Please take your time and write carefully your solutions. There is no NO PARTIAL CREDIT.

You get 0 pts for a solution with a formula that is NOT a well formed formula of the given language.

QUESTION 1 (15pts)
T1 (5pts) Write the following natural language statement:

One likes to eat apples, or from the fact that the apples are expensive we conclude the following: one does not like eat apples or one likes not to eat apples

as a formula $A_1 \in \mathcal{F}_\infty$ of a language $\mathcal{L}_{\{\neg, \land, \lor, \Rightarrow\}}$, where $A$ represents statement "one likes $A$", "A is liked".

Solution Propositional Variables are: (use a, b, ... and you must write which variables denote which sentences)
a denotes statement: eat apples,
b denotes a statement: the apples are expensive

Translation $A_1 = (La \lor (b \Rightarrow (\neg La \lor L\neg a)))$

T2 (10 pts)

Here is a mathematical statement $S$:

*For all rational numbers $x \in Q$ the following holds: If $x \neq 0$, then there is a natural number $n \in N$, such that $x + n \neq 0*

1. (2pts) Re-write $S$ as a symbolic mathematical statement $SM$ that only uses mathematical and logical symbols.

Solution $S$ becomes a symbolic mathematical statement

$SM : \forall x \in Q (x \neq 0 \Rightarrow \exists n \in N \ x + n \neq 0)$

2. (5pts) Translate the symbolic statement $SM$ into a corresponding formula of the predicate language $\mathcal{L}$ with restricted quantifiers. Use SYMBOLS: $Q(x)$ for $x \in Q$, $N(y)$ for $y \in N$, $c$ for the number $0$. Use $E \in \mathcal{F}$ to denote the relation $=$ and use symbol $f \in \mathcal{F}$ to denote the function $+$.

Solution

The statement $x \neq 0$ becomes a formula $\neg E(x, c)$. The statement $x + n \neq 0$ becomes a formula $\neg E(f(x, y), c)$.

The symbolic mathematical statement $SM$ becomes a restricted quantifiers formula

$\forall x \in Q (\neg E(x, c) \Rightarrow \exists y \in N \neg E(f(x, y), c))$

3. (3pts) Translate your restricted domain quantifiers logical formula into a correct formula $A$ of $\mathcal{L}$.

Solution We apply now the transformation rules and get a corresponding formula $A \in \mathcal{F}$:

$\forall x (Q(x) \Rightarrow (\neg E(x, c) \Rightarrow \exists y (N(y) \cap \neg E(f(x, y), c))))$
QUESTION 2  (20 pts)

We define a 3 valued extensional semantics $M$ for the language $\mathcal{L}_{\neg, \cup, \Rightarrow}$ by defining the connectives $\neg$, $\cup$, $\Rightarrow$ on a set $\{F, \bot, T\}$ of logical values as the following functions.

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<th>L Connective</th>
<th>Negation</th>
<th>Implication</th>
<th>Disjunction</th>
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1. (5pts) Verify whether $|\equiv_M (LA \cup \neg LA)$. Use shorthand notation.

Solution

We verify

$LT \cup \neg LT = T \cup F = T$, $L \bot \cup \neg L \bot = F \cup \neg F = F \cup T = T$, $LF \cup \neg LF = F \cup \neg F = T$

2. (5pts) Verify whether set $G = \{La, (a \cup \neg Lb), (a \Rightarrow b), b\}$ is $M$-consistent. Use shorthand notation.

Solution

Any $v$, such that $v(a) = T$, $v(b) = T$ is a $M$ model for $G$

$LT = T$, $(T \cup \neg LT) = T$, $(T \Rightarrow T) = T$, $b = T$

We define: a formula $A \in \mathcal{F}$ is called $M$-independent from a set $G \subseteq \mathcal{F}$ if and only if

the sets $G \cup \{A\}$ and $G \cup \{\neg A\}$ are both $M$-consistent. I.e. when there are truth assignments $v_1$, $v_2$ such that $v_1|_M G \cup \{A\}$ and $v_2|_M G \cup \{\neg A\}$.

