \vdash \neg A_1$$

From **Lemma** formula **4.** and by **monotonicity** we have

$$B_1,\ B_2, ...,\ B_n \vdash (\neg A_1 \Rightarrow (A_1 \Rightarrow A_2))$$

Proof of the Main Lemma

By Modus Ponens we get that

$$B_1, B_2, ..., B_n \vdash (A_1 \Rightarrow A_2)$$

that is

$$B_1, B_2, ..., B_n \vdash A^{'}$$

We have covered **all cases** and, by **mathematical induction** on the degree of the formula A we got

$$B_1, B_2, ..., B_n \vdash A^{'}$$

This **ends** the proof of the **Main Lemma**

Proof One of Completeness Theorem

Proof of Completeness Theorem

Now we use the **Main Lemma** to prove the following

Completeness Theorem (Completeness Part)
For any formula $A \in \mathcal{F}$

$$\text{if} \quad \models A \quad \text{then} \quad \vdash A$$

Proof
Assume that $\models A$
Let $b_1, b_2, ..., b_n$ be all propositional variables that occur in the formula A, i.e.

$$A = A(b_1, b_2, ..., b_n)$$

By the **Main Lemma** we know that, for any truth assignment v, the corresponding formulas $A^{'}$, B_1, B_2, ..., B_n can be found such that

$$B_1, B_2, ..., B_n \vdash A^{'}$$

Proof Completeness Theorem

Note that in this case $A' = A$ for any v since $\models A$

We have two cases.

1. If v is such that $v(b_n) = T$, then $B_n = b_n$ and

$$B_1, B_2, ..., b_n \vdash A$$

2. If v is such that $v(b_n) = F$, then $B_n = \neg b_n$ and by the **Main Lemma**

$$B_1, B_2, ..., \neg b_n \vdash A$$

So, by the **Deduction Theorem** we have

$$B_1, B_2, ..., B_{n-1} \vdash (b_n \Rightarrow A)$$

and

$$B_1, B_2, ..., B_{n-1} \vdash (\neg b_n \Rightarrow A)$$

Proof of Completeness Theorem

By **Lemma** formula **8.**

$$\vdash ((A \Rightarrow B) \Rightarrow ((\neg A \Rightarrow B) \Rightarrow B))$$

for $A = b_n,\ B = A$
By **monotonicity** we have that

$$B_1, B_2, ..., B_{n-1} \vdash ((b_n \Rightarrow A) \Rightarrow ((\neg b_n \Rightarrow A) \Rightarrow A))$$

Applying Modus Ponens twice we get that

$$B_1, B_2, ..., B_{n-1} \vdash A$$

Similarly, $v^*(B_{n-1})$ may be T or F
Applying the **Main Lemma** , the **Deduction Theorem**, **monotonicity**, **Lemma** formula **8.** and Modus Ponens twice we can eliminate B_{n-1} just as we have eliminated B_n
After n steps, we finally obtain proof of A in H_2, i.e. we proved that

$$\vdash A$$

Constructiveness of the Proof

Observe that the proof of the **Completeness Theorem** is **constructive**

Moreover, we have used in it only **Main Lemma** and **Deduction Theorem** which both have **constructive** proofs

We **can** hence **reconstruct** proofs in each case when we apply these theorems back to the **original axioms** of H_2

Constructiveness of the Proof

The same applies to the proofs in H_2 of all formulas **1.** - **9.** of the **Lemma**

It means that for any A, such that

$$\models A$$

the set V_A of all v **restricted** to A provides a method of a **construction** of the formal proof of A in H_2

Example

Example

The proof of **Completeness Theorem** defines a **method** of efficiently combining truth assignments $v \in V_A$ restricted to A while **constructing** the proof of A

Let's consider a **tautology** A, where the formula A is

$$A(a,b,c) = ((\neg a \Rightarrow b) \Rightarrow (\neg(\neg a \Rightarrow b) \Rightarrow c)$$

We **present** on the next slides all steps of the **Proof One** as applied to A

Example

Given

$$A(a,b,c) = ((\neg a \Rightarrow b) \Rightarrow (\neg(\neg a \Rightarrow b) \Rightarrow c))$$

By the **Main Lemma** and the assumption that

$$\models A(a,b,c)$$

any $v \in V_A$ **defines** formulas B_a , B_b, B_c such that

$$B_a, B_b, B_c \vdash A$$

The proof is based on a method of using all $v \in V_A$ (there are 8 of them) to **define** a process of elimination of all hypothesis B_a, B_b, B_c to **construct** the proof of A , i.e. to prove that

$$\vdash A$$

Example

Step 1: elimination of B_c

Observe that by definition, B_c is c or $\neg c$ depending on the **choice** of $v \in V_A$

We **choose** two truth assignments $v_1 \neq v_2 \in V_A$ such that

$$v_1 \mid \{a, b\} = v_2 \mid \{a, b\} \quad \text{and} \quad v_1(c) = T, \;\; v_2(c) = F$$

Case 1: $v_1(c) = T$

By by definition $B_c = c$

By our choice, the assumption that $\models A$ and the **Main Lemma** applied to v_1

$$B_a, B_b, c \vdash A$$

By **Deduction Theorem** we have that

$$B_a, B_b \vdash (c \Rightarrow A)$$

Example

Case 2: $v_2(c) = F$

By definition $B_c = \neg c$

By our **choice**, assumption that $\models A$, and the **Main Lemma** applied to v_2

$$B_a, B_b, \neg c \vdash A$$

By the **Deduction Theorem** we have that

$$B_a, B_b \vdash (\neg c \Rightarrow A)$$

Example

By **Lemma** formula **8.** for $A = c$, $B = A$ we have that

$$\vdash ((c \Rightarrow A) \Rightarrow ((\neg c \Rightarrow A) \Rightarrow A))$$

By **monotonicity** we have that

$$B_a, B_b \vdash ((c \Rightarrow A) \Rightarrow ((\neg c \Rightarrow A) \Rightarrow A))$$

Applying Modus Ponens twice to the above property and properties on the previous slide we get that

$$B_a, B_b \vdash A$$

We have **eliminated** B_c

Example

Step 2: elimination of B_b from $B_a, B_b \vdash A$

We repeat the **Step 1**

As before we have **2 cases** to consider: $B_b = b$ or $B_b = \neg b$

We **choose** two truth assignments $w_1 \neq w_2 \in V_A$ such that

$w_1 | \{a\} = w_2 | \{a\} = v_1 | \{a\} = v_2 | \{a\}$ and $w_1(b) = T$, $w_2(b) = F$

Case 1: $w_1(b) = T$ and by definition $B_b = b$

By our choice, assumption that $\models A$ and the **Main Lemma** applied to w_1

$$B_a, b \vdash A$$

By **Deduction Theorem** we have that

$$B_a \vdash (b \Rightarrow A)$$

Example

Case 2: $w_2(b) = F$ and by definition $B_b = \neg b$

By choice, assumption that $\models A$ and the **Main Lemma** applied to w_2

$$B_a, \neg b \vdash A$$

By the **Deduction Theorem** we have that

$$B_a \vdash (\neg b \Rightarrow A)$$

Example

By **Lemma** formula **8.** for $A = b,\ B = A$ we have that

$$\vdash ((b \Rightarrow A) \Rightarrow ((\neg b \Rightarrow A) \Rightarrow A))$$

