CSE541 EXAMPLE 2: PRACTICE MIDTERM SOLUTIONS
submitted by a student

QUESTION 1

Write the following natural language statement:

One likes to play bridge, or from the fact that the weather is good we conclude the following: one does not like to play bridge or one likes not to play bridge

as a formula of 2 different languages

1. Formula $A_1 \in \mathcal{F}_1$ of a language $\mathcal{L}_{\{\neg,\ L,\ \cup,\ \Rightarrow\}}$, where LA represents statement "one likes A", "A is liked".

2. Formula $A_2 \in \mathcal{F}_2$ of a language $\mathcal{L}_{\{\neg,\ \cup,\ \Rightarrow\}}$.

Solution. 1. We translate the statement into a formula $A_1 \in \mathcal{F}$ of $\mathcal{L}_{\{\neg,\mathcal{L},\cup,\Rightarrow\}}$ as follows:
Propositional variables: a, b where

- a denotes the statement: play bridge
- b denotes the statement: the weather is good.

Propositional model connectives: L, $\neg$, $\cup$, $\Rightarrow$ where

- $\neg$ denotes the statement: not
- L denotes the statement: one likes, it is liked
- $\cup$ denotes the statement: and
- $\Rightarrow$ denotes the statement: from the fact ... we conclude ...

Now A_1 becomes

$$A_1 = (La \cup (b \Rightarrow (\neg La \cup L\neg a))) \qquad (1)$$

2. We translate the statement into a formula $A_2 \in \mathcal{F}$ of $\mathcal{L}_{\{\neg,\cup,\Rightarrow\}}$ as follows:
Propositional variables: a, b, c where

- a denotes that one likes to play bridge
- b denotes that one likes not to play bridge
- c denotes that the weather is good

Propositional model connectives: $\neg, \cup, \Rightarrow$ where

- $\neg$ denotes not
- $\cup$ denotes and

- $\Rightarrow$ denotes from the fact of ... we conclude that ...

Then

$$A_2 = (a \cup (c \Rightarrow (\neg a \cup b))) \tag{2}$$

□

QUESTION 2

Write the formal definition of the language $\mathcal{L}_{\{\neg,\ \mathbf{L},\ \cup,\ \Rightarrow\}}$ and give examples of its formulas of the degrees 0, 1, 2, 3, and 4.

Solution. 1. We give the definition of language $\mathcal{L}_{\{\neg,\ \mathbf{L},\ \cup,\ \Rightarrow\}}$ in following steps.

- $\mathcal{L}_{\{\neg,\mathcal{L},\cup,\Rightarrow\}} = \{\mathcal{A}, \mathcal{F}\}$ where $\mathcal{A} = VAR \cup CON \cup PAR$ and $\mathcal{F}$ is the set of formulae. $CON = \{\neg, L\} \cup \{\cup, \Rightarrow\}$. VAR, PAR are defined the same as in classical semantics and $\mathcal{F}$ is defined to be the smallest set such that
  (a) $VAR \subseteq \mathcal{F}$,
  (b) For all $A \in \mathcal{F}$, $\neg A \in \mathcal{F}$ and $LA \in \mathcal{F}$,
  (c) For all $A \in \mathcal{F}$ and $B \in \mathcal{F}$, $(A \cup B) \in \mathcal{F}$ and $(A \Rightarrow B) \in \mathcal{F}$.

