Chapter 2: Introduction to Propositional Logic

PART ONE: History and Motivation

Origins: Stoic school of philosophy (3rd century B.C.), with the most eminent representative was **Chryssipus.**

Modern Origins: Mid-19th century - English mathematician **G. Boole**, who is sometimes regarded as the founder of mathematical logic.

First Axiomatic System: 1879 by German logician **G. Frege.**

The first assumption underlying the formalization of classical propositional logic (calculus) is the following.

We assume that sentences are always evaluated as **true** or **false**. Such sentences are called **logical sentences** or **propositions**.

Hence the name **propositional logic**.

A statement: $2+2=4$ is a proposition (true).

A statement: $2+2=5$ is also a proposition (false).

A statement: *I am pretty* is modeled as a logical sentence (proposition). We assume that it is false, or true.

A statement: $2+n=5$ **is not** a proposition; it might be true for some n, for example n=3, false for other n, for example n= 2, and moreover, we don't know what n is. Sentences of this kind are called **propositional functions**.

We model propositional functions within propositional logic by treating propositional functions as propositions.

The classical logic reflects the **black** and **white** qualities of mathematics.

We expect from mathematical theorems to be always either true or false and the reasonings leading to them should guarantee this without any ambiguity.

Formulas

We combine logical sentences to form more complicated sentences, called **formulas**.

We combine them using the following words or phrases:

not; *and*; *or*; *if ..., then*; *if and only if*.

We use only **symbols** do denote both *logical sentences* and the phrases: *not*; *and*; *or*; *if ..., then*; *if and only if*.

Hence the name **symbolic logic**.

Logical sentences are denoted by **symbols**

$$a, b, c, p, r, q, ..$$

Symbols for logical connectives are: $\neg$ for "not", $\cap$ for "and", $\cup$ for "or", $\Rightarrow$ for *"if ..., then"*, and $\Leftrightarrow$ for *"if and only if"*.

Translate a natural language sentence:

The fact that it is not true that at the same time $2+2=4$ *and* $2+2=5$ *implies that* $2+2=4$

into its **propositional symbolic logic** formula.

First we write it in a form:

If not $(2+2=4$ *and* $2+2=5)$ *then* $2+2=4$

Second we write it in a symbolic formula:

$(\neg(a \cap b) \Rightarrow a)$.

Translate a natural language sentence:

The fact that it is not true that at the same time $2 + n = 4$ *and some numbers are pretty implies that* $2 + n = 4$

into its **propositional symbolic logic** formula.

First we write it in a form:

If not $(2 + n = 4$ *and* some numbers are pretty) *then* $2 + n = 4$

Second we write it in a symbolic formula:

$(\neg(a \cap b) \Rightarrow a)$.

Syntax of **a symbolic language** is the formal description of the symbols we use and the way we construct the **formulas.**

A formal language, or just a language, is another word for the symbolic language.

Propositional languages are the syntax of **propositional logics.**

Predicate languages are the syntax of more complex logics, called **predicate logics** or **predicate calculi**.

GENERAL REMARK: The formal language symbols and formulas i.e. the established syntax don't directly carry with them any *logical value*.

We assign them their logical value in a separate step.

This next step is called **a semantics** of the given language.

A given language can have different semantics and the different semantics will define different logics.

Propositional Language

Any symbolic language consists of *an alphabet* and *a set of formulas*.

Propositional language consists of *a propositional alphabet* and *a set of formulas* (propositional).

Propositional Alphabet consists of a set of *variables* and a set of *propositional connectives*.

Variables are the symbols denoting logical sentences (propositions) are called **propositional variables**.

We denote the propositional variables by letters *a, b, c,....* , with indices if necessary.

We also use $a_1, a_2, ..., b_1, b_2$, ... etc... as symbols for propositional variables.

The symbols for connectives are: $\neg$, $\cap$, $\cup$, $\Rightarrow$, $\Leftrightarrow$ and their names are: *a negation*, *a conjunction*, *a disjunction*, *an implication*,and *an equivalence*, respectively.

Formulas are expressions build by means of logical connectives and variables and are be denoted by $A, B, C, ...$, with indices, if necessary.

The propositional variables are formulas and are called **atomic formulas**.