By **monotonicity**

$$B_a \ \vdash \ ((b \Rightarrow A) \Rightarrow ((\neg b \Rightarrow A) \Rightarrow A))$$

Applying Modus Ponens twice to the above property and properties from the previous slide we get that

$$B_a \ \vdash \ A$$

We have **eliminated** B_b

Example

Step 3: elimination] of B_a from $B_a \vdash A$

We repeat the **Step 2**

As before we have **2 cases** to consider: $B_a = a$ or $B_a = \neg a$

We choose two truth assignments $g_1 \neq g_2 \in V_A$ such that

$$g_1(a) = T \quad \text{and} \quad g_2(a) = F$$

Case 1: $g_1(a) = T$, and by definition $B_a = a$

By the choice, assumption that $\models A$, and the **Main Lemma** applied to g_1

$$a \vdash A$$

By **Deduction Theorem** we have that

$$\vdash (a \Rightarrow A)$$

Example

Case 2: $g_2(a) = F$ and by definition $B_a = \neg a$

By the choice, assumption that $\models A$, and the **Main Lemma** applied to g_2

$$\neg a \vdash A$$

By the **Deduction Theorem** we have that

$$\vdash (\neg a \Rightarrow A)$$

Example

By **Lemma** formula **8.** for $A = a,\ B = A$ we have that

$$\vdash ((a \Rightarrow A) \Rightarrow ((\neg a \Rightarrow A) \Rightarrow A))$$

Applying Modus Ponens twice to the above property and properties from previous slides we get that

$$\vdash\ A$$

We have **eliminated** $B_a,\ B_b,\ B_c$ and constructed the **proof** of A in S

Exercises

Exercise 1

The **Lemma** listed formulas **1.** - **9.** that we said they were needed for **both** proofs of the **Completeness Theorem**.

List all the formulas from t**Lemma** that are are **needed** for the **Proof One** alone

Exercises

Exercise 2

The system H_2 was defined and the **Proof One** was carried out for the language $\mathcal{L}_{\{\Rightarrow,\neg\}}$

Extend the system H_2 and the **Proof One** to the language $\mathcal{L}_{\{\Rightarrow,\cup,\neg\}}$ by **adding** all new cases concerning the new connective $\cup$

List all new formulas needed to be **added** as new Axioms to H_2 to be able to follow the methods of the original **Proof One**

Exercise 3

Repeat the **Exercise 2** for he language

$$\mathcal{L}_{\{\Rightarrow,\ \cup,\ \cap\ \ \neg\}}$$

Chapter 5
Hilbert Proof Systems
Completeness of Classical Propositional Logic

Slides Set 4

PART 6: **Completeness Theorem Proof Two:**

A Counter- Model Existence Method

Completeness Theorem Proof Two

Our goal now is to prove the following

Completeness Theorem (Completeness Part)

For any formula $A \in \mathcal{F}$ of H_2

$$\text{if } \models A \text{ then } \vdash A$$

We do so by **proving** its logically equivalent **opposite implication:**

$$\text{If } \nvdash A, \text{ then } \not\models A$$

Hence the **Proof Two** consists of using the information that a formula A is not provable to show the **existence** of a counter-model for A

Completeness Theorem Proof Two

The **Proof Two** is much more complicated then the **Proof One**

The **main point** of the proof is a general, non- constructive method for proving **existence** of a counter-model for any non-provable formula A

The **generality** of the method makes it possible to **adopt** it for other cases of predicate and some non-classical logics

This is why we call the **Proof Two** a counter-model **existence** method

Proof Two Steps

The construction of a **counter-model** for any **non-provable** formula A presented in this proof is abstract, not constructive, as it was in the **Proof One**

It can be generalized to the case of predicate logic, and many of non-classical logics; propositional and predicate.

This is the reason we present it here

Proof Two Steps

We remind that $\not\models A$ means that there is a truth assignment $v : VAR \longrightarrow \{T, F\}$, such that (as we are in classical semantics) $v^*(A) = F$

We assume that A **does not** have a proof i.e. $\nvdash A$ we use this information in order to define a general method of constructing v, such that $v^*(A) = F$

This is done in the following steps.

Proof Two Steps

Step 1

Definition of a special set of formulas Δ^*

We use the information $\nvdash A$ to define a set of formulas Δ^* such that $\neg A \in \Delta^*$

Step 2

Definition of the counter - model

We define the variable truth assignment $v : VAR \longrightarrow \{T, F\}$ as follows:

$$v(a) = \begin{cases} T & \text{if} \quad \Delta^* \vdash a \\ F & \text{if} \quad \Delta^* \vdash \neg a \end{cases}$$

Proof 2 Steps

Step 3

We prove that v is a **counter-model** for A

We first prove a following more general property of v

Property

The set Δ^* and v defined in the Steps 1 and 2 are such that for every formula $B \in \mathcal{F}$

$$v^*(B) = \begin{cases} T & \text{if } \Delta^* \vdash B \\ F & \text{if } \Delta^* \vdash \neg B \end{cases}$$

We then use the **Step 3** to prove that $v^*(A) = F$

Main Notions

The definition, construction and the properties of the set Δ^* and hence the **Step 1**, are the most essential for the Proof Two

The other steps have mainly technical character

The **main notions** involved in the proof are: consistent set, complete set and a consistent complete extension of a set of formulas

We are going **prove** some essential facts about them.

Consistent and Inconsistent Sets

There exist two definitions of consistency; semantical and syntactical

Semantical definition uses the notion of a model and says:

A set is **consistent** if it has a **model**

Syntactical definition uses the notion of provability and says:

A set is **consistent** if one **can't prove** a contradiction from it

Consistent and Inconsistent Sets

In our proof of the **Completeness Theorem** we use the following formal syntactical definition of consistency of a set of formulas

Definition of a **consistent set**

We say that a set $\Delta \subseteq \mathcal{F}$ of formulas is **consistent** if and only if
there is no a formula $A \in \mathcal{F}$ such that

$$\Delta \vdash A \quad \text{and} \quad \Delta \vdash \neg A$$

Consistent and Inconsistent Sets

Definition of an **inconsistent set**

A set $\Delta \subseteq \mathcal{F}$ is **inconsistent** if and only if **there is** a formula $A \in \mathcal{F}$ such that

$$\Delta \vdash A \quad \text{and} \quad \Delta \vdash \neg A$$

The notion of consistency, as defined above, is characterized by the following **Consistency Lemma**

Consistency Condition Lemma

Lemma Consistency Condition

For every set $\Delta \subseteq \mathcal{F}$ of formulas, the following conditions are equivalent

(i) Δ is **consistent**

(ii) **there is** a formula $A \in \mathcal{F}$ such that $\Delta \nvdash A$

Proof of Consistency Lemma

Proof

To establish the equivalence of **(i)** and **(ii)** we prove the corresponding opposite implications

We prove the following two cases

Case 1 not **(ii)** implies not **(i)**

Case 2 not **(i)** implies not **(ii)**

Proof of Consistency Lemma

Case 1

Assume that not **(ii)**

It means that **for all formulas** $A \in \mathcal{F}$ we have that

$$\Delta \vdash A$$

In particular it is true for a certain $A = B$ and for a certain $A = \neg B$ i.e.

