CSE541 QUIZ 2 SOLUTIONS Fall 2022
25pts + 5extra credit

Construct your decomposition trees carefully. A wrong application of a decomposition rule results in **(0 pts)** for the tree.

QUESTION 1 (5pts)

Consider a proof system **RS'** obtained from **RS** by **changing** the sequence Γ' into Γ in **all** of the rules of inference of **RS**.

1. Prove that the rule $(\neg\cap)$ of **RS'** is strongly sound. You can use the shorthand notation.

2. Explain shortly how strong soundness of **RS** implies strong soundness of **RS'**.

Solution

Consider the rule $(\neg\cap)$.

$$(\neg\cap)\ \frac{\Gamma,\ \neg A,\ \neg B,\ \Delta}{\Gamma,\ \neg(A\cap B),\ \Delta}$$

1. By the **definition** we have that

$v^*(\Gamma, \neg A, \neg B, \Delta) = v^*(\delta_{\{\Gamma,\neg A,\neg B,\Delta\}}) = v^*(\Gamma) \cup v^*(\neg A) \cup v^*(\neg B) \cup v^*(\Delta) = v^*(\Gamma) \cup (\neg \mathbf{v}^*(\mathbf{A}) \cup \neg \mathbf{v}^*(\mathbf{B})) \cup v^*(\Delta)$

$= v^*(\Gamma) \cup \neg(\mathbf{v}^*(\mathbf{A}) \cap \mathbf{v}^*(\mathbf{B})) \cup v^*(\Delta) = v^*(\Gamma) \cup \mathbf{v}^*(\neg(\mathbf{A} \cap \mathbf{B})) \cup v^*(\Delta) = v^*(\delta_{\{\Gamma,\neg(A\cap B),\Delta\}}) = v^*(\Gamma', \neg(A \cap B), \Delta)$

Shorthand Notation

$v^*(\Gamma, \neg A, \neg B, \Delta) = \Gamma \cup (\neg A \cup \neg B) \cup \Delta) = \Gamma \cup \neg(A \cap B) \cup \Delta) = v^*(\Gamma, \neg(A \cap B), \Delta)$

2. The proof of strong soundness of rules of **RS'** is obtained directly from corresponding proof in **RS** only by changing the sequence Γ' into Γ.

QUESTION 2 (5pts)

Let **GL** be the Gentzen style proof system for classical logic.

1. Prove, by constructing a proper decomposition tree that

$\vdash_{\mathbf{GL}}((\neg(a \cap b) \Rightarrow b) \Rightarrow (\neg b \Rightarrow (\neg a \cup \neg b)))$.

Solution Consider the following tree.

$$\mathbf{T}_{\rightarrow A}$$

$$\longrightarrow ((\neg(a \cap b) \Rightarrow b) \Rightarrow (\neg b \Rightarrow (\neg a \cup \neg b)))$$

$$\mid (\rightarrow\Rightarrow)$$

$$(\neg(a \cap b) \Rightarrow b) \longrightarrow (\neg b \Rightarrow (\neg a \cup \neg b))$$

$$\mid (\rightarrow\Rightarrow)$$

$$\neg b, (\neg(a \cap b) \Rightarrow b) \longrightarrow (\neg a \cup \neg b)$$

$$\mid (\rightarrow \cup)$$

$$\neg b, (\neg(a \cap b) \Rightarrow b) \longrightarrow \neg a, \neg b$$

$$\mid (\rightarrow \neg)$$

$$b, \neg b, (\neg(a \cap b) \Rightarrow b) \longrightarrow \neg a$$

$$\mid (\rightarrow \neg)$$

$$b, a, \neg b, (\neg(a \cap b) \Rightarrow b) \longrightarrow$$

$$\mid (\neg \rightarrow)$$

$$b, a, (\neg(a \cap b) \Rightarrow b) \longrightarrow b$$

$$\bigwedge (\Rightarrow\rightarrow$$

$b, a \longrightarrow \neg(a \cap b), b$	$b, a, b \longrightarrow b$
$\mid (\rightarrow \neg)$	*axiom*
$b, a, (a \cap b) \longrightarrow b$	
$\mid (\cap \rightarrow)$	
$b, a, a, b \longrightarrow b$	
axiom	

All leaves of the decomposition tree are axioms, hence the proof has been found.

2. Use the **completeness theorem** for **GL** to prove that

$\nvdash_{\mathbf{GL}} ((a \Rightarrow b) \Rightarrow (\neg b \Rightarrow a))$.