Recursive step: if we already have two formulas A, B, then we adopt the expression: $(A \cap B), (A \cup B), (A \Rightarrow B), (A \Leftrightarrow B)$ and also $\neg A$ as formulas.

Example

By the definition, any propositional variable is a formula. For example, a, b are formulas (atomic).

By the recursive step we get that

$$(a \cap b), (a \cup b), (a \Rightarrow b), (a \Leftrightarrow b), \neg a, \neg b$$

are formulas.

Recursive step applied again produces for example the following formulas :

$$\neg(a \cap b), ((a \Leftrightarrow b) \cup \neg b), \neg\neg a, \neg\neg(a \cap b).$$

We didn't list all formulas we obtained in the first recursive step.

Moreover , the recursive process continue. The set of all formulas is *(countably) infinite.*

Remark that we put parenthesis within the formulas in a way to avoid *ambiguity.*

The expression: $a \cap b \cup a$, is ambiguous. We don't know whether it represents $(a \cap b) \cup a$ or $a \cap (b \cup a)$. So, it is not a formula.

Introduction to Semantics for Classical Propositional Connectives

We present here the definition of propositional connectives in terms of logical values (*true* or *false*) and discussed the motivations for presented definitions.

The resulting definitions are called *a semantics for the classical propositional connectives.*

The formal description of a process of assigning a logical value (*true* or *false*) to all formulas is called *a semantics of the classical propositional logic.*

CONJUNCTION - Motivation and definition.

A conjunction $(A \cap B)$ is a *true* formula if both A and B are *true* formulas. If one of the formulas, or both, are *false*, then the conjunction is a *false* formula.

Denote *A is false* by $v(A) = F$ and *A is true* by $v(A) = T$.

The logical value of a conjunction depends on the logical values of its factors in a way which is express in the form of the following table (truth table).

Conjunction Table :

$v(A)$	$v(B)$	$v(A \cap B)$
T	T	T
T	F	F
F	T	F
F	F	F

DISJUNCTION - Motivation and definition.

The word *or* is used in two different senses.

First: *A or B* is *true* if at least one of the statements A and B is true.

Second: *A or B* is *true* if one of the statements A and B is true, and the other is false.

In mathematics and hence in logic, the word *or* is used in the first sense.

Hence, we adopt the convention that a *disjunction* $(A \cup B)$ is *true* if at least one of the formulas A and B is true.

We write in a form of the following

Disjunction Table :

$v(A)$	$v(B)$	$v(A \cup B)$
T	T	T
T	F	T
F	T	T
F	F	F

NEGATION - Motivation and definition.

The negation of a true formula is a false formula, and the negation of a false formula is a true formula. This is expressed in the following

Negation Table :

$v(A)$	$v(\neg A)$
T	F
F	T

IMPLICATION - Motivation and definition.

The semantics of the statements in the form *if A, then B* needs a little but more od discussion.

In everyday language a statement
if A, then B
is interpreted to mean that B can be **inferred** from A.

In mathematics its interpretation *differs* from that in natural language.

Consider the following arithmetical theorem:

For every natural number n,

if 6 DIVIDES n, then 3 DIVIDES n.

The theorem is true for any natural number, hence, in particular, it is true for numbers **2,3,6.**

Consider number 2.

The following proposition **is true**.

if 6 DIVIDES 2, then 3 DIVIDES 2.

It means an implication $(A \Rightarrow B)$ in which A and B are *false* statements is interpreted as a **true** statement.

Consider now a number 3.

The following proposition **is true**.

if 6 DIVIDES 3, then 3 DIVIDES 3,

It means an implication $(A \Rightarrow B)$ in which A is *false* and B is *true* is interpreted as a **true** statement.

Consider now a number 6.

The following proposition **is true**.

if 6 DIVIDES 6, then 3 DIVIDES 6.

It means an implication $(A \Rightarrow B)$ in which A and B are *true* is interpreted as a **true** statement.

One more case : what happens when in the implication $(A \Rightarrow B)$, A is *true* and B is *false*.

Example : consider a sentence

if 6 DIVIDES 12, then 6 DIVIDES 5.

Obviously, this is a **false** statement.

The above examples justify adopting the following semantics of an implication $(A \Rightarrow B)$.

Implication Table :

$v(A)$	$v(B)$	$v(A \Rightarrow B)$
T	T	T
T	F	F
F	T	T
F	F	T

EQUIVALENCE - Motivation and definition.