  To define a notion of tautology **tautology** for $\mathcal{L}_{\{\neg,\mathcal{L},\cup,\Rightarrow\}}$ in the following steps.
- Given the nonempty set of logical values V, we can define a mapping $v : VAR \to V$, which is called a truth assignment. Now we define the extension $v^* : \mathcal{F} \to V$ of v by
  (a) for any $a \in VAR$,
  $$v^*(a) = v(a) \tag{3}$$
  (b) for any $A, B \in \mathcal{F}$,
  $$\begin{aligned} v^*(\neg A) &= \neg v^*(A) \\ v^*(LA) &= Lv^*(A) \\ v^*((A \cup B)) &= \cup(v^*(A), v^*(B)) \\ v^*((A \Rightarrow B)) &= \Rightarrow (v^*(A), v^*(B)) \end{aligned} \tag{4}$$
- Since the set V is nonempty, we can pick one and denote it as T, the value of True. Given a truth assignment $v : VAR \to V$ and a formula $A \in \mathcal{F}$, if $v^*(A) = T$ then we say v satisfies A, denoted as $v \models A$. And if $v^*(A) \neq T$ then we say v does not satisfy A. In addition, if v satisfies A we say v is a model for A, and if v does not satisfy A then v is a counter-model for A.
- Given $A \in \mathcal{F}$, we say it is a tautology if for all truth assignment v,
  $$v \models A. \tag{5}$$
  And we denote this by $\models A$.

2. To write formulae of degree 0,1,2,3,4 we can set A_0,A_1,A_2,A_3,A_4 as follows: Suppose $a \in VAR$,

   (a) $A_0 = a$
   (b) $A_1 = \neg a$
   (c) $A_2 = \neg\neg a$
   (d) $A_3 = \neg\neg\neg a$
   (e) $A_4 = \neg\neg\neg\neg a$

   are five formulae that satisfy the desired property.

$\square$

QUESTION 3

Define formally your OWN 3 valued extensional semantics **M** for the language $\mathcal{L}_{\{\neg,\ \mathbf{L},\ \cup,\ \Rightarrow\}}$ under the following assumptions

1. Assume that the third value is **intermediate** between truth and falsity, i.e. the set of logical values is **ordered** and we have the following

Assumption 1 $F < \perp < T$

Assumption 2 T is the **designated value**

2. Model the situation in which one "likes" only truth; i.e. in which $\mathbf{L}T = T$ and $\mathbf{L}\perp = F$, $\mathbf{L}F = F$

3. The connectives $\neg$, $\cup$, $\Rightarrow$ can be defined as you wish, but you have to define them in such a way to make sure that

$$\models_{\mathbf{M}} (\mathbf{L}A \cup \neg\mathbf{L}A)$$

REMINDER

Formal definition of many valued extensional semantics follows the pattern of the classical case and consists of giving **definitions** of the following main components:

1. Logical Connectives

2. Truth Assignment

3. Satisfaction Relation, Model, Counter-Model

4. Tautology

Solution. $\mathcal{L}_{\{\neg,\mathcal{L},\cup,\Rightarrow\}} = \{\mathcal{A}, \mathcal{F}\}$ where $\mathcal{A} = VAR \cup CON \cup PAR$ and $\mathcal{F}$ is the set of formulae. $CON = \{\neg, L\} \cup \{\cup, \Rightarrow\}$, VAR, PAR are defined same as the classical semantics and $\mathcal{F}$ is defined to be the smallest set such that

1. $VAR \subseteq \mathcal{F}$,

2. For all $A \in \mathcal{F}$, $\neg A \in \mathcal{F}$ and $LA \in \mathcal{F}$,
3. For all $A \in \mathcal{F}$ and $B \in \mathcal{F}$, $(A \cup B) \in \mathcal{F}$ and $(A \Rightarrow B) \in \mathcal{F}$.

Given the nonempty set of logical values V, we can define a mapping $v : VAR \to V$, which is called a truth assignment. Now we define the extension $v^* : \mathcal{F} \to V$ of v by

1. for any $a \in VAR$,
$$v^*(a) = v(a) \tag{6}$$
2. for any $A, B \in \mathcal{F}$,
$$\begin{aligned} v^*(\neg A) &= \neg v^*(A) \\ v^*(LA) &= Lv^*(A) \\ v^*((A \cup B)) &= \cup(v^*(A), v^*(B)) \\ v^*((A \Rightarrow B)) &= \Rightarrow (v^*(A), v^*(B)) \end{aligned} \tag{7}$$
where on the right-hand side $\neg$ and L are mappings $V \to V$ and $\cup$, $\Rightarrow$ are mappings $V \times V \to V$.
In particular if x, y are two arbitrary elements in V we define
$$\neg F = T,\ \neg \bot = T,\ \neg T = F$$
$$LT = T,\ L\bot = F,\ LF = F$$
$$x \cup y = T$$
$$x \Rightarrow y = T \quad if\ x \leq y$$
$$x \Rightarrow y = F \quad if\ x > y$$

