Please take your time and write carefully your solutions. There is no NO PARTIAL CREDIT.
You get 0 pts for a solution with a formula that is NOT a well formed formula of the given language.

QUESTION 1  (5pts)

Write the following natural language statement:

From the fact that there is a bird that does not fly and 4 + 4 = 4, we deduce the following:

it is not possible that all birds fly OR it is not necessary that 4 + 4 = 4.

in the THREE ways.

1. (1pts) As a formula $A_1 \in F_1$ of a language $L_{\{\neg, \cap, \cup, \Rightarrow\}}$

Solution  Propositional Variables are: $a, b, c, d$

$a$ denotes statement: there is a bird that does not fly,

$b$ denotes statement: 4 + 4 = 4,

$c$ denotes statement: possible that all birds fly,

$d$ denotes statement: necessary that 4 + 4 = 4.

Formula $A_1 \in F_1$ is

$((a \cap b) \Rightarrow (\neg c \cup \neg d))$

2. (1pts) As a formula $A_2 \in F_2$ of a language $L_{\{\neg, \diamond, \lozenge, \cap, \cup, \Rightarrow\}}$

Solution  Propositional Variables are: $a, b, c, d$

$a$ denotes statement: there is a bird that does not fly,

$b$ denotes statement: 4 + 4 = 4,

$c$ denotes statement: all birds fly

Formula $A_2 \in F_2$ is:

$((a \cap b) \Rightarrow (\neg \lozenge c \cup \neg \diamond b))$

3. (3pts) As a formula $A_3 \in F_3$ of a PREDICATE LANGUAGE language $L_{\{\neg, \diamond, \lozenge, \cap, \cup, \Rightarrow\}}(P, F, V)$, i.e. as a formula of the predicate language $L(P, F, V)$ with the set $\{\neg, \diamond, \lozenge, \cap, \cup, \Rightarrow\}$ of propositional connectives.

Solution  Use the following Predicates, Functions and Constants

$B(x)$ for $x$ is a bird, $F(x)$ for $x$ can fly, $E(x, y)$ for $x = y$, $f(x, y)$ for $+$, and $c$ for 4.

(1pts) Restricted domain formula is:

$((\exists B(X) \neg F(x) \cap E(f(c, c), c)) \Rightarrow (\neg \lozenge \forall B(X) F(x) \cup \neg \diamond E(f(c, c), c)))$

(2pts) Formula $A_3 \in F_3$ is:

$((\exists x(B(X) \cap \neg F(x)) \cap E(f(c, c), c)) \Rightarrow (\neg \lozenge \forall x(B(X) \Rightarrow F(x)) \cup \neg \diamond E(f(c, c), c)))$

1
QUESTION 2 (5pts)

1. (2pts)  
Circle formulas that are propositional tautologies

\[ S_1 = \{ (\neg (c \land c) \Rightarrow (\neg b \Rightarrow (d \lor e))), \quad ((a \Rightarrow b) \land (a \Rightarrow \neg b)), \quad ((a \land \neg b) \Rightarrow ((a \land \neg b) \Rightarrow (d \lor e))), \quad (\neg a \Rightarrow (a \lor \neg b)) \} \]

Solution  \[ \not\models (\neg a \Rightarrow (a \lor \neg b)) \quad \text{and} \quad \not\models ((a \land \neg b) \Rightarrow ((a \land \neg b) \Rightarrow (d \lor e))), \] all other formulas are tautologies

2. (3pts)  
Circle formulas that are predicate tautologies

\[ S_2 = \{ (\exists x A(x) \Rightarrow \forall x A(x)), \quad (\forall x (P(x, y) \land Q(y)) \Rightarrow \exists y (P(x, y) \land Q(y))), \quad ((\exists A(x) \land \exists B(x)) \Rightarrow \forall x (A(x) \land B(x))), \quad (\exists A(x) \Rightarrow B) \Rightarrow (\exists A(x) \Rightarrow B) \} \]

Solution  \[ \not\models ((\exists A(x) \land \exists B(x)) \Rightarrow \exists y (A(x) \land B(x))), \] all other formulas are tautologies

QUESTION 3 (5pts)

Let \( A(x), B(x) \) be any formulas with a free variable \( x \).  Prove that

\[ \forall x (A(x) \lor B(x)) \not\models (\forall x A(x) \lor \forall x B(x)) \]

Solution  
Distributivity of universal quantifier over disjunction holds only on one direction, namely for any formulas \( A(x), B(x) \) with a free variable \( x \), we have that

\[ \models_p ((\forall x A(x) \lor \forall x B(x)) \Rightarrow \forall x (A(x) \lor B(x))). \]

The inverse implication is not a predicate tautology

\[ \not\models_p (\forall x (A(x) \lor B(x)) \Rightarrow (\forall x A(x) \lor \forall x B(x))). \]

It means that we have to find a concrete formula \( A(x), B(x) \in \mathcal{F} \) and a model structure \( M = (U, I) \) that is a counter-model for a concrete formula \( F \).