$$\Delta \vdash B \quad \text{and} \quad \Delta \vdash \neg B$$

and hence it proves that Δ is **inconsistent**

i.e. not **(i)** holds

Proof of Consistency Lemma

Case 2

Assume that not **(i)**, i.e that Δ is **inconsistent**

Then there is a formula A such that $\Delta \vdash A$ and $\Delta \vdash \neg A$

Let B be any formula

We proved (**Lemma** formula **6.**) that $\vdash (\neg A \Rightarrow (A \Rightarrow B))$

By monotonicity

$$\Delta \vdash (\neg A \Rightarrow (A \Rightarrow B))$$

Applying Modus Ponens twice to $\neg A$ first, and to A next we get that $\Delta \vdash B$ for any formula B

Thus not **(ii)** and it ends the proof of the Consistency Condition **Lemma**

Inconsistency Condition Lemma

Inconsistent sets are hence characterized by the following fact

Lemma Inconsistency Condition

For every set $\Delta \subseteq \mathcal{F}$ of formulas, the following conditions are equivalent:

(i) Δ is **inconsistent**,

(i) for **any formula** $A \in \mathcal{F}$ $\Delta \vdash A$

Finite Consequence Lemma

We remind here property of the finiteness of the **consequence** operation.

Lemma Finite Consequence

For every set Δ of formulas and for every formula $A \in \mathcal{F}$

$\Delta \vdash A$ if and only if there is a **finite** set $\Delta_0 \subseteq \Delta$ such that $\Delta_0 \vdash A$

Proof

If $\Delta_0 \vdash A$ for a certain $\Delta_0 \subseteq \Delta$,
hence by the monotonicity of the consequence, also $\Delta \vdash A$

Finite Consequence Lemma

Assume now that $\Delta \vdash A$ and let

$$A_1, A_2, ..., A_n$$

be a formal proof of A from Δ

Let

$$\Delta_0 = \{A_1, A_2, ..., A_n\} \cap \Delta$$

Obviously, Δ_0 is finite and $A_1, A_2, ..., A_n$ is a formal proof of A from Δ_0

Finite Inconsistency Theorem

The following theorem is a simple corollary of just proved Finite Consequence Lemma

Theorem Finite Inconsistency

(1.) If a set Δ is **inconsistent**, then it has a finite **inconsistent** subset Δ_0

(2.) If every finite subset of a set Δ is **consistent** then the set Δ is also **consistent**

Finite Inconsistency Theorem

Proof

If Δ is **inconsistent**, then for some formula A,

$$\Delta \vdash A \text{ and } \Delta \vdash \neg A$$

By the Finite Consequence Lemma , there are finite subsets Δ_1 and Δ_2 of Δ such that

$$\Delta_1 \vdash A \quad \text{and} \quad \Delta_2 \vdash \neg A$$

The union $\Delta_1 \cup \Delta_2$ is a finite subset of Δ and by monotonicity

$$\Delta_1 \cup \Delta_2 \vdash A \quad \text{and} \quad \Delta_1 \cup \Delta_2 \vdash \neg A$$

Hence we proved that $\Delta_1 \cup \Delta_2$ is a **finite inconsistent subset** of Δ

The second implication **(2.)** is the opposite to the one just proved and hence also holds

Consistency Lemma

The following **Lemma** links the notion of non-provability and consistency

It will be used as an important step in our **Proof Two** of the **Completeness Theorem**

Lemma

For any formula $A \in \mathcal{F}$,

if $\nvdash A$ then the set $\{\neg A\}$ is **consistent**

Consistency Lemma

Proof We prove the opposite implication

If $\{\neg A\}$ is **inconsistent**, then $\vdash A$

Assume that $\{\neg A\}$ is **inconsistent**

By the Inconsistency Condition **Lemma** we have that $\{\neg A\} \vdash B$ for **any formula** B, and hence in particular

$$\{\neg A\} \vdash A$$

By **Deduction Theorem** we get

$$\vdash (\neg A \Rightarrow A)$$

We proved (**Lemma** formula **9.**) that

$$\vdash ((\neg A \Rightarrow A) \Rightarrow A)$$

By Modus Ponens we get

$$\vdash A$$

This **ends the proof**

Complete and Incomplete Sets

Another important notion, is that of a **complete set** of formulas.

Complete sets, as defined here are sometimes called **maximal**, but we use the first name for them.

They are defined as follows.

Definition Complete set

A set Δ of formulas is called **complete** if for every formula $A \in \mathcal{F}$

$$\Delta \vdash A \quad \text{or} \quad \Delta \vdash \neg A$$

Godel used this notion of complete sets in his **Incompleteness of Arithmetic Theorem**

The complete sets are characterized by the following fact.

Complete and Incomplete Sets

Complete Set Condition Lemma

For every set $\Delta \subseteq \mathcal{F}$ of formulas, the following conditions are equivalent

(i) The set Δ is **complete**

(ii) For every formula $A \in \mathcal{F}$,

if $\Delta \nvdash A$ then then the set $\Delta \cup \{A\}$ is **inconsistent**

Proof

We consider two cases

Case 1 We show that **(i)** implies **(ii)** and

Case 2 we show that **(ii)** implies **(i)**

Complete Set Condition Lemma

Proof of **Case 1**

Assume **(i)** and not**(ii)** i.e.

assume that Δ is **complete** and there is a formula $A \in \mathcal{F}$ such that $\Delta \nvdash A$ and the set $\Delta \cup \{A\}$ is **consistent**

We have to show that we get a **contradiction**

But if $\Delta \nvdash A$, then from the assumption that Δ is **complete** we get that

$$\Delta \vdash \neg A$$

By the monotonicity of the consequence we have that

$$\Delta \cup \{A\} \vdash \neg A$$

Complete Set Condition Lemma

We proved (**Lemma** formula **4.**) $\vdash (A \Rightarrow A)$

By monotonicity $\Delta \vdash (A \Rightarrow A)$ and by **Deduction Theorem**

$$\Delta \cup \{A\} \vdash A$$

We hence proved that that there is a formula $A \in \mathcal{F}$ such that

$$\Delta \cup \{A\} \quad \text{and} \quad \Delta \cup \{A\} \vdash \neg A$$

i.e. that the set $\Delta \cup \{A\}$ is **inconsistent**

Contradiction

Complete Set Condition Lemma

Proof of **Case 2**

Assume **(ii)**, i.e. that for every formula $A \in \mathcal{F}$

if $\Delta \nvdash A$ then the set $\Delta \cup \{A\}$ is **inconsistent**

Let A be any formula.

We want to show **(i)**, i.e. to show that the following condition

$$\mathbf{C}: \quad \Delta \vdash A \quad \text{or} \quad \Delta \vdash \neg A$$

is satisfied.