Since the set V is nonempty, we can pick one and denote it T, the value of true. Given a truth assignment $v : VAR \to V$ and a formula $A \in \mathcal{F}$, if $v^*(A) = T$ then we say v satisfies A, denoted as $v \models_M A$. Similarly if $v^*(A) \neq T$ then we say v does not satisfy A. In addition, we say that if v satisfies A then v is a model for A, and if v does not satisfy A then v is a counter-model for A. Given $A \in \mathcal{F}$, we say it is a tautology if for all truth assignment v,
$$v \models_M A. \tag{8}$$
And we denote this by $\models_M A$. From the above definition we can see the three valued semantics M for $\mathcal{L}_{\{\neg,\mathcal{L},\cup,\Rightarrow\}}$satisfies the requirement in the questions, especially
$$\models_M (LA \cup \neg LA)$$
since no matter what values $v^*(LA)$ and $v^*(\neg LA)$ are, the combination of them by $\cup$ will always be T. □

QUESTION 4

1. Verify whether the formulas A_1 and A_2 from the **QUESTION 1** have a model/ counter model under your semantics **M**. You can use **shorthand notation**

2. Verify whether the following set **G** is **M**-consistent. You can use **shorthand notation**

$$\mathbf{G} = \{ \ \mathbf{L}a, \ \ (a \cup \neg \mathbf{L}b), \ \ (a \Rightarrow b), \ b \ \}$$

3. Give an **example** on an infinite, **M - consistent** set of formulas of the language $\mathcal{L}_{\{\neg, \ \mathbf{L}, \ \cup, \ \Rightarrow\}}$

Solution. 1. Recall that

$$A_1 = (La \cup (b \Rightarrow (\neg La \cup L\neg a))) \tag{9}$$

and

$$A_2 = (a \cup (c \Rightarrow (\neg a \cup b))) \tag{10}$$

In A_1 if we set (using shorthand notation) $a = T$, $b = T$ then

$$A_1 = (LT \cup (T \Rightarrow (\neg LT \cup L\neg T))) = (LT \cup (T \Rightarrow T)) = (T \cup T) = T \tag{11}$$

Thus A_1 has a model. Similarly in A_2

$$v^*(A_2) = v^*(a \cup (c \Rightarrow (\neg a \cup b))) = v^*(a) \cup v^*(c \Rightarrow (\neg a \cup b)) = T \tag{12}$$

since no matter what values $v^*(a)$ and $v^*(c \Rightarrow (\neg a \cup b))$ take the result of their $\cup$ is always T under M.

2. This set has a model if we set $v^*(a) = T$ and $v^*(b) = T$. Actually (using shorthand notation)

$$\begin{aligned} La &= LT = T \\ (a \cup \neg Lb) &= T \cup F = T \\ (a \Rightarrow b) &= T \Rightarrow T = T \\ b &= T. \end{aligned} \tag{13}$$

3. Consider the set **G** of formulae

$$\mathbf{G} = \{(a \cup b) : \ \ a, b \in VAR\}$$

It is **M - consistent** since whatever logical value a and b takes, $v^*(a \cup b) = v^*(a) \cup v^*(b) = T$ by the definition of $\cup$. Also this set is infinite since the set VAR is infinite.

