CSE541 MIDTERM SOLUTIONS Fall 2022
(75pts + 15extra credit)

Please take your time and write **carefully** your solutions. There is no NO PARTIAL CREDIT.

You get **0 pts** for a solution with a formula that is NOT a well formed formula of the given language.

QUESTION 1 (15pts)

T1 (5pts) Write the following natural language statement:

> **One likes to eat apples, or from the fact that the apples are expensive we conclude the following: one does not like eat apples or one likes not to eat apples**
>
> as a formula $A_1 \in \mathcal{F}_\infty$ of a language $\mathcal{L}_{\{\neg, \mathbf{L}, \cup, \Rightarrow\}}$, where $\mathbf{L}A$ represents statement "one likes A", "A is liked".

Solution Propositional Variables are: (use a, b, ... and you must write which variables denote which sentences)
a denotes statement: *eat apples*,
b denotes a statement: *the apples are expensive*

Translation $A_1 = (\mathbf{L}a \cup (b \Rightarrow (\neg\mathbf{L}a \cup \mathbf{L}\neg a)))$

T2 (10 pts)

Here is a mathematical statement **S**:

For all rational numbers $x \in Q$ the following holds: If $x \neq 0$, then there is a natural number $n \in N$, such that $x + n \neq 0$

1. (2pts). Re-write **S** as a symbolic mathematical statement **SM** that only uses mathematical and logical symbols.

Solution **S** becomes a symbolic mathematical statement

$$\mathbf{SM}: \quad \forall_{x \in Q}(x \neq 0 \Rightarrow \exists_{n \in N}\ x + n \neq 0)$$

2. (5pts) Translate the symbolic statement **SM** into to a corresponding formula of the predicate language $\mathcal{L}$ with **restricted quantifiers**. Use SYMBOLS: Q(x) for $x \in Q$, N(y) for $y \in N$, c for the number 0. Use $E \in \mathbf{P}$ to denote the relation = and use symbol $f \in \mathbf{F}$ to denote the function +.

Solution

The statement $x \neq 0$ becomes a **formula** $\neg E(x, c)$. The statement $x + n \neq 0$ becomes a **formula** $\neg E(f(x, y), c)$.

The symbolic mathematical statement **SM** becomes a **restricted quantifiers** formula

$$\forall_{Q(x)}(\neg E(x, c) \Rightarrow \exists_{N(y)}\neg E(f(x, y), c))$$

3. (3pts) Translate your **restricted domain** quantifiers logical formula into a correct formula A of $\mathcal{L}$.

Solution We apply now the **transformation rules** and get a corresponding formula $A \in \mathcal{F}$:

$$\forall x\ (Q(x) \Rightarrow (\neg E(x, c) \Rightarrow \exists y\ (N(y) \cap \neg E(f(x, y), c))))$$

QUESTION 2 (20 pts)

We define a 3 valued extensional semantics **M** for the language $\mathcal{L}_{\{\neg, \mathbf{L}, \cup, \Rightarrow\}}$ by **defining the connectives** $\neg$, **L** $\cup$, $\Rightarrow$ on a set $\{F, \perp, T\}$ of logical values as the following functions.

L Connective

L	F	$\perp$	T
	F	F	T

Negation :

$\neg$	F	$\perp$	T
	T	F	F

Implication

$\Rightarrow$	F	$\perp$	T
F	T	T	T
$\perp$	T	$\perp$	T
T	F	F	T

Disjunction :

$\cup$	F	$\perp$	T
F	F	$\perp$	T
$\perp$	$\perp$	T	T
T	T	T	T

1. (5pts) Verify whether $\models_{\mathbf{M}} (\mathbf{L}A \cup \neg\mathbf{L}A)$. Use **shorthand** notation.

Solution

We verify

$\mathbf{L}T \cup \neg\mathbf{L}T = T \cup F = T$, $\quad \mathbf{L} \perp \cup \neg\mathbf{L} \perp = F \cup \neg F = F \cup T = T$, $\quad \mathbf{L}F \cup \neg\mathbf{L}F = F \cup \neg F = T$

2. (5pts) Verify whether set $\mathbf{G} = \{ \mathbf{L}a, \ (a \cup \neg\mathbf{L}b), \ (a \Rightarrow b), \ b \}$ is **M**-consistent. Use **shorthand notation**