Observe that if

$$\Delta \vdash \neg A$$

then the condition **C** is obviously satisfied

Complete Set Condition Lemma

If, on the other hand,

$$\Delta \nvdash \neg A$$

then we are going to show now that it must be, under the assumption of **(ii)**, that $\Delta \vdash A$ i.e. that **(i)** holds

Assume that

$$\Delta \nvdash \neg A$$

then by **(ii)** the set $\Delta \cup \{\neg A\}$ is **inconsistent**

Complete Set Condition Lemma

The Inconsistency Condition Lemma says

For every set $\Delta \subseteq \mathcal{F}$ of formulas, the following conditions are equivalent:

(i) Δ is **inconsistent**,

(i) for any formula $A \in \mathcal{F}$, $\Delta \vdash A$

We just proved that the set $\Delta \cup \{\neg A\}$ is **inconsistent**

So by the the above Lemma we get

$$\Delta \cup \{\neg A\} \vdash A$$

Complete Set Condition Lemma

By the **Deduction Theorem** $\Delta \cup \{\neg A\} \vdash A$ implies that

$$\Delta \vdash (\neg A \Rightarrow A)$$

Observe that by **Lemma** formula **4.**

$$\vdash ((\neg A \Rightarrow A) \Rightarrow A)$$

By monotonicity

$$\Delta \vdash ((\neg A \Rightarrow A) \Rightarrow A)$$

Detaching, by MP the formula $(\neg A \Rightarrow A)$ we obtain that

$$\Delta \vdash A$$

This **ends** the proof that **(i)** holds.

Incomplete Sets

Definition Incomplete Set

A set Δ of formulas is called **incomplete** if it is **not complete** i.e. when the following condition holds

There exists a formula $A \in \mathcal{F}$ such that

$$\Delta \nvdash A \quad \text{and} \quad \Delta \nvdash \neg A$$

Incomplete Set Condition Lemma

We get as a direct consequence of the Complete Set Condition Lemma the following characterization of **incomplete sets**

Lemma Incomplete Set Condition

For every set $\Delta \subseteq \mathcal{F}$ of formulas, the following conditions are equivalent:

(i) Δ is **incomplete**,

(ii) there is formula $A \in \mathcal{F}$ such that $\Delta \nvdash A$ and the set $\Delta \cup \{A\}$ is **consistent**.

Main Lemma: Complete Consistent Extension

Now we are going to prove a **Main Lemma** that is essential to the construction of the special set Δ^* mentioned in the **Step 1** of the proof of the **Completeness Theorem** and hence to the **proof of the theorem** itself

Let's first introduce one more notion

Complete Consistent Extension

Definition Extension Δ^* of the set Δ

A set Δ^* of formulas is called an **extension** of a set Δ of formulas if the following **condition holds**

$$\{A \in \mathcal{F} : \Delta \vdash A\} \subseteq \{A \in \mathcal{F} : \Delta^* \vdash A\}$$

i.e.

$$Cn(\Delta) \subseteq Cn(\Delta^*)$$

In this case **we say** also that Δ **extends** to the set of formulas Δ^*

Main Lemma

Main Lemma

Main Lemma Complete Consistent Extension

Every **consistent** set Δ of formulas can be **extended** to a **complete consistent** set Δ^* of formulas
i. e

For every **consistent** set Δ there is a set Δ^* that is **complete** and **consistent** and is an **extension** of Δ i.e.

$$Cn(\Delta) \subseteq Cn(\Delta^*)$$

Proof of the Main Lemma

Proof

Assume that the lemma does not hold, i.e. that there is a **consistent** set Δ, such that all its consistent extensions are **not complete**

In particular, as Δ is an consistent extension of itself, we have that Δ is **not complete**

The proof consists of a **construction** of a particular set Δ^* and **proving** that it forms a **complete** consistent extension of Δ

This is **contrary** to the assumption that all its consistent extensions are **not complete**

Construction of Δ^*

As we know, the set $\mathcal{F}$ of all formulas is enumerable; they can hence be put in an infinite sequence

$$\mathbf{F} \qquad A_1, A_2, \ \ldots \ , A_n, \ \ldots$$

such that every formula of $\mathcal{F}$ occurs in that sequence exactly once

We define, by **mathematical induction**, an infinite sequence

$$\mathbf{D} \qquad \{\Delta_n\}_{n \in N}$$

of **consistent** subsets of formulas together with a sequence

$$\mathbf{B} \qquad \{B_n\}_{n \in N}$$

of formulas as follows

Construction of Δ^*

Initial Step

In this step we define the sets

$$\Delta_1, \Delta_2 \quad \text{and the formula} \quad B_1$$

and **prove** that

$$\Delta_1 \quad \text{and} \quad \Delta_2$$

are **consistent, incomplete** extensions of Δ

We take as the first set in **D** the set Δ, i.e. we define

$$\Delta_1 = \Delta$$

Construction of Δ^*

By assumption the set Δ, and hence also Δ_1 is **not complete**.

From the Incomplete Set Condition **Lemma** we get that there is a formula $B \in \mathcal{F}$ such that

$$\Delta_1 \nvdash B \quad \text{and} \quad \Delta_1 \cup \{B\} \quad \text{is } \textbf{consistent}$$

Let B_1 be the **first formula** with this property in the sequence **F** of all formulas

We **define**

$$\Delta_2 = \Delta_1 \cup \{B_1\}$$

Construction of Δ^*

Observe that the set Δ_2 is consistent and

$$\Delta_1 = \Delta \subseteq \Delta_2$$

By monotonicity Δ_2 is a **consistent extension** of Δ

Hence, as we assumed that all consistent extensions of Δ are **not complete**, we get that Δ_2 cannot be complete, i.e.

Δ_2 is **incomplete**

Construction of Δ^*

Inductive Step

Suppose that we have defined a sequence

$$\Delta_1, \; \Delta_2, \; \ldots, \; \Delta_n$$

of **incomplete, consistent extensions** of Δ and a sequence

$$B_1, B_2, \; \ldots, \; B_{n-1}$$

of formulas, for $n \geq 2$

Construction of Δ^*

Since Δ_n is **incomplete**, it follows from the Incomplete Set Condition **Lemma** that

there is a formula $B \in \mathcal{F}$ such that

$$\Delta_n \nvdash B \quad \text{and} \quad \Delta_n \cup \{B\} \quad \text{is } \textbf{consistent}$$

Construction of Δ^*

Let B_n be the first formula with this property in the sequence **F** of all formulas.

We **define**

$$\Delta_{n+1} = \Delta_n \cup \{B_n\}$$

By the definition

$$\Delta \subseteq \Delta_n \subseteq \Delta_{n+1}$$

and the set Δ_{n+1} is a **consistent** extension of Δ

Hence by our assumption that all all consistent extensions o f Δ are **incomplete** we get that

$$\Delta_{n+1}$$

is an **incomplete** consistent extension of Δ

Construction of Δ^*

By the principle of mathematical induction we have defined an infinite sequence

$$\mathbf{D} \quad \Delta = \Delta_1 \subseteq \Delta_2 \subseteq ..., \subseteq \Delta_n \subseteq \Delta_{n+1} \subseteq ....$$

such that for all $n \in N$, Δ_n is **consistent**, and each Δ_n an **incomplete consistent extension** of Δ

Moreover, we have also defined a sequence

$$\mathbf{B} \qquad B_1, B_2, \ \ldots, B_n, \ldots$$

of formulas, such that for all $n \in N$,

$$\Delta_n \nvdash B_n \quad \text{and} \quad \Delta_n \cup \{B_n\} \quad \text{is } \textbf{consistent}$$

Observe that $B_n \in \Delta_{n+1}$ for all $n \geq 1$

Definition of Δ^*

Now we are ready to define Δ^*

Definition of Δ^*

$$\Delta^* = \bigcup_{n \in N} \Delta_n$$

To complete the proof our theorem we have now to prove that Δ^* is a **complete consistent extension** of Δ

Δ^* Consistent

Obviously directly from the definition $\Delta \subseteq \Delta^*$ and hence we have the following

Fact 1 Δ^* is an **extension** of Δ

By Monotonicity of Consequence $Cn(\Delta) \subseteq Cn(\Delta^*)$, hence extension

As the next step we prove

Fact 2 The set Δ^* is **consistent**

Δ^* Consistent

Proof that Δ^* is **consistent**

Assume that Δ^* is **inconsistent**

By the Finite Inconsistency Theorem there is a finite subset Δ_0 of Δ^* that is **inconsistent**, i.e.