An equivalence $(A \Leftrightarrow B)$ is *true* if both formulas A and B have the same logical value.

Equivalence Table :

$v(A)$	$v(B)$	$v(A \Leftrightarrow B)$
T	T	T
T	F	F
F	T	F
F	F	T

Extensional connectives are the connectives that have the following property:

the logical value of the formulas form by means of these connectives and certain given formulas depends only on the logical value(s) of the given formulas.

All classical connectives are *extensional* .

Binary connectives are such connectives that they enable us to form a new formula from *two* formulas.

The classical connectives: $\cup, \cap, \Rightarrow$, and $\Leftrightarrow$ are *binary propositional connectives*.

Unary connectives are such connectives that they enable us to form a new formula from *one* formula.

The classical connective $\neg$ is a *unary propositional connective*.

Remark that in everyday language there are expressions which are propositional connectives but are not extensional. They do not play any role in mathematics and so are not discussed in classical logic.

Other Notations :

Negation	Disjunction	Conjunction	Implication	Equivalence
$-$A	$A \cup B$	$A \cap B$	$A \Rightarrow B$	$A \Leftrightarrow B$
NA	DAB	CAB	IAB	EAB
$\overline{A}$	$A \vee B$	A & B	$A \rightarrow B$	$A \leftrightarrow B$
$\sim A$	$A \vee B$	$A \cdot B$	$A \supset B$	$A \equiv B$
A'	$A + B$	$A \cdot B$	$A \rightarrow B$	$A \equiv B$

The first notation is the closest to ours and is drawn mainly from the algebra of sets and lattice theory.

The second comes from the Polish logician *J. Łukasiewicz* and is called the *Polish notation*.

The third was used by D. Hilbert.

The fourth comes from Peano and Russell.

The fifth goes back to Schröder and Pierce.

There are many other propositional connectives!

Table of all unary connectives :

$v(A)$	$v(\bigtriangledown_1 A)$	$v(\bigtriangledown_2 A)$	$v(\neg A)$	$v(\bigtriangledown_4 A)$
T	F	T	F	T
F	F	F	T	T

Table of all binary connectives :

$v(A)$	$v(B)$	$v(A \circ_1 B)$	$v(A \cap B)$	$v(A \circ_3 B)$	$v(A \circ_4 B)$
T	T	F	T	F	F
T	F	F	F	T	F
F	T	F	F	F	T
F	F	F	F	F	F
$v(A)$	$v(B)$	$v(A \downarrow B)$	$v(A \circ_6 B)$	$v(A \circ_7 B)$	$v(A \leftrightarrow B)$
T	T	F	T	T	T
T	F	F	T	F	F
F	T	F	F	T	F
F	F	T	F	F	T
$v(A)$	$v(B)$	$v(A \circ_9 B)$	$v(A \circ_{10} B)$	$v(A \circ_{11} B)$	$v(A \cup B)$
T	T	F	F	F	T
T	F	T	T	F	T
F	T	T	F	T	T
F	F	F	T	T	F
$v(A)$	$v(B)$	$v(A \circ_{13} B)$	$v(A \Rightarrow B)$	$v(A \uparrow B)$	$v(A \circ_{16} B)$
T	T	T	T	F	T
T	F	T	F	T	T
F	T	F	T	T	T
F	F	T	T	T	T

FUNCTIONAL DEPENDENCY is the ability of defining some connectives in terms of some others.

All propositional connectives can be defined in terms of *disjunction* and *negation*.

Two binary connectives: $\downarrow$ and $\uparrow$ suffice, each of them separately, to define **all** connectives, whether unary or binary.

The connective $\uparrow$ was discovered in 1913 by H.M. Sheffer, who called it **alternative negation**. Now it is often called a **Sheffer's connective**.

The formula $A \uparrow B$ reads: *not both A and B*.

Negation $\neg A$ is defined as $A \uparrow A$.

Disjunction is defined as $A \cup B$, as $(A \uparrow A) \uparrow (B \uparrow B)$.

The connective $\downarrow$ was termed by J. Łukasiewics **joint negation**.

The formula $A \downarrow B$ reads: *neither A nor B*.

It was proved in 1925 by E. Żyliński that no propositional connective other than $\uparrow$ and $\downarrow$ suffices to define all the remaining connectives.