Solution

Any v, such that $v(a) = T$, $v(b) = T$ is a **M model** for **G** as

$\mathbf{L}T = T$, $\quad (T \cup \neg\mathbf{L}T) = T$, $\quad (T \Rightarrow T) = T$, $\quad b = T$

We define: a formula $A \in \mathcal{F}$ is called **M- independent** from a set $\mathcal{G} \subseteq \mathcal{F}$ if and only if

the sets $\mathcal{G} \cup \{A\}$ and $\mathcal{G} \cup \{\neg A\}$ are both **M-consistent**. I.e. when there are truth assignments v_1, v_2 such that

$v_1 \models_{\mathbf{M}} \mathcal{G} \cup \{A\}$ and $v_2 \models_{\mathbf{M}} \mathcal{G} \cup \{\neg A\}$.

3. (5pts) FIND a formula A that is **M- independent** of a set **G**. Use shorthand notation to prove it.

Solution

This is the simplest solution. You can have a different solution- but the idea must be similar.

Remark: always look for the simples example possible!

Let A be any atomic formula $c \in VAR - \{a, b\}$.

Any v, such that a=T, b= T, and c = T is a model for $\mathcal{G} \cup \{d\}$.

Any v, such that a=T, b= T, and c = F is a model for $\mathcal{G} \cup \{\neg d\}$.

4. (5pts) Find infinitely many formulas that are **M- independent** of a set **G**. Justify your answer

Solution

This is a generalization of solution above. You can have a different solution- but the idea must be similar.

Remark: always look for the simples example possible!

Let A be any atomic formula $d \in VAR - \{a, b\}$.

Any v, such that a=T, b= T, and d= T is a model for $\mathcal{G} \cup \{d\}$.

Any v, such that a=T, b= T, and d= F is a model for $\mathcal{G} \cup \{\neg d\}$.

There is countably infinitely many atomic formulas A=d, where $d \in VAR - \{a, b\}$.

QUESTION 3 (15pts)

Let S be the following **proof system** $S = (\mathcal{L}_{\{\neg, \mathbf{L}, \cup, \Rightarrow\}}, \mathcal{F}, \{\mathbf{A1}, \mathbf{A2}\}, \{r1, r2\})$

for the logical axioms and rules of inference defined for any formulas $A, B \in \mathcal{F}$ as follows

Logical Axioms

A1 $(\mathbf{L}A \cup \neg\mathbf{L}A)$, **A2** $(A \Rightarrow \mathbf{L}A)$

Rules of inference:

$$(r1)\ \frac{A\ ;B}{(A \cup B)}, \qquad (r2)\ \frac{A}{\mathbf{L}(A \Rightarrow B)}$$

1. (10pts) Show, by constructing a proper **formal proof** that

$$\vdash_S ((\mathbf{L}b \cup \neg\mathbf{L}b) \cup \mathbf{L}((\mathbf{L}a \cup \neg\mathbf{L}a) \Rightarrow b)))$$

Write all steps of the **formal proof** with **comments** how each step was obtained.

Solution

Here is the proof B_1, B_2, B_3, B_4

B_1: $(\mathbf{L}a \cup \neg\mathbf{L}a)$ Axiom A_1 for A= a

B_2: $\mathbf{L}((\mathbf{L}a \cup \neg\mathbf{L}a) \Rightarrow b)$ rule r2 for B=b applied to B_1

B_3: $(\mathbf{L}b \cup \neg\mathbf{L}b)$ Axiom A_1 for A=b

B_4: $((\mathbf{L}b \cup \neg\mathbf{L}b) \cup \mathbf{L}((\mathbf{L}a \cup \neg\mathbf{L}a) \Rightarrow b))$ r1 applied to B_3 and B_2

2. (5pts) Does the above point **1.** PROVE that $\models_{\mathbf{M}} ((\mathbf{L}b \cup \neg\mathbf{L}b) \cup \mathbf{L}((\mathbf{L}a \cup \neg\mathbf{L}a) \Rightarrow b)))$? for the semantics **M** defined in QUESTION 2 JUSTIFY your answer.

Solution

No, it doesn't because the system S is not sound.

Rule 2 is **not sound** because when $A = T$ and $B = F$ (or $B = \perp$) we get $\mathbf{L}(A \Rightarrow B) = \mathbf{L}(T \Rightarrow F) = \mathbf{L}F = F$ or $\mathbf{L}(T \Rightarrow \perp) = \mathbf{L} \perp = F$

Observe that both logical axioms of S are **M tautologies**

A1 is **M** tautology as we proved in **1.**, **A2** is **M** tautology by direct evaluation.