$\Delta_0 \subseteq \bigcup_{n \in N} \Delta_n$, $\Delta_0 = \{C_1, ..., C_n\}$, Δ_0 is **inconsistent**

Proof of Δ^* Consistent

We have $\Delta_0 = \{C_1, \ldots, C_n\}$

By the definition of Δ^* for each formula $C_i \in \Delta_0$

$$C_i \in \Delta_{k_i}$$

for certain Δ_{k_i} in the sequence

$$\mathbf{D} \quad \Delta = \Delta_1 \subseteq \Delta_2 \subseteq ..., \subseteq \Delta_n \subseteq \Delta_{n+1} \subseteq ....$$

Hence $\Delta_0 \subseteq \Delta_m$ for $m = max\{k_1, k_2, ..k_n\}$

Proof of Δ^* Consistent

But we proved that all sets of the sequence **D** are **consistent**

This contradicts the fact that Δ_m is **consistent** as it contains an **inconsistent** subset Δ_0

This contradiction ends the proof that Δ^* is **consistent**

Proof of Δ^* Complete

Fact 3 The set Δ^* is **complete**

Proof Assume that Δ^* is **not complete**.

By the Incomplete Set Condition, there is a formula $B \in \mathcal{F}$ such that

$\Delta^* \nvdash B$, and the set $\Delta^* \cup \{B\}$ is **consistent**

By definition of the sequence **D** and the sequence **B** of formulas we have that for every $n \in N$

$$\Delta_n \nvdash B_n \quad \text{and the set} \quad \Delta_n \cup \{B_n\} \quad \text{is } \textbf{consistent}$$

Moreover $B_n \in \Delta_{n+1}$ for all $n \geq 1$

Proof of Δ^* Complete

Since the formula B is one of the formulas of the sequence **B** so we get that $B = B_j$ for certain j

By definition, $B_j \in \Delta_{j+1}$ and it proves that

$$B \in \Delta^* = \bigcup_{n \in N} \Delta_n$$

But this means that $\Delta^* \vdash B$

This is a contradiction with the assumption $\Delta^* \nvdash B$ and it **ends the proof** of the **Fact 3**

Main Lemma

Facts 1- 3 prove that that Δ^* is a **complete consistent extension** of Δ

We hence **completed** the proof of the Main Lemma

Main Lemma

Every **consistent** set Δ of formulas can be **extended** to a **complete consistent** set Δ^* of formulas

Proof Two of Completeness Theorem

Proof Two of Completeness Theorem

We proved already that H_2 is **sound**, so we have to prove only the Completeness part of the **Completeness Theorem**:

For any formula $A \in \mathcal{F}$,

$$\text{If} \quad \models A, \quad \text{then} \quad \vdash A$$

We prove it by **proving** its logically equivalent opposite implication form, i.e we prove now the following

Completeness Theorem

For any formula $A \in \mathcal{F}$,

$$\text{If} \quad \nvdash A, \quad \text{then} \quad \not\models A$$

Proof Two of Completeness Theorem

Proof

Assume that A **does not** have a proof, we want to define a **counter-model** for A

But if $\nvdash A$, then by the Inconsistency Lemma the set $\{\neg A\}$ is **consistent**

By the **Main Lemma** there is a complete, consistent extension of the set $\{\neg A\}$

This means that there is a set Δ^* such that $\{\neg A\} \subseteq \Delta^*$, i.e.

E $\quad \neg A \in \Delta^*$ and Δ^* is **complete** and **consistent**

Proof Two of Completeness Theorem

Since Δ^* is a **consistent, complete** set, it satisfies the following form of

Consistency Condition

For any $A \in \mathcal{F}$,

$$\Delta^* \nvdash A \quad \text{or} \quad \Delta^* \nvdash \neg A$$

Δ^* i s also **complete** i.e. satisfies

Completeness Condition

For any $A \in \mathcal{F}$,

$$\Delta^* \vdash A \quad \text{or} \quad \Delta^* \vdash \neg A$$

Proof Two of Completeness Theorem

Directly from the **Completeness** and **Consistency** Conditions we get the following

Separation Condition

For any $A \in \mathcal{F}$, **exactly one** of the following conditions is satisfied:

(1) $\Delta^* \vdash A$, or **(2)** $\Delta^* \vdash \neg A$

In particular case we have that for every propositional variable $a \in VAR$ **exactly one** of the following conditions is satisfied:

(1) $\Delta^* \vdash a$, or **(2)** $\Delta^* \vdash \neg a$

This justifies the **correctness** of the following definition

Proof Two of Completeness Theorem

Definition

We define the variable truth assignment

$$v : VAR \longrightarrow \{T, F\}$$

as follows:

$$v(a) = \begin{cases} T & \text{if } \Delta^* \vdash a \\ F & \text{if } \Delta^* \vdash \neg a. \end{cases}$$

We show, as a separate Lemma below, that such defined variable assignment v has the following property

Property of v Lemma

Lemma Property of v

Let v be the variable assignment defined above and v^* its extension to the set $\mathcal{F}$ of all formulas $B \in \mathcal{F}$, the following is true

$$v^*(B) = \begin{cases} T & \text{if } \Delta^* \vdash B \\ F & \text{if } \Delta^* \vdash \neg B \end{cases}$$

Proof 2 of Completeness Theorem

Given the Property of v Lemma (still to be proved)

we now **prove** that the v is in fact, a **counter model** for any formula A, such that $\nvdash A$

Let A be such that $\nvdash A$

By the Property **E** we have that $\neg A \in \Delta^*$

So obviously

$$\Delta^* \vdash \neg A$$

Hence by the Property of v Lemma

$$v^*(A) = F$$

what **proves** that v is a **counter-model** for A and it **ends the proof** of the Completeness Theorem

Proof of Property of v Lemma

Proof of the Property of v **Lemma**

The proof is conducted by the induction on the degree of the formula A

Initial step A is a propositional variable so the **Lemma** holds by definition of v

Inductive Step

If A is not a propositional variable, then A is of the form $\neg C$ or $(C \Rightarrow D)$, for certain formulas C, D

By the inductive assumption the **Lemma** holds for the formulas C and D

Proof of Property of v Lemma

Case $A = \neg C$

By the **Separation Condition** for Δ^* we consider two possibilities

1. $\Delta^* \vdash A$

2. $\Delta^* \vdash \neg A$

Consider case **1.** i.e. we assume that $\Delta^* \vdash A$

It means that

$$\Delta^* \vdash \neg C$$

Then from the fact that Δ^* is **consistent** it must be that

$$\Delta^* \nvdash C$$

Proof of Property of v Lemma

By the inductive assumption we have that $v^*(C) = F$ and accordingly $v^*(A) = v^*(\neg C) = \neg v^*(C) = \neg F = T$

