CSE541 QUIZ 1 SOLUTIONS Fall 2022
(25pts)

Please take your time and write **carefully** your solutions. There is no NO PARTIAL CREDIT.

You get **0 pts** for a solution with a formula that is NOT a well formed formula of the given language.

QUESTION 1 (5pts)

Write the following natural language statement:

From the fact that there is a bird that does not fly and $4 + 4 = 4$*, we deduce the following:*

it is not possible that all birds fly OR it is not necessary that $4 + 4 = 4$.

in the **THREE** ways.

1. (1pts) As a formula $A_1 \in \mathcal{F}_1$ of a language $\mathcal{L}_{\{\neg, \cap, \cup, \Rightarrow\}}$

Solution Propositional Variables are: a, b, c, d
a denotes statement: *there is a bird that does not fly*,
b denotes statement: $4 + 4 = 4$,
c denotes statement: *possible that all birds fly*,
d denotes statement: *necessary that* $4 + 4 = 4$.

Formula $A_1 \in \mathcal{F}_1$ is

$$((a \cap b) \Rightarrow (\neg c \cup \neg d))$$

2. (1pts) As a formula $A_2 \in \mathcal{F}_2$ of a language $\mathcal{L}_{\{\neg, \Box, \Diamond, \cap, \cup, \Rightarrow\}}$

Solution Propositional Variables are: a, b, c, d
a denotes statement: *there is a bird that does not fly*,
b denotes statement: $4 + 4 = 4$,
c denotes statement: *all birds fly*

Formula $A_2 \in \mathcal{F}_2$ is:

$$((a \cap b) \Rightarrow (\neg\Diamond c \cup \neg\Box b))$$

3. (3pts) As a formula $A_3 \in \mathcal{F}_3$ of a PREDICATE LANGUAGE language $\mathcal{L}_{\{\neg, \Box, \Diamond, \cap, \cup, \Rightarrow\}}$**(P, F, V)**,

i.e. as a formula of the **predicate language** $\mathcal{L}$(**P, F, V**) with the set $\{\neg, \Box, \Diamond, \cap, \cup, \Rightarrow\}$ of propositional connectives.

Solution Use the following Predicates, Functions and Constants

$B(x)$ for x is a bird, $F(x)$ for x can fly, $E(x, y)$ for $x = y$, $f(x, y)$ for $+$, and c for 4.

(1pts) Restricted domain formula is:

$$((\exists_{B(X)}\neg F(x) \cap E(f(c, c), c)) \Rightarrow (\neg\Diamond\forall_{B(X)}F(x) \cup \neg\Box E(f(c, c), c)))$$

(2pts) Formula $A_3 \in \mathcal{F}_3$ is:

$$((\exists x(B(X) \cap \neg F(x)) \cap E(f(c, c), c)) \Rightarrow (\neg\Diamond\forall x(B(X) \Rightarrow F(x)) \cup \neg\Box E(f(c, c), c)))$$

QUESTION 2 (5pts)

1. (2pts) **Circle** formulas that are **propositional tautologies**

$$\mathcal{S}_1 = \{ ((\neg c \cap c) \Rightarrow (\neg b \Rightarrow (d \cap e))), \quad ((a \Rightarrow b) \cup (a \cap \neg b)), \quad ((a \cap \neg b) \Rightarrow ((a \cap \neg b) \Rightarrow (\neg d \cup e))), \quad (\neg a \Rightarrow (a \cup \neg b)) \}$$

Solution $\not\models (\neg a \Rightarrow (a \cup \neg b))$ and $\not\models ((a \cap \neg b) \Rightarrow ((a \cap \neg b) \Rightarrow (\neg d \cup e)))$, all other formulas are tautologies

2. (3pts) **Circle** formulas that are **predicate tautologies**

$$\mathcal{S}_2 = \{ (\exists x\, A(x) \Rightarrow \neg\forall x \neg\, A(x)), \quad (\forall x\, (P(x, y) \cap Q(y)) \Rightarrow \neg\exists x\, \neg(P(x, y) \cap Q(y))),$$
$$((\exists x A(x) \cap \exists x B(x)) \Rightarrow \exists x\, (A(x) \cap B(x))), \quad (\forall x(A(x) \Rightarrow B) \;\Rightarrow\; (\exists x A(x) \Rightarrow B)) \;\}$$

Solution $\not\models ((\exists x A(x) \cap \exists x B(x)) \Rightarrow \exists x\, (A(x) \cap B(x)))$, all other formulas are tautologies

QUESTION 3 (5pts)

Let $A(x), B(x)$ be any formulas with a free variable x. Prove that

$$\forall x\, (A(x) \cup B(x)) \quad \not\equiv \quad (\forall x A(x) \cup \forall x B(x))$$

Solution Distributivity of universal quantifier over disjunction holds only on one direction, namely for any formulas $A(x), B(x)$ with a free variable x, we have that

$$\models_p \; ((\forall x A(x) \cup \forall x B(x)) \Rightarrow \forall x\, (A(x) \cup B(x))).$$

The inverse implication is not a predicate tautology

$$\not\models_p \; (\forall x\, (A(x) \cup B(x)) \Rightarrow (\forall x A(x) \cup \forall x B(x))).$$

It means that we have to find a concrete formula $A(x), B(x) \in \mathcal{F}$ and a model structure $\mathbf{M} = (U, I)$ that is a **counter-model** for a concrete formula F.