Rule r1 is **sound** because when $A = T$ and $B = T$ we get $A \cup B = T \cup T = T$

PROBLEM 4 (15pts)

Consider the Hilbert system $H1 = (\mathcal{L}_{\{\Rightarrow\}},\ \mathcal{F},\ \{A1, A2\},\ (MP) \frac{A\ ;\ (A \Rightarrow B)}{B})$ where for any $A, B \in \mathcal{F}$

$A1;\ (A \Rightarrow (B \Rightarrow A)),\quad A2:\ ((A \Rightarrow (B \Rightarrow C)) \Rightarrow ((A \Rightarrow B) \Rightarrow (A \Rightarrow C)))$.

1. (5pts) The **Deduction Theorem** holds for $H1$. Use the **Deduction Theorem** to show that

$$(A \Rightarrow (C \Rightarrow B)) \vdash_{H1} (C \Rightarrow (A \Rightarrow B))$$

Solution

We apply the **Deduction Theorem** twice, i.e. we get

$(A \Rightarrow (C \Rightarrow B)) \vdash_H (C \Rightarrow (A \Rightarrow B))$ if and only if

$(A \Rightarrow (C \Rightarrow B)),\ C \vdash_H (A \Rightarrow B)$ if and only if

$(A \Rightarrow (C \Rightarrow B)),\ C,\ A \vdash_H B$

We now construct a proof of $(A \Rightarrow (C \Rightarrow B)), C, A \vdash_H B$ as follows

$B_1 :\ (A \Rightarrow (C \Rightarrow B))$ hypothesis

$B_2 :\ C$ hypothesis

$B_3 :\ A$ hypothesis

$B_4 :\ (C \Rightarrow B)$ B_1, B_3 and (MP)

$B_5 :\ B$ B_2, B_4 and (MP)

2. (5pts) Explain why **1.** proves that $(\neg a \Rightarrow ((b \Rightarrow \neg a) \Rightarrow b)) \vdash_{H1} ((b \Rightarrow \neg a) \Rightarrow (\neg a \Rightarrow b))$.

Solution This is **1.** for $A = \neg a$, $C = (b \Rightarrow \neg a)$, and $B = b$.

3. (5pts) Let $H2$ be the proof system obtained from the system $H1$ by **extending the language** to contain the negation $\neg$ and **adding** one additional axiom:

A3 $((\neg B \Rightarrow \neg A) \Rightarrow ((\neg B \Rightarrow A) \Rightarrow B)))$.

We know that $H2$ is **complete**. Let $H3$ be the proof system obtained from the system $H2$ **adding** additional axiom

A4 $(\neg(A \Rightarrow B) \Rightarrow \neg(A \Rightarrow \neg B))$

Does **Completeness Theorem** hold for $H3$? JUSTIFY.

Solution

No, **it doesn't**. The system H3 is **not sound**. Axiom **A4** is not a tautology.

Any v such that A=T and B=F is a **counter model** for $(\neg(A \Rightarrow B) \Rightarrow \neg(A \Rightarrow \neg B))$.

QUESTION 5 (15pts)

Remark This question is designed to check if you understand the notion of completeness, monotonicity, application of Deduction Theorem and use of some basic tautologies.

Consider any proof system $S = (\mathcal{L}_{\{\cap,\cup,\Rightarrow,\neg\}},\ \mathcal{F},\ LA,\ (MP)\frac{A,(A\Rightarrow B)}{B})$

We assume that S **complete** under **classical** semantics and **Deduction Theorem** holds in S.

Given any $\Gamma \subseteq F$, we define $Cn(\Gamma) = \{A \in F :\ \Gamma \vdash_S A\}$.

Prove that for any $A, B \in F$, $\quad Cn(\{A, B\}) \subseteq Cn(\{(A \cap B)\})$

Hint: Use Deduction Theorem and Completeness of S and the fact that $\models (((A \Rightarrow (B \Rightarrow C)) \Rightarrow ((A \cap B) \Rightarrow C)))$

Solution

Assume $C \in Cn(\{A, B\})$.