Consider case **2.** i.e. we assume that $\Delta^* \vdash \neg A$

Then from the fact that Δ^* is **consistent** it must be that $\Delta^* \nvdash A$ and

$$\Delta^* \nvdash \neg C$$

If so, then $\Delta^* \vdash C$, as the set Δ^* is **complete**

By the **inductive assumption**, $v^*(C) = T$, and accordingly

$$v^*(A) = v^*(\neg C) = \neg v^*(C) = \neg T = F$$

Thus A **satisfies** the Property of v Lemma

Proof of Property of v Lemma

Case $A = (C \Rightarrow D)$

As in the previous case, we assume that the Lemma holds for the formulas C, D and we consider by the **Separation Condition** for Δ^* two possibilities:

1. $\Delta^* \vdash A$ and **2.** $\Delta^* \vdash \neg A$

Case 1. Assume $\Delta^* \vdash A$

It means that $\Delta^* \vdash (C \Rightarrow D)$

If at the same time $\Delta^* \nvdash C$, then $v^*(C) = F$, and accordingly

$$v^*(A) = v^*(C \Rightarrow D) =$$

$$v^*(C) \Rightarrow v^*(D) = F \Rightarrow v^*(D) = T$$

Proof of Property of v Lemma

If at the same time $\Delta^* \vdash C$, then since $\Delta^* \vdash (C \Rightarrow D)$, we infer, by Modus Ponens, that

$$\Delta^* \vdash D$$

If so, then $v^*(C) = v^*(D) = T$

and accordingly

$$v^*(A) = v^*(C \Rightarrow D) =$$

$$v^*(C) \Rightarrow v^*(D) = T \Rightarrow T = T$$

Thus if $\Delta^* \vdash A$, then $v^*(A) = T$

Proof of Property of v Lemma

Case 2. Assume now, as before, that $\Delta^* \vdash \neg A$,

Then from the fact that Δ^* is **consistent** it must be that $\Delta^* \nvdash A$, i.e.,

$$\Delta^* \nvdash (C \Rightarrow D)$$

It follows from this that $\Delta^* \nvdash D$

For if $\Delta^* \vdash D$, then, as $(D \Rightarrow (C \Rightarrow D))$ is provable formula **1.** in S, by monotonicity also

$$\Delta^* \vdash (D \Rightarrow (C \Rightarrow D))$$

Applying Modus Ponens we obtain

$$\Delta^* \vdash (C \Rightarrow D)$$

which is contrary to the assumption, so it must be $\Delta^* \nvdash D$

Proof of Property of v Lemma

Also we must have

$$\Delta^* \vdash C$$

for otherwise, as Δ^* is **complete** we would have $\Delta^* \vdash \neg C$
This this is **impossible** since by **Lemma** formula **9.**

$$\vdash (\neg C \Rightarrow (C \Rightarrow D))$$

By **monotonicity**

$$\Delta^* \vdash (\neg C \Rightarrow (C \Rightarrow D))$$

Applying Modus Ponens we would get

$$\Delta^* \vdash (C \Rightarrow D)$$

which is **contrary** to the assumption $\Delta^* \nvdash (C \Rightarrow D)$

Proof Two of Completeness Theorem

This **ends** the proof of the Property of v **Lemma** and the Proof Two of the **Completeness Theorem** is also **completed**

Chapter 5
Hilbert Proof Systems
Completeness of Classical Propositional Logic

Slides Set 5

PART 6: Some **Other Axiomatizations** and
Examples and Exercises

Some Other Axiomatizations

We present here some of the most **known**, and historically important **axiomatizations** of classical propositional logic

It means the **Hilbert** proof systems that are proven to be **complete** under classical semantics

Lukasiewicz (1929)

The Lukasiewicz proof system (axiomatization) is

$$L = (\ \mathcal{L}_{\{\neg,\ \Rightarrow\}},\ \ \mathcal{F},\ \ A1, A2, A3,\ \ MP\ )$$

where

A1 $((\neg A \Rightarrow A) \Rightarrow A)$

A2 $(A \Rightarrow (\neg A \Rightarrow B))$

A3 $((A \Rightarrow B) \Rightarrow ((B \Rightarrow C) \Rightarrow (A \Rightarrow C)))$

for any formulas $A, B, C \in \mathcal{F}$

Hilbert and Ackermann (1928)

$$HA = (\ \mathcal{L}_{\{\neg,\cup\}},\ \ \mathcal{F},\ \ A1 - A4,\ \ MP\ )$$

where for any $A, B, C \in \mathcal{F}$

A1 $(\neg(A \cup A) \cup A)$

A2 $(\neg A \cup (A \cup B))$

A3 $(\neg(A \cup B) \cup (B \cup A))$

A4 $(\neg(\neg B \cup C) \cup (\neg(A \cup B) \cup (A \cup C)))$

The Modus Ponens rule in the language $\mathcal{L}_{\{\neg,\cup\}}$ has a form

$$MP\ \ \frac{A\ ;\ \ (\neg A \cup B)}{B}$$

Hilbert and Ackermann

Observe that also the **Deduction Theorem** is now formulated as follow.

Deduction Theorem for *HA*

For any subset Γ of the set of formulas $\mathcal{F}$ of HA and for any formulas $A, B \in \mathcal{F}$,

$$\Gamma,\ A \vdash_{HA} B \quad \text{if and only if} \quad \Gamma \vdash_{HA} (\neg A \cup B)$$

In particular,

$$A \vdash_{HA} B \quad \text{if and only if} \quad \vdash_{HA} (\neg A \cup B)$$

Hilbert (1928)

$$H = (\ \mathcal{L}_{\{\neg,\cup,\cap,\Rightarrow\}},\ \ \mathcal{F},\ \ A1 - A15,\ \ MP\ )$$

where for any $A, B, C \in \mathcal{F}$

A1 $(A \Rightarrow A)$

A2 $(A \Rightarrow (B \Rightarrow A))$

A3 $((A \Rightarrow B) \Rightarrow ((B \Rightarrow C) \Rightarrow (A \Rightarrow C)))$

A4 $((A \Rightarrow (A \Rightarrow B)) \Rightarrow (A \Rightarrow B))$

A5 $((A \Rightarrow (B \Rightarrow C)) \Rightarrow (B \Rightarrow (A \Rightarrow C)))$

A6 $((A \Rightarrow B) \Rightarrow ((B \Rightarrow C) \Rightarrow (A \Rightarrow C)))$

A7 $((A \cap B) \Rightarrow A)$

A8 $((A \cap B) \Rightarrow B)$

Hilbert

A9 $((A \Rightarrow B) \Rightarrow ((A \Rightarrow C) \Rightarrow (A \Rightarrow (B \cap C)))$

A10 $(A \Rightarrow (A \cup B))$

A11 $(B \Rightarrow (A \cup B))$

A12 $((A \Rightarrow C) \Rightarrow ((B \Rightarrow C) \Rightarrow ((A \cup B) \Rightarrow C)))$

A13 $((A \Rightarrow B) \Rightarrow ((A \Rightarrow \neg B) \Rightarrow \neg A))$

A14 $(\neg A \Rightarrow (A \Rightarrow B))$

A1 - A14 are the axioms **Hilbert** proposed and were accepted as axioms defining Intuitionistic logic

