CSE541 PRACTICE FINAL SOLUTIONS FALL 2022
(15pts extra credit)

PROBLEM 1 (5pts)

1. Let $\mathbf{M} = [U, I]$ be a structure such that

$U = N - \{0\}$ and $P_I := =$, $f_I(x, y)$ is x^y

For the following formula A

$$\forall x \forall y \forall z P(f(x, f(y, z)), f(f(x, y), z))$$

decide whether $\mathbf{M} \models A$ or $\mathbf{M} \not\models A$.

Do so by examining the corresponding mathematical statement defined by $\mathbf{M}$.

Solution

The corresponding mathematical statement defined by $\mathbf{M}$ (written with logical symbols) is

$$\forall n \forall m \forall k \; n^{m^k} = (n^m)^k$$

It is a FALSE statement in the set $N - \{0\}$

For example for $n = 2,\ m = 3,\ k = 2$ we get

$$2^{3^2} = 2^9 \text{ and } (2^3)^2 = 2^6 \quad \text{and} \quad 2^9 \neq 2^6$$

The mathematical statement can also be written as

$$\forall_{n \in N-\{0\}} \forall_{m \in N-\{0\}} \forall_{k \in N-\{0\}} \; n^{m^k} = (n^m)^k$$

2. Prove that

$$\not\models ((\exists x A(x) \cap \exists x B(x)) \Rightarrow \exists x\, (A(x) \cap B(x)))$$

Do so by providing an example of particular formulas $A(x), B(x)$ and construct a counter model $\mathbf{M}$ for these particular formulas.

Use the "shorthand" solution, i.e. examine the corresponding mathematical statement defined by $\mathbf{M}$.

Solution

Let $A(x), B(x)$ be atomic formulas $Q(x, c),\ P(x, c)$.

We consider the formula F

$$F : \quad ((\exists x Q(x, c) \;\cap\; \exists x P(x, c)) \Rightarrow \exists x\, (Q(x, c) \cap P(x, c)))$$

We define $\mathbf{M} = (R, I)$, where R is the set of real numbers, and the interpretation I is

$$Q_I : >, \quad P_I : <, \quad c_I : 0$$

The formula F becomes an obviously **false** mathematical statement

$$F_I : \quad ((\exists_{x \in R}\, x > 0 \cap \exists_{x \in R}\, x < 0) \Rightarrow \exists_{x \in R}\, (x > 0 \cap x < 0))$$

PROBLEM 2 (5pts)

Given a predicate (first order) language . $\mathcal{L} = \mathcal{L}_{\{\cap,\cup,\Rightarrow,\neg\}}(\mathbf{P},\mathbf{F},\mathbf{C})$.

1. Show that for any formulas $A(x), B(x)$ with a free variable x the following holds.

$$\vdash_{QRS} \ (\neg\forall_{B(x)} A(x) \Rightarrow \exists_{B(x)}\neg A(x)\ )$$

You must write comments at each step of decomposition that uses the rules ($\exists$) and ($\forall$).

You treat $A(x), B(x)$ as **atomic formulas** .

.

Solution

By definition of Restricted Quantifiers,

$$\vdash_{QRS} \ (\neg\forall_{B(x)} A(x) \Rightarrow \exists_{B(x)}\neg A(x)) \ \text{ if and only if } \ \vdash_{QRS} \ (\neg\forall x(B(x) \Rightarrow A(x)) \ \Rightarrow \ \exists x(B(x)\cap\neg A(x))\ )$$

To construct the decomposition tree $\mathcal