Let $A(x), B(x)$ be atomic formulas $Q(x, c), R(x, c)$, we get that

$$F : \; (\forall x\, (Q(x, c) \cup R(x, c)) \Rightarrow (\forall x(Q(x, c) \cup \forall x R(x, c))).$$

Take $\mathbf{M} = (R, I)$, where R is the set of real numbers.

We define $Q_I :\geq,\; R_I :<,\; c_I : 0.$

The formula F becomes an obviously **false** mathematical statement

$$F_I : (\forall_{x \in R}\, (x \geq 0 \cup x < 0) \Rightarrow (\forall_{x \in R}\; x \geq 0 \cup \forall_{x \in R}\; x < 0)).$$

QUESTION 4 (10 pts)

Let $\mathcal{L} = \mathcal{L}_{\{\neg,\, \sim,\, \Rightarrow,\, \rightarrow\}}$ be a language with one argument connectives $\neg,\ \sim$ called *negation* and *weak negation*, and with two arguments connectives $\Rightarrow,\ \rightarrow$ called *implication* and *weak implication*.

We define a 3 valued extensional semantics **M** for the language $\mathcal{L}_{\{\neg,\, \sim,\, \Rightarrow,\, \rightarrow\}}$ by **defining the connectives** $\neg,\ \sim,\ \Rightarrow,\ \rightarrow$ as functions on a set $\{F, \bot, T\}$ of 3 logical values as follows.

The functions $\neg$, $\Rightarrow$ **restricted** to the set $\{F, T\}$ are the same as in classical case.

We extend them to the full set $\{F, \perp, T\}$ for *negation* as $\neg \perp = F$,

and for *implication* as

$$\perp \Rightarrow y = \begin{cases} \perp & \text{if } y = \perp \\ T & \text{otherwise} \end{cases}$$

and $x \Rightarrow \perp = F$ for $x = T, F$.

We define the *weak negation* $\sim$: $\{T, \perp, F\} \longrightarrow \{T, \perp, F\}$ as

$$\sim x = \begin{cases} T & \text{if } x = \perp \\ x & \text{for } x \in \{T, F\} \end{cases}$$

We define the *weak implication* $\rightarrow$: $\{T, \perp, F\} \times \{T, \perp, F\} \longrightarrow \{T, \perp, F\}$ as

$$x \rightarrow y = \sim (x \Rightarrow y)$$

for any $x, y \in \{T, \perp, F\}$.

1. (3pts) Fill in the connectives tables

$\neg$	F	$\perp$	T
	T	F	F

$\sim$	F	$\perp$	T
	F	T	T

$\Rightarrow$	F	$\perp$	T
F	T	F	T
$\perp$	T	$\perp$	T
T	F	F	T

$\rightarrow$	F	$\perp$	T
F	T	F	T
$\perp$	T	T	T
T	F	F	T

2. (2pts) Write the following natural language statement:

The fact that 3 is not a number weakly implies that if 3 is weakly not a number then it is not true that 3+0= 5

as a formula A of the language $\mathcal{L} = \mathcal{L}_{\{\neg, \sim, \Rightarrow, \rightarrow\}}$ and show that it has a **M model**. You can use shorthand notation.

Solution

The formula A is: $(\neg a \rightarrow (\sim a \Rightarrow \neg b))$,

where a denotes statement: *3 is a number*, b denotes statement: $3 + 0 = 5$.

The **M model** is any $v : VAR \longrightarrow \{F, \perp, T\}$ such that $v(a) = v(b) = \perp$. We use shorthand notation to evaluate $v^*(A)$.

$v^*((\neg a \rightarrow (\sim a \Rightarrow \neg b))) = \neg \perp \rightarrow (\sim \perp \Rightarrow \neg \perp)) = F \rightarrow (T \Rightarrow F) = F \rightarrow F = T$.

This is not the only model.

3. (3pts) Let **T** be a set of classical tautologies and **MT** be a set of all **M** tautologies.

Prove that $\mathbf{T} \cap \mathbf{MT} \neq \emptyset$.

Solution

Observe, that a formula $A \in \mathbf{T} \cap \mathbf{MT}$ can not conain connectives $\sim$, $\rightarrow$, i.e. A must belong to the language $\mathcal{L}_{\{\neg, \Rightarrow\}}$.

There are countably infinitely many candidates to consider. In this situation the best strategy is to start with the simplest, one variable formulas of the language $\mathcal{L}_{\{\neg, \ \Rightarrow\}}$ that one knowns to be classical tautologies.

For example, lets consider the formulas $(a \Rightarrow a)$, $(\neg\neg a \Rightarrow)$, $(a \Rightarrow \neg\neg a)$.

We have that $\not\models_{\mathbf{M}} (a \Rightarrow a)$ because

[any $v : VAR \longrightarrow \{F, \bot, T\}$ such that $v(a) = \bot$ is its counter model as $\bot \Rightarrow \bot = F$.

We have that $\not\models_{\mathbf{M}} (\neg\neg a \Rightarrow a)$ because

$\neg\neg \bot \Rightarrow \bot = \neg F \Rightarrow \bot = T \Rightarrow \bot = F$

We have that $\models_{\mathbf{M}} (a \Rightarrow \neg\neg a)$.

We evaluate $\bot \Rightarrow \neg\neg \bot = \bot \Rightarrow \neg F = \bot \Rightarrow T = T$.

4. (2pts) Prove that the semantics **M** is **well defined**

Solution . By definition, semantics **M** is **well defined** if and only if $\mathbf{MT} \neq \emptyset$.

This is true as as we proved above that $\mathbf{T} \cap \mathbf{MT} \neq \emptyset$.