They were later **proved** to be **complete** when the intuitionistic **semantics** was discovered

Hilbert obtained his classical axiomatization by adding as the last axiom the **excluded middle** law rejected by intuitionists

A15 $(A \cup \neg A)$

Kleene

Kleene (1952)

$$K = (\ \mathcal{L}_{\{\neg,\cup,\cap,\Rightarrow\}},\ \ \mathcal{F},\ \ A1 - A10,\ \ MP\ )$$

where for any $A, B, C \in \mathcal{F}$

A1 $(A \Rightarrow (B \Rightarrow A))$

A2 $((A \Rightarrow (B \Rightarrow C)) \Rightarrow (B \Rightarrow (A \Rightarrow C)))$

A3 $((A \cap B) \Rightarrow A)$

A4 $((A \cap B) \Rightarrow B)$

A5 $(A \Rightarrow (B \Rightarrow (A \cap B)))$

Kleene

A6 $(A \Rightarrow (A \cup B))$

A7 $(B \Rightarrow (A \cup B))$

A8 $((A \Rightarrow C) \Rightarrow ((B \Rightarrow C) \Rightarrow ((A \cup B) \Rightarrow C)))$

A9 $((A \Rightarrow B) \Rightarrow ((A \Rightarrow \neg B) \Rightarrow \neg A))$

A10 $(\neg\neg A \Rightarrow A)$

Kleene proved that when A10 is **replaced** by

A10' $(\neg A \Rightarrow (A \Rightarrow B))$

the **resulting** system is a **complete** axiomatization of Intuitionistic Logic

Rasiowa-Sikorski (1950)

$$RS = (\ \mathcal{L}_{\{\neg,\cup,\cap,\Rightarrow\}},\ \ \mathcal{F},\ \ A1 - A12,\ \ MP\ )$$

where for any $A, B, C \in \mathcal{F}$

A1 $((A \Rightarrow B) \Rightarrow ((B \Rightarrow C) \Rightarrow (A \Rightarrow C)))$

A2 $(A \Rightarrow (A \cup B))$

A3 $(B \Rightarrow (A \cup B))$

A4 $((A \Rightarrow C) \Rightarrow ((B \Rightarrow C) \Rightarrow ((A \cup B) \Rightarrow C)))$

Rasiowa-Sikorski

A5 $((A \cap B) \Rightarrow A)$

A6 $((A \cap B) \Rightarrow B)$

A7 $((C \Rightarrow A) \Rightarrow ((C \Rightarrow B) \Rightarrow (C \Rightarrow (A \cap B)))$

A8 $((A \Rightarrow (B \Rightarrow C)) \Rightarrow ((A \cap B) \Rightarrow C))$

A9 $(((A \cap B) \Rightarrow C) \Rightarrow (A \Rightarrow (B \Rightarrow C))$

A10 $(A \cap \neg A) \Rightarrow B)$

A11 $((A \Rightarrow (A \cap \neg A)) \Rightarrow \neg A)$

A12 $(A \cup \neg A)$

Rasiowa-Sikorski

Rasiowa - Sikorski proved A1 - A11 to be a **complete** axiomatization for the Intuitionistic Logic

They obtained the classical axiomatization by adding A12, the **excluded middle** law rejected by intuitionists, as **Hilbert** did

Both classical and intuitionistic **completeness** proofs were carried under respective Boolean and Pseudo-Boolean algebras semantics what is reflected in the **choice** of axioms A1 - A12

Shortest Axiomatizations

Here is the shortest axiomatization for the language

$$\mathcal{L}_{\{\neg,\ \Rightarrow\}}$$

It contains just one axiom

Meredith (1953)

$$M = (\ \mathcal{L}_{\{\neg,\ \Rightarrow\}},\ \ \mathcal{F},\ \ A1\ \ MP\ )$$

where

A1 $((((((A \Rightarrow B) \Rightarrow (\neg C \Rightarrow \neg D)) \Rightarrow C) \Rightarrow E)) \Rightarrow ((E \Rightarrow A) \Rightarrow (D \Rightarrow A)))$

Shortest Axiomatizations

Here is another axiomatization that uses only one axiom

Nicod (1917)

$$N = (\ \mathcal{L}_{\{\uparrow\}},\ \ \mathcal{F},\ \ A1,\ \ (r)\ )$$

where

A1 $(((A \uparrow (B \uparrow C)) \uparrow ((D \uparrow (D \uparrow D)) \uparrow ((E \uparrow B) \uparrow ((A \uparrow E) \uparrow (A \uparrow E))))))$

and

$$(r)\ \frac{A \uparrow (B \uparrow C)}{A}$$

Reminder

We have proved in chapter 3 that

$$\mathcal{L}_{\{\neg,\cup,\cap,\Rightarrow\}} \equiv \mathcal{L}_{\{\uparrow\}}$$

Exercises

Here are few exercises designed to help with understanding the notions of **completeness**, monotonicity of the **consequence operation**, the role of the **deduction theorem** and the importance of somc basic **tautologies**

Complete Hilbert System S

Let S be any **Hilbert** proof system

$$S = (\mathcal{L}_{\{\cap,\cup,\Rightarrow,\neg\}},\ \mathcal{F},\ \ LA,\ \ MP\ \frac{A, (A \Rightarrow B)}{B}\ )$$

with the set LA of logical axioms such that S is **complete** under classical semantics

Let $X \subseteq \mathcal{F}$ be any subset of the set $\mathcal{F}$ of formulas of the language

$$\mathcal{L}_{\{\cap,\cup,\Rightarrow,\neg\}}$$

We **define**, as we did in chapter 4, a set $Cn(X)$ of all **consequences** of the set X as

$$Cn(X) = \{A \in \mathcal{F} :\ \ X \vdash_S A\}$$

Exercises

Reminder

The proof system

$$S = (\mathcal{L}_{\{\cap,\cup,\Rightarrow,\neg\}},\ \mathcal{F},\ \ LA,\ \ MP\ \ \frac{A, (A \Rightarrow B)}{B}\ )$$

in **all exercises** is **complete**

Exercises

Exercise 1

1. Prove that for any subsets X, Y of the set $\mathcal{F}$ of formulas of S the following **monotonicity property** holds

$$\text{If } X \subseteq Y, \quad \text{then} \quad Cn(X) \subseteq Cn(Y)$$

Solution

1. Let $A \in \mathcal{F}$ be any formula such that $A \in Cn(X)$

By the consequence definition, we have that $X \vdash_S A$ and A has a formal proof from the set $X \cup LA$

But $X \subseteq Y$, hence this proof is also a proof from the set $Y \cup LA$, i.e . $Y \vdash_S A$ and $A \in Cn(Y)$

This proves that $Cn(X) \subseteq Cn(Y)$

Exercises

Exercise 1

2. Do we need the **completeness** of S to prove that the **monotonicity** property holds for S?

Solution

2. No, we do not need the **completeness** of S for the **monotonicity** property to hold

We have used only the **definition** of a formal proof from the hypothesis X and the definition of the consequence operation