□

QUESTION 5

Let S be the following **proof system**

$$S = (\ \mathcal{L}_{\{\neg, \mathbf{L}, \cup, \Rightarrow\}},\ \mathcal{F},\ \ \{\mathbf{A1}, \mathbf{A2}\},\ \ \{r1,\ r2\}\ )$$

for the logical axioms and rules of inference defined for any formulas $A, B \in \mathcal{F}$ as follows

Logical Axioms

A1 $(\mathbf{L}A \cup \neg\mathbf{L}A)$

A2 $(A \Rightarrow \mathbf{L}A)$

Rules of inference:

$$(r1)\ \frac{A\ ;B}{(A \cup B)}, \qquad\qquad (r2)\ \frac{A}{\mathbf{L}(A \Rightarrow B)}$$

1. Write a proof in S with 2 applications of rule (r1) and one application of rule (r2)

You must write comments how each step pot the proof was obtained

2. Show, by constructing a formal proof that

$$\vdash_S\ ((\mathbf{L}b \cup \neg\mathbf{L}b) \cup \mathbf{L}((\mathbf{L}a \cup \neg\mathbf{L}a) \Rightarrow b)))$$

3. Verify whether the inference rules r1, r2 are **M**-sound. You can use **shorthand notation**

4. Verify whether the system S is **M**-sound. You can use **shorthand notation**

EXTRA QUESTION

If the system S **is not sound** under your semantics **M** then **re-define the connectives** in a way that such obtained new semantics **N** would make S sound.

You can use **shorthand notation**

Here are the **solutions**

1. Write a proof in S with 2 applications of rule (r1) and one application of rule (r2)

You must write comments how each step pot the proof was obtained

Solution. 1. Below we present a proof S1, S2, S3, S4 with two application of rule (r1) and one application of rule (r2), where $A \in \mathcal{F}$ is a formula.

S1: $(A \Rightarrow LA)$

Axiom A1

S2: $((A \Rightarrow LA) \cup (A \Rightarrow LA))$

Application of rule (r1) to S1 and S1

S3: $(L((A \Rightarrow LA) \cup (A \Rightarrow LA)) \Rightarrow (A \Rightarrow LA))$

Application rule (r2) to S2 and B = S1

S4: $((L((A \Rightarrow LA) \cup (A \Rightarrow LA)) \Rightarrow (A \Rightarrow LA)) \cup (LA \cup \neg LA))$

Application of rule (r1) to S3 and S1

2. Show, by constructing a formal proof that

$$\vdash_S ((\mathbf{L}b \cup \neg \mathbf{L}b) \cup \mathbf{L}((\mathbf{L}a \cup \neg \mathbf{L}a) \Rightarrow b)))$$

We construct a proof S1, S2, S3, S4 as follows:

S1: $(Lb \cup \neg Lb)$

Axiom A1 for $A = b$

S2: $(La \cup \neg La)$

Axiom A1 for $A = a$

S3: $L((La \cup \neg La) \Rightarrow b)$

Application of rule (r2) to S2 and S2 for B=b

S4: $((Lb \cup \neg Lb) \cup L((La \cup \neg La) \Rightarrow b))$

Application of rule (r1) to S1 and S3

3. Verify whether the inference rules r1, r2 are **M**-sound. You can use **shorthand notation**

To verify (r1) is sound we first assume all its premises, i.e $A = T$ and $B = T$ and observe that

$$(A \cup B) = T \cup T = T.$$

To prove (r2) is not sound, first we assume its premises, $A = T$, but also assume $B = F$, then we have

$$L(A \Rightarrow B) = L(T \Rightarrow F) = LF = F.$$

which means although we assume all its premises true, the conclusion of it could still not be true.

2. If the system S is M-sound then all its axioms must be tautologies and all its rules must be sound. In previous question we have seen that rule (r2) is not sound. Thus S is **not sound**.

EXTRA Credit We redefine the binary connective "$\Rightarrow$" to be a mapping

$$V \times V \to V$$

such that for any $x, y \in V$

$$x \Rightarrow y = T.$$

Now we can verify both axioms A1 and A2 are tautologies and both rules (r1) and (r2) are sound. For A1 we see that

$$(LA \cup \neg LA) = T$$

by the definition of $\cup$. For A2 we see that

$$(A \Rightarrow LA) = T$$

by the definition of $\Rightarrow$. For (r1) we have

$$A \cup B = T$$

and for (r2) we have

$$L(A \Rightarrow B) = LT = T.$$

Therefore under this new definition, system S is a sound system.

$\square$