Solution

By the **Completeness Theorem** we have that

$$\nvdash_{\mathbf{GL}} ((a \Rightarrow b) \Rightarrow (\neg b \Rightarrow a)) \quad \text{if and only if} \quad \not\models ((a \Rightarrow b) \Rightarrow (\neg b \Rightarrow a))$$

Any v, such that $v(a) = v(b) = F$ is a counter-model for $((a \Rightarrow b) \Rightarrow (\neg b \Rightarrow a))$, hence By the **Completeness Theorem** $\nvdash_{\mathbf{GL}} ((a \Rightarrow b) \Rightarrow (\neg b \Rightarrow a))$

QUESTION 3 (5pts) Let **GL** be the Gentzen style proof system for classical logic.

1. Define SHORTLY Decomposition Tree for any A in **GL**.

Solution

Here is my short definition.

Decomposition Tree $\mathbf{T}_A$

For each formula $A \in \mathcal{F}$, a decomposition tree $\mathbf{T}_A$ is a tree build as follows.

Step 1. The sequent $\longrightarrow A$ is the **root** of $\mathbf{T}_A$ and for any node $\Gamma \longrightarrow \Delta$ of the tree we follow the steps below.

Step 2. If $\Gamma \longrightarrow \Delta$ is indecomposable, then $\Gamma \longrightarrow \Delta$ becomes a **leaf** of the tree.

Step 3. If $\Gamma \longrightarrow \Delta$ is decomposable, then we pick one rule that applies by matching the sequent of the current node with the domain of the rules. Then we apply this rule as decomposition rule and put its left and right premises as the left and right leaves, or as one leaf in case of one premiss rule.

Step 4. We repeat steps 2 and 3 until we obtain only indecomposable leaves.

2. Prove Completeness Theorem for **GL**. We assume that the STRONG soundness has been proved.

Solution

Formula Completeness Theorem

For any formula $A \in \mathcal{F}$,

$$\vdash_{\mathbf{GL}} A \;\; if\ and\ only\ if \;\; \models A.$$

We prove the logically equivalent form of the Completeness part: for any $A \in \mathcal{F}$

$$\text{If} \quad \nvdash_{\mathbf{GL}} \longrightarrow A \;\; \text{then} \;\; \not\models A.$$

Assume $\nvdash_{\mathbf{GL}} \longrightarrow A$, i.e. $\longrightarrow A$ does not have a proof in **GL**. Let $\mathcal{T}_{\mathcal{A}}$ be a set of all decomposition trees of $\longrightarrow A$. As $\nvdash_{\mathbf{GL}} \longrightarrow A$, each $\mathcal{T} \in \mathcal{T}_{\mathcal{A}}$ has a non-axiom leaf. We choose an arbitrary $T_A \in \mathcal{T}_{\mathcal{A}}$. Let $\Gamma' \longrightarrow \Delta'$, Γ' be an non-axiom leaf of T_A, for $\Delta' \in VAR^*$ such that $\{\Gamma'\} \cap \{\Delta'\} = \emptyset$.

The non-axiom leaf $L = \Gamma' \longrightarrow \Delta'$ *defines* a truth assignment $v : VAR \leftarrow \{T, F\}$ which falsifies A as follows:

$$v(a) = \begin{cases} T & \text{if a appears in } \Gamma' \\ F & \text{if } a \text{ appears in} \Delta' \\ any\ value & \text{if a does not appear in } L \end{cases}$$

This proves, by **strong soundness** of the rules of inference of **GL** that $\not\models A$.

QUESTION 4 (5pts)

We know that a classical tautology $(\neg(a \cap b) \cup (a \cap b))$ is NOT Intuitionistic tautology and we know by **Tarski Theorem** that $\neg\neg(\neg(a \cap b) \cup (a \cap b))$ is intuitionistically PROVABLE

FIND the proof of the formula

$$\neg\neg(\neg(a \cap b) \cup (a \cap b))$$

in the Gentzen system **LI** for Intuitionistic Logic.