3. (5pts) FIND a formula $A$ that is $M$-independent of a set $G$. Use shorthand notation to prove it.

Solution

This is the simplest solution. You can have a different solution- but the idea must be similar.

Remark: always look for the simplest example possible!

Let $A$ be any atomic formula $c \in VAR - \{a, b\}$.

Any $v$, such that $a = T$, $b = T$, and $c = T$ is a model for $G \cup \{d\}$.

Any $v$, such that $a = T$, $b = T$, and $c = F$ is a model for $G \cup \{\neg d\}$.

4. (5pts) Find infinitely many formulas that are $M$-independent of a set $G$. Justify your answer.
Solution

This is a generalization of solution above. You can have a different solution- but the idea must be similar.

Remark: always look for the simples example possible!

Let A be any atomic formula \( d \in \text{VAR} - \{a, b\} \).

Any v, such that a=T, b=T, and d= T is a model for \( G \cup \{d\} \).

Any v, such that a=T, b=T, and d= F is a model for \( G \cup \{\neg d\} \).

There is countably infinitely many atomic formulas A=d, where \( d \in \text{VAR} - \{a, b\} \).

QUESTION 3 (15pts)

Let \( S \) be the following proof system \( S = ( \mathcal{L}_{L, \lor, \rightarrow}, \mathcal{F}, \{A1, A2\}, \{r1, r2\}) \)

for the logical axioms and rules of inference defined for any formulas \( A, B \in \mathcal{F} \) as follows

**Logical Axioms**

A1 \( (LA \cup \neg LA) \)

A2 \( (A \rightarrow LA) \)

**Rules** of inference:

\[
\begin{align*}
(r1) & \quad \frac{A, B}{A \lor B}, \\
(r2) & \quad \frac{A}{L(A \rightarrow B)}
\end{align*}
\]

1. (10pts) Show, by constructing a proper formal proof that

\[ \vdash_{S} ((Lb \cup \neg Lb) \cup L((La \cup \neg La) \rightarrow b)) \]

Write all steps of the formal proof with comments how each step was obtained.

**Solution**

Here is the proof \( B1, B2, B3, B4 \)

\( B1: \quad (La \cup \neg La) \quad \text{Axiom A1 for A= a} \)

\( B2: \quad L((La \cup \neg La) \rightarrow b) \quad \text{rule r2 for B= b applied to B1} \)

\( B3: \quad (Lb \cup \neg Lb) \quad \text{Axiom A1 for A= b} \)

\( B4: \quad ((Lb \cup \neg Lb) \cup L((La \cup \neg La) \rightarrow b)) \quad \text{r1 applied to B3 and B2} \)

2. (5pts) Does the above point 1. PROVE that \( \models_{M} ((Lb \cup \neg Lb) \cup L((La \cup \neg La) \rightarrow b)) \) for the semantics \( M \) defined in QUESTION 2 JUSTIFY your answer.

**Solution**

**No, it doesn’t** because the system \( S \) is not sound.

Rule 2 is **not sound** because when \( A = T \) and \( B = F \) (or \( B = \bot \) ) we get \( L(A \rightarrow B) = L(T \rightarrow F) = LF = F \) or \( L(T \rightarrow \bot) = L \bot = F \)

**Observe** that both logical axioms of \( S \) are **M tautologies**

A1 is **M tautology** as we proved in 1, A2 is **M tautology** by direct evaluation.
Rule r1 is sound because when $A = T$ and $B = T$ we get $A \cup B = T \cup T = T$

**PROBLEM 4 (15pts)**

Consider the Hilbert system $H_1 = (L_{\{\Rightarrow\}}, F, \{A_1, A_2\}, (MP) \frac{A \Rightarrow (A \Rightarrow B)}{B} )$ where for any $A, B \in F$

$A_1; (A \Rightarrow (B \Rightarrow A)), A_2 : ((A \Rightarrow (B \Rightarrow C)) \Rightarrow ((A \Rightarrow B) \Rightarrow (A \Rightarrow C)))$.