Exercises

Exercise 2

1. Prove that for any set $X \subseteq \mathcal{F}$, the set $\mathbf{T} \subseteq \mathcal{F}$ of all classical **tautologies** of the language $\mathcal{L}_{\{\cap, \cup, \Rightarrow, \neg\}}$ of the system S is a **subset** of $Cn(X)$; i.e. prove that

$$\mathbf{T} \subseteq Cn(X)$$

2. Do we need the **completeness** of S to prove that the property $\mathbf{T} \subseteq Cn(X)$ holds for S?

Solution

1. The proof system S is **complete**, so by the **completeness theorem** we have that

$$\mathbf{T} = \{\in \mathcal{F} : \vdash_S A\}$$

By definition of the consequence,

$$\{A \in \mathcal{F} : \vdash_S A\} = Cn(\emptyset)$$

and hence $Cn(\emptyset) = \mathbf{T}$

But $\emptyset \subseteq X$ for any set X, so by **monotonicity** property

$$\mathbf{T} \subseteq Cn(X)$$

2. Yes, the **completeness** of S in the main property used in the proof of **1.**

The other property is the **monotonicity**

Exercises

Exercise 3

Prove that for any formulas $A, B \in \mathcal{F}$, and for any set $X \subseteq \mathcal{F}$,

$$(A \cap B) \in Cn(X) \text{ if and only if } A \in Cn(X) \text{ and } B \in Cn(X)$$

List all properties essential to the proof

Solution

(1) Proof of the implication:

if $(A \cap B) \in Cn(X)$, then $A \in Cn(X)$ and $B \in Cn(X)$

Assume $(A \cap B) \in Cn(X)$, i.e. $X \vdash_S (A \cap B)$
From **monotonicity** property proved in **Exercise 1**, **completeness** of S, and the fact that

$$\models ((A \cap B) \Rightarrow A) \quad \text{and} \quad \models ((A \cap B) \Rightarrow B)$$

we get that

$$X \vdash_S ((A \cap B) \Rightarrow A) \quad \text{and} \quad X \vdash_S ((A \cap B) \Rightarrow B)$$

From the **assumption** $X \vdash_S (A \cap B)$ and the above

$$X \vdash_S ((A \cap B) \Rightarrow A)$$

we get by Modus Ponens

$$X \vdash_S A$$

Exercises

Similarly, from the **assumption** $X \vdash_S (A \cap B)$ and the above property

$$X \vdash_S ((A \cap B) \Rightarrow B)$$

we get by Modus Ponens

$$X \vdash_S B$$

This proves that $A \in Cn(X)$ and $B \in Cn(X)$ and **ends** the **proof** of the implication **(1)**

Exercises

(2) Proof of the implication:

if $A \in Cn(X)$ and $B \in Cn(X)$, then $(A \cap B) \in Cn(X)$

Assume now $A \in Cn(X)$ and $B \in Cn(X)$, i.e.

$$X \vdash_S A \quad \text{and} \quad X \vdash_S B$$

By the **monotonicity** property, **completeness** of S, and **tautology**

$$(A \Rightarrow (B \Rightarrow (A \cap B)))$$

we get that

$$X \vdash_S (A \Rightarrow (B \Rightarrow (A \cap B)))$$

Exercises

By the **assumption** we have that

$$X \vdash_S A, \quad X \vdash_S B$$

and the above

$$X \vdash_S (A \Rightarrow (B \Rightarrow (A \cap B)))$$

we get by Modus Ponens

$$X \vdash_S (B \Rightarrow (A \cap B))$$

Applying Modus Ponens again we obtain

$$X \vdash_S (A \cap B)$$

This proves

$$(A \cap B) \in Cn(X)$$

and **ends** the **proof** and the implication **(2)** and the **proof** of **Exercise 3**

Exercises

Exercise 4

Prove that classical completeness of a **Hilbert** proof system **implies** the **Deduction Theorem**, i.e prove that the following theorem holds for the system S

Deduction Theorem

For any subset Γ of the set of formulas $\mathcal{F}$ of S and for any formulas $A, B \in \mathcal{F}$,

$$\Gamma,\ A \vdash_S B \quad \text{if and only if} \quad \Gamma \vdash_S (A \Rightarrow B)$$

Solution

The formulas

$$A1 = (A \Rightarrow (B \Rightarrow A)) \quad \text{and}$$

$$A2 = ((A \Rightarrow (B \Rightarrow C)) \Rightarrow ((A \Rightarrow B) \Rightarrow (A \Rightarrow C)))$$

are basic classical autologies

By the **completeness** of S we have that

$$\vdash_S (A \Rightarrow (B \Rightarrow A)) \text{ and}$$

$$\vdash_S ((A \Rightarrow (B \Rightarrow C)) \Rightarrow ((A \Rightarrow B) \Rightarrow (A \Rightarrow C)))$$

The formulas $A1, A2$ are the **axioms** of the Hilbert system H_1

By the **completeness** of S, we have that both axioms of H_1 are **provable** in S

These axioms were **sufficient** for the proof of the **Deduction Theorem** for H_1 and so the H_1 proof can be **repeated** for the system S

Exercise 5

Prove that for any $A, B \in \mathcal{F}$

$$Cn(\{A, B\}) = Cn(\{(A \cap B)\})$$

Solution

(1) Proof of the inclusion

$$Cn(\{A, B\}) \subseteq Cn(\{(A \cap B)\})$$

Assume $C \in Cn(\{A, B\})$, i.e. we assume $A,\ B \vdash_S C$

By **Exercise 4** the **Deduction Theorem** holds for S and we apply it **twice** to get an equivalent form

$$\vdash_S (A \Rightarrow (B \Rightarrow C))$$

of the **assumption**

Exercises

We use **completeness** of S, the fact that the formula

$$(((A \Rightarrow (B \Rightarrow C)) \Rightarrow ((A \cap B) \Rightarrow C)))$$

is a **tautology** and get that

$$\vdash_S (((A \Rightarrow (B \Rightarrow C)) \Rightarrow ((A \cap B) \Rightarrow C)))$$

Applying Modus Ponens to the above and the assumption

$$\vdash_S (A \Rightarrow (B \Rightarrow C))$$

we get

$$\vdash_S ((A \cap B) \Rightarrow C)$$

This is equivalent by **Deduction Theorem** to

$$(A \cap B) \vdash_S C$$

We have proved that

$$C \in Cn(\{(A \cap B)\})$$

and this **ends** the proof of the inclusion **(1)**

Exercises

(2) Proof of the inclusion

$$Cn(\{(A \cap B)\}) \subseteq Cn(\{A, B\})\})$$

Assume that $C \in Cn(\{(A \cap B)\})$, i.e.

$$(A \cap B) \vdash_S C$$

By **Deduction Theorem**

$$\vdash_S ((A \cap B) \Rightarrow C)$$

We want to prove that $C \in Cn(\{A, B\})$

This is equivalent, by **Deduction Theorem** applied **twice** to proving that

$$\vdash_S (A \Rightarrow (B \Rightarrow C))$$

Exercises

The proof is similar to the previous case

We use **completeness** of S, the fact that the formula

$$(((A \cap B) \Rightarrow C) \Rightarrow (A \Rightarrow (B \Rightarrow C)))$$

is a **tautology** to get

$$\vdash_S (((A \cap B) \Rightarrow C) \Rightarrow (A \Rightarrow (B \Rightarrow C)$$

Applying Modus Ponens to above and the the assumption

$$\vdash_S ((A \cap B) \Rightarrow C)$$

we get

$$\vdash_S (A \Rightarrow (B \Rightarrow C))$$

what **ends** the **proof**