Solution

$$\longrightarrow \neg\neg(\neg(a \cap b) \cup (a \cap b))$$

$$\mid (\longrightarrow \neg)$$

$$\neg(\neg(a \cap b) \cup (a \cap b)) \longrightarrow$$

$$\mid (contr \longrightarrow)$$

$$\neg(\neg(a \cap b) \cup (a \cap b)),\ \neg(\neg(a \cap b) \cup (a \cap b)) \longrightarrow$$

$$\mid (\neg \longrightarrow)$$

$$\neg(\neg(a \cap b) \cup (a \cap b)) \longrightarrow\ (\neg(a \cap b) \cup (a \cap b))$$

$$\mid (\longrightarrow \cup_1)$$

$$\neg(\neg(a \cap b) \cup (a \cap b)) \longrightarrow\ \neg(a \cap b)$$

$$\mid (\longrightarrow \neg)$$

$$(a \cap b),\ \neg(\neg(a \cap b) \cup (a \cap b)) \longrightarrow$$

$$\mid (exch \longrightarrow)$$

$$\neg(\neg(a \cap b) \cup (a \cap b)),\ (a \cap b),\ \longrightarrow$$

$$\mid (\neg \longrightarrow)$$

$$(a \cap b)\ \longrightarrow\ (\neg(a \cap b) \cup (a \cap b))$$

$$\mid (\longrightarrow \cup_2)$$

$$(a \cap b)\ \longrightarrow\ (a \cap b)$$

$$axiom$$

QUESTION 5 (10pts)

Use the **QRS** proof system to prove that $\not\models A$ for

$$A = (\exists x((\neg P(x, y) \cap R(x)) \cap Q(x)) \Rightarrow \forall x((\neg P(x, y) \cap R(x)) \cap Q(x))$$

where P is a two argument predicate symbol and R, Q one argument predicate symbols

1. (5pts) Build the Decomposition Tree $\mathcal{T}_A$.

You must write comments at each step of decomposition that uses the rules $(\exists)$ and $(\forall)$.

No comments (or wrong comments), or a wrong application of any rule results in (0 pts) for the tree.

2. (5pts) Define a counter model for A determined by a non-axiom leaf of the tree $\mathcal{T}_A$. Justify why it proves that $\not\models A$.

Solution

1. (5pts)

The Decomposition Tree $\mathcal{T}_A$ is:

$$\mathbf{T}_A$$

$$(\exists x((\neg P(x,y) \cap R(x)) \cap Q(x)) \Rightarrow \forall x((\neg P(x,y) \cap R(x)) \cap Q(x))$$

$$\mid (\Rightarrow)$$

$$\neg\exists x((\neg P(x,y) \cap R(x)) \cap Q(x)),\ \forall x((\neg P(x,y) \cap R(x)) \cap Q(x))$$

$$\mid (\neg\exists)$$

$$\forall x\neg((\neg P(x,y) \cap R(x)) \cap Q(x)),\ \forall x((\neg P(x,y) \cap R(x)) \cap Q(x))$$

$$\mid (\forall)$$

$$\neg((\neg P(x_1,y) \cap R(x_1)) \cap Q(x_1)),\ \forall x((\neg P(x,y) \cap R(x)) \cap Q(x))$$

where x_1 is a first free variable in the sequence of all terms such that x_1 does not appear in $\neg((\neg P(x_1,y) \cap R(x_1)) \cap Q(x_1)),\ \forall x((\neg P(x,y) \cap R(x)) \cap Q(x))$

$$\mid (\neg\cap)$$

$$\neg(\neg P(x_1,y) \cap R(x_1)), \neg Q(x_1),\ \forall x((\neg(P(x,y) \cap R(x)) \cap Q(x))$$

$$\mid (\neg\cap)$$

$$\neg\neg P(x_1,y), \neg R(x_1), \neg Q(x_1),\ \forall x((\neg P(x,y) \cap R(x)) \cap Q(x))$$

$$\mid (\neg\neg)$$

$$P(x_1,y), \neg R(x_1), \neg Q(x_1),\ \forall x((\neg(Px,y) \cap R(x)) \cap Q(x))$$

$$\mid (\forall)$$

$$P(x_1,y), \neg R(x_1), \neg Q(x_1),\ ((\neg(P(x_2,y) \cap R(x_2)) \cap Q(x_2))$$

where x_2 is a first free variable in the sequence of all terms such that x_2 does not appear in $P(x_1,y), \neg R(x_1), \neg Q(x_1),\ \forall x((\neg(Px,y) \cap R(x)) \cap Q(x))$.

The sequence is one-to- one, hence $x_1 \neq x_2$

$$\bigwedge(\cap)$$

$$P(x_1,y), \neg R(x_1), \neg Q(x_1), (\neg(P(x_2,y) \cap R(x_2))$$

We can stop here, as we have found a non-axiom leaf

$$P(x_1,y), \neg R(x_1), \neg Q(x_1), Q(x_2)$$

$x_1 \neq x_2$, Non-axiom

2. (5pts) **Short Solution**

Given the **non-axiom leaf**

$$L_A = P(x_1,y), \neg R(x_1), \neg Q(x_1), Q(x_2)$$

We define a structure $\mathcal{M} = [M, I]$ and an assignment v, such that $(\mathcal{M}, v) \not\models L_A$ as follows.