1. (5pts) The Deduction Theorem holds for $H_1$. Use the Deduction Theorem to show that $(A \Rightarrow (C \Rightarrow B)) \vdash_{H_1} (C \Rightarrow (A \Rightarrow B))$

**Solution**

We apply the Deduction Theorem twice, i.e. we get

$(A \Rightarrow (C \Rightarrow B)) \vdash_{H} (C \Rightarrow (A \Rightarrow B))$ if and only if

$(A \Rightarrow (C \Rightarrow B)), C \vdash_{H} (A \Rightarrow B)$ if and only if

$(A \Rightarrow (C \Rightarrow B)), C, A \vdash_{H} B$

We now construct a proof of $(A \Rightarrow (C \Rightarrow B)), C, A \vdash_{H} B$ as follows

$B_1 : (A \Rightarrow (C \Rightarrow B))$ hypothesis

$B_2 : C$ hypothesis

$B_3 : A$ hypothesis

$B_4 : (C \Rightarrow B) \quad B_1, B_3$ and (MP)

$B_5 : B \quad B_2, B_4$ and (MP)

2. (5pts) Explain why 1. proves that $(\neg a \Rightarrow ((b \Rightarrow \neg a) \Rightarrow b)) \vdash_{H_1} ((b \Rightarrow \neg a) \Rightarrow (\neg a \Rightarrow b))$.

**Solution** This is 1. for $A = \neg a, C = (b \Rightarrow \neg a)$, and $B = b$.

3. (5pts) Let $H_2$ be the proof system obtained from the system $H_1$ by extending the language to contain the negation $\neg$ and adding one additional axiom:

$A_3 \quad ((\neg B \Rightarrow \neg A) \Rightarrow ((\neg B \Rightarrow A) \Rightarrow B)))$.

We know that $H_2$ is complete. Let $H_3$ be the proof system obtained from the system $H_2$ adding additional axiom

$A_4 \quad (\neg(A \Rightarrow B) \Rightarrow \neg(A \Rightarrow \neg B))$

Does Completeness Theorem hold for $H_3$? JUSTIFY.

**Solution**

No, it doesn’t. The system $H_3$ is not sound. Axiom $A_4$ is not a tautology.

Any $v$ such that $A=T$ and $B=F$ is a counter model for $(\neg(A \Rightarrow B) \Rightarrow \neg(A \Rightarrow \neg B))$.

**QUESTION 5 (15pts)**

**Remark** This question is designed to check if you understand the notion of completeness, monotonicity, application of Deduction Theorem and use of some basic tautologies.
Consider any proof system $S = \langle \mathcal{L}_{\lor, \land, \lnot}, \mathcal{T}, \text{LA}, (MP)^{\Delta(A \equiv B)} \rangle$

We assume that $S$ complete under classical semantics and Deduction Theorem holds in $S$.

Given any $\Gamma \subseteq F$, we define $Cn(\Gamma) = \{ A \in F : \Gamma \vdash_S A \}$.

**Prove** that for any $A, B \in F$, $Cn(\{A, B\}) \subseteq Cn(\{A \cap B\})$

**Hint:** Use Deduction Theorem and Completeness of $S$ and the fact that $|= (((A \Rightarrow (B \Rightarrow C)) \Rightarrow ((A \cap B) \Rightarrow C))$.

**Solution**

Assume $C \in Cn(\{A, B\})$.

This means $A, B \vdash_S C$. We apply Deduction Theorem and we get $\vdash_S (A \Rightarrow (B \Rightarrow C))$.

By the completeness of $S$ and the fact that the formula $((A \Rightarrow (B \Rightarrow C)) \Rightarrow ((A \cap B) \Rightarrow C))$ is a tautology, we get that $\vdash_S ((A \Rightarrow (B \Rightarrow C)) \Rightarrow ((A \cap B) \Rightarrow C))$.

Applying Modus Ponens to the above we get $\vdash_S ((A \cap B) \Rightarrow C)$.

This is equivalent to $(A \cap B) \vdash_S C$ by Deduction Theorem and we hence have proved that $C \in Cn((A \cap B))$.