Let \( A(x), B(x) \) be atomic formulas \( Q(x, c), R(x, c) \), we get that

\[ F : (\forall x (Q(x, c) \lor R(x, c)) \Rightarrow (\forall x (Q(x, c) \lor \forall x R(x, c))). \]

Take \( M = (R, I) \), where \( R \) is the set of real numbers.

We define \( Q_I : \geq, \quad R_I : <, \quad c_I : 0. \)

The formula \( F \) becomes an obviously false mathematical statement

\[ F_I : (\forall x \in R (x \geq 0 \lor x < 0) \Rightarrow (\forall x \in R x \geq 0 \lor \forall x \in R x < 0)). \]

QUESTION 4 (10 pts)

Let \( \mathcal{L} = \mathcal{L}_{\{\neg, \Rightarrow, \rightarrow\}} \) be a language with one argument connectives \( \neg, \) called negation and weak negation, and with two arguments connectives \( \Rightarrow, \) called implication and weak implication.

We define a 3 valued extensional semantics \( \mathbf{M} \) for the language \( \mathcal{L}_{\{\neg, \Rightarrow, \rightarrow\}} \) by defining the connectives \( \neg, \Rightarrow, \rightarrow \) as functions on a set \( \{T, \bot, F\} \) of 3 logical values as follows.
The functions \(\neg, \Rightarrow\) restricted to the set \(\{F, T\}\) are the same as in classical case.

We extend them to the full set \(\{F, \bot, T\}\) for negation as \(\neg \bot = F\), and for implication as

\[
\bot \Rightarrow y = \begin{cases} 
\bot & \text{if } y = \bot \\
T & \text{otherwise}
\end{cases}
\]

and \(x \Rightarrow \bot = F\) for \(x = T, F\).

We define the weak negation \(\sim: \{T, \bot, F\} \rightarrow \{T, \bot, F\}\) as

\[
\sim x = \begin{cases} 
T & \text{if } x = \bot \\
x & \text{for } x \in \{T, F\}
\end{cases}
\]

We define the weak implication \(\rightarrow: \{T, \bot, F\} \times \{T, \bot, F\} \rightarrow \{T, \bot, F\}\) as

\[
x \rightarrow y = \sim (x \Rightarrow y)
\]

for any \(x, y \in \{T, \bot, F\}\).

1. (3pts) Fill in the connectives tables

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2. (2pts) Write the following natural language statement:

*The fact that 3 is not a number weakly implies that if 3 is weakly not a number then it is not true that 3+0=5*

as a formula \(A\) of the language \(\mathcal{L} = \mathcal{L}_{\neg, \Rightarrow, \rightarrow}\) and show that it has a M model. You can use shorthand notation.

**Solution**

The formula \(A\) is:

\(\neg a \rightarrow (\neg a \Rightarrow \neg b)\),

where \(a\) denotes statement: *3 is a number*, \(b\) denotes statement: *3 + 0 = 5*.

The M model is any \(\nu: \text{VAR} \rightarrow \{F, \bot, T\}\) such that \(\nu(a) = \nu(b) = \bot\). We use shorthand notation to evaluate \(\nu^*(A)\).

\(\nu^*((\neg a \rightarrow (\neg a \Rightarrow \neg b))) = \neg \bot \rightarrow (\neg \bot \Rightarrow \neg \bot) = F \rightarrow (T \Rightarrow F) = F \rightarrow F = T\).

This is not the only model.

3. (3pts) Let \(T\) be a set of classical tautologies and \(\text{MT}\) be a set of all M tautologies.

Prove that \(T \cap \text{MT} \neq \emptyset\).

**Solution**

Observe, that a formula \(A \in T \cap \text{MT}\) can not contain connectives \(\neg, \rightarrow\), i.e. \(A\) must belong to the language \(\mathcal{L}_{\neg, \Rightarrow}\).
There are countably infinitely many candidates to consider. In this situation the best strategy is to start with the simplest, one variable formulas of the language \( L_{\land \lor} \) that one knows to be classical tautologies.

For example, let's consider the formulas \((a \Rightarrow a), (\neg a \Rightarrow a), (a \Rightarrow \neg a)\).

We have that \( \not{\models} M (a \Rightarrow a) \) because
\[
\text{any } v : \text{VAR} \rightarrow \{F, \bot, T\} \text{ such that } v(a) = \bot \text{ is its counter model as } \bot \Rightarrow \bot = F.
\]

We have that \( \not{\models} M (\neg a \Rightarrow a) \) because
\[
\neg \bot \Rightarrow \bot = \neg F \Rightarrow \bot = T \Rightarrow \bot = F
\]

We have that \( \models M (a \Rightarrow \neg a) \).

We evaluate \( \bot \Rightarrow \neg \bot = \bot \Rightarrow \neg F = \bot \Rightarrow T = T \).

4. (2pts) Prove that the semantics \( M \) is well defined

\textbf{Solution} \hspace{1em} By definition, semantics \( M \) is well defined if and only if \( MT \neq \emptyset \).

This is true as we proved above that \( T \cap MT \neq \emptyset \).