This means $A,\ B \vdash_S C$. We apply Deduction Theorem and we get

$$\vdash_S (A \Rightarrow (B \Rightarrow C)).$$

By the completeness of S and the fact that the formula $(((A \Rightarrow (B \Rightarrow C)) \Rightarrow ((A \cap B) \Rightarrow C)))$ is a tautology, we get that

$$\vdash_S (((A \Rightarrow (B \Rightarrow C)) \Rightarrow ((A \cap B) \Rightarrow C))).$$

Applying Modus Ponens to the above we get

$$\vdash_S ((A \cap B) \Rightarrow C).$$

This is equivalent to $(A \cap B) \vdash_S C$ by Deduction Theorem and we hence have proved that

$$C \in Cn(\{(A \cap B)\}).$$

QUESTION 6 (10pts)

1. For any formula $A = A(b_1, b_2, ..., b_n)$ and any truth assignment v we define, a corresponding formulas A', $B_1, B_2, ..., B_n$ as follows:

$$A' = \begin{cases} A & \text{if } v^*(A) = T \\ \neg A & \text{if } v^*(A) = F \end{cases} \qquad B_i = \begin{cases} b_i & \text{if } v(b_i) = T \\ \neg b_i & \text{if } v(b_i) = F \end{cases}$$

We proved the following Lemma for H_2.

Main Lemma

For any formula $A = A(b_1, b_2, ..., b_n)$ and any truth assignment v,
if A', B_1, B_2, ..., B_n are corresponding formulas defined above, then $B_1, B_2, ..., B_n \vdash A'$.

1. (2pts) Let A be a formula $((\neg a \Rightarrow \neg b) \Rightarrow (b \Rightarrow a))$.

Write what **Main Lemma** asserts for the formula A and v such that v(a)= T, v(b) = F.

Solution

Observe that the formula A is a basic tautology, hence A' =A.

A = A(a, b) and we get $B_1 = a$, $B_2 = \neg b$ and **Main Lemma** asserts

$$a,\ \neg b \vdash ((\neg a \Rightarrow \neg b) \Rightarrow (b \Rightarrow a)).$$

2. The proof of **Completeness Theorem** for H_2 defines a **method** of efficiently combining $v \in VAR$ and the **Main Lemma** to describe a construction of the proof of any tautology in H_2.

Here are the steps of the **Proof** as applied to the basic tautology

$$A(a,b) = ((\neg a \Rightarrow \neg b) \Rightarrow (b \Rightarrow a))$$

s1. By the **Main Lemma** and the assumption that $\models A(a,b)$ **any** $v \in V_A$ **defines** formulas $B_a,\ B_b$ such that

$$B_a,\ B_b \ \vdash\ A.$$

The proof is based on a method of **elimination** of $B_a,\ B_b$ to obtain $\vdash\ A$ by the use of Deduction Theorem, monotonicity of consequence, and provability of the formula

$$(*) :\ ((A \Rightarrow B) \Rightarrow ((\neg A \Rightarrow B) \Rightarrow B)).$$

s2 (8pts) **Perform** the elimination of $B_a,\ B_b$ to construct the proof of A.

Solution

We know that **any** $v \in V_A$ **defines** formulas $B_a,\ B_b$ such that

$$B_a,\ B_b \ \vdash\ A.$$

We construct the proof of A as follows.

Elimination of B_b.

We have to cases: $v(b) = T$ or $v(b) = F$.
Let $v(b) = T$, so $B_a,\ b \vdash A$, and by Deduction Theorem we get $B_a \vdash (b \Rightarrow A)$.
Let $v(b) = F$, so $B_a,\ \neg b \vdash A$, and by Deduction Theorem we get $B_a \vdash (\neg b \Rightarrow A)$.

By the provability of the formula $(*)$ for $A = b, B = A$ and monotonicity

$$B_a \ \vdash\ ((b \Rightarrow A) \Rightarrow ((\neg b \Rightarrow A) \Rightarrow A))$$

By MP applied twice twice we eliminated B_b and got $B_a \vdash A$.

Elimination of B_a.

We consider $B_a \vdash A$.

We have to cases: $v(a) = T$ or $v(a) = F$.
Let $v(a) = T$, so $a \vdash A$, and by Deduction Theorem we get $\vdash (a \Rightarrow A)$.
Let $v(a) = F$, so $\neg a \vdash A$, and by Deduction Theorem we get $\vdash (\neg a \Rightarrow A)$.

By the provability of the formula $(*)$ for $A = a, B = A$

$$\vdash\ ((a \Rightarrow A) \Rightarrow ((\neg a \Rightarrow A) \Rightarrow A))$$

By MP applied twice twice and get

$$\vdash A,$$

i.e. we eliminated B_a and got the proof of A.