We take a the universe of $\mathcal{M}$ the set $\mathbf{T}$ of all terms of our language $\mathcal{L}$, i.e. we put $M = \mathbf{T}$.

We define the interpretation I as follows.

$P_I(x_1, y)$ is false (does not hold) for x_1 and for any $y \in VAR$,

$R_I(x_1)$ is true (holds) for x_1 and $Q_I(x_2)$ is false (does not hold) for for x_2, and

$Q_I(x_i)$ is true (holds) for x_2.

We define the assignment $v : VAR \longrightarrow T$ as *identity*, i.e., we put $v(x) = x$ for any $x \in VAR$.

Longer Solution- you can add this for explanation

Obviously, for such defined structure $[M, I]$ and the assignment v we have that

$$([\mathbf{T}, I],\ v) \not\models P(x_1, y),\ \ ([\mathbf{T}, I],\ v) \models R(x_1),\ ([\mathbf{T}, I],\ v) \models Q(x_1),\ \text{and}\ \ ([\mathbf{T}, I],\ v) \not\models Q(x_2).$$

We hence obtain that

$$([\mathbf{T}, I],\ v) \not\models P(x_1, y), \neg R(x_1), \neg Q(x_1), Q(x_2)$$

This **proves** that such defined structure $[\mathbf{T}, I]$ is a counter model for the non-axiom leaf

$$L_A = P(x_1, y), \neg R(x_1), \neg Q(x_1), Q(x_2)$$

3. (3pts)

By the **strong soundness** of **QRS** the structure $\mathcal{M} = [\mathbf{T}, I]$ is also a counter- model for the formula Ai.e. we proved that

$$\not\models\ (\exists x((\neg P(x, y) \cap R(x)) \cap Q(x)) \Rightarrow \forall x((\neg(Px, y) \cap R(x)) \cap Q(x))$$

REMARK 1

We STOPED the decomposition process at the right branch of the $(\cap)$ rule on the node

$$P(x_1, y), \neg R(x_1), \neg Q(x_1),\ ((\neg(P(x_2, y) \cap R(x_2)) \cap Q(x_2))$$

If we decompose on the left branch of the $(\cap)$ rule we get other leaves as follows

$$P(x_1, y), \neg R(x_1), \neg Q(x_1),\ ((\neg(P(x_2, y) \cap R(x_2)))$$

$$\bigwedge (\cap)$$

$$P(x_1, y), \neg R(x_1), \neg Q(x_1), \neg P(x_2, y) \qquad\qquad P(x_1, y), \neg R(x_1), \neg Q(x_1), R(x_2)$$

$x_1 \neq x_2$, Non-axiom $\qquad\qquad$ $x_1 \neq x_2$, Non-axiom

REMARK 2

We define the counter models for the non-axiom leaves

$$L1_A = P(x_1, y), \neg R(x_1), \neg Q(x_1), \neg P(x_2, y)\ \text{ and }\ \ L2_A = P(x_1, y), \neg R(x_1), \neg Q(x_1), R(x_2)$$

following a general definition below.

Definition

Given a **non-axiom leaf** L_A of a decomposition tree $\mathcal{T}_A$ we define a structure $\mathcal{M} = [M, I]$ and an assignment v,

such that $(\mathcal{M}, v) \not\models L_A$ as follows.

We take a the universe of $\mathcal{M}$ the set $\mathbf{T}$ of all terms of our language $\mathcal{L}$, i.e. we put $M = \mathbf{T}$.

We define the interpretation I as follows.

For any predicate symbol $Q \in \mathbf{P}, \#Q = n$ we put that $Q_I(t_1, \ldots t_n)$ is **true** (holds) for terms $t_1, \ldots t_n$ if and only if the negation $\neg Q_I(t_1, \ldots t_n)$ of the formula $Q(t_1, \ldots t_n)$ appears on the leaf L_A and we put $Q_I(t_1, \ldots t_n)$ is **false** (does not hold) for terms $t_1, \ldots t_n$ otherwise.

For any functional symbol $f \in \mathbf{F}, \#f = n$ we put $f_I(t_1, \ldots t_n) = f(t_1, \ldots t_n)$.