**QUESTION 6 (10pts)**

1. For any formula $A = A(b_1, b_2, ..., b_n)$ and any truth assignment $v$ we define, a corresponding formulas $A', B_1, B_2, ..., B_n$ as follows:

   $$A' = \begin{cases} A & \text{if } v^*(A) = T \\ \lnot A & \text{if } v^*(A) = F \end{cases}$$

   $$B_i = \begin{cases} b_i & \text{if } v(b_i) = T \\ \lnot b_i & \text{if } v(b_i) = F \end{cases}$$

   We proved the following Lemma for $H_2$.

**Main Lemma**

For any formula $A = A(b_1, b_2, ..., b_n)$ and any truth assignment $v$,

if $A', B_1, B_2, ..., B_n$ are corresponding formulas defined above, then $B_1, B_2, ..., B_n \vdash A'$.

1. (2pts) Let $A$ be a formula $((\lnot a \Rightarrow \lnot b) \Rightarrow (b \Rightarrow a))$.

   Write what **Main Lemma** asserts for the formula $A$ and $v$ such that $v(a) = T$, $v(b) = F$.

**Solution**

Observe that the formula $A$ is a basic tautology, hence $A' = A$.

$A = A(a, b)$ and we get $B_1 = a$, $B_2 = \lnot b$ and **Main Lemma** asserts

$$a, \lnot b \vdash ((\lnot a \Rightarrow \lnot b) \Rightarrow (b \Rightarrow a)).$$

2. The proof of **Completeness Theorem** for $H_2$ defines a method of efficiently combining $v \in VAR$ and the **Main Lemma** to describe a construction of the proof of any tautology in $H_2$.  

5
Here are the steps of the Proof as applied to the basic tautology

\[ A(a, b) = ((\neg a \Rightarrow \neg b) \Rightarrow (b \Rightarrow a)) \]

s1. By the Main Lemma and the assumption that \( \models A(a, b) \) any \( v \in V_A \) defines formulas \( B_a, B_b \) such that

\[ B_a, B_b \vdash A. \]

The proof is based on a method of elimination of \( B_a, B_b \) to obtain \( \vdash A \) by the use of Deduction Theorem, monotonicity of consequence, and provability of the formula

\[ (\ast) : ((A \Rightarrow B) \Rightarrow ((\neg A \Rightarrow B) \Rightarrow B)). \]

s2 (8pts) Perform the elimination of \( B_a, B_b \) to construct the proof of \( A \).

Solution

We know that any \( v \in V_A \) defines formulas \( B_a, B_b \) such that

\[ B_a, B_b \vdash A. \]

We construct the proof of \( A \) as follows.

Elimination of \( B_b \).

We have to cases: \( v(b) = T \) or \( v(b) = F \).

Let \( v(b) = T \), so \( B_a, b \vdash A \), and by Deduction Theorem we get \( B_a \vdash (b \Rightarrow A) \).

Let \( v(b) = F \), so \( B_a, \neg b \vdash A \), and by Deduction Theorem we get \( B_a \vdash (\neg b \Rightarrow A) \).

By the provability of the formula \((\ast)\) for \( A = b, B = A \) and monotonicity

\[ B_a \vdash ((b \Rightarrow A) \Rightarrow ((\neg b \Rightarrow A) \Rightarrow A)) \]

By MP applied twice twice we eliminated \( B_b \) and got \( B_a \vdash A \).

Elimination of \( B_a \).

We consider \( B_a \vdash A \).

We have to cases: \( v(a) = T \) or \( v(a) = F \).

Let \( v(a) = T \), so \( a \vdash A \), and by Deduction Theorem we get \( \vdash (a \Rightarrow A) \).

Let \( v(a) = F \), so \( \neg a \vdash A \), and by Deduction Theorem we get \( \vdash (\neg a \Rightarrow A) \).

By the provability of the formula \((\ast)\) for \( A = a, B = A \)

\[ \vdash ((a \Rightarrow A) \Rightarrow ((\neg a \Rightarrow A) \Rightarrow A)) \]

By MP applied twice twice and get

\[ \vdash A, \]

i.e. we eliminated \( B_a \) and got the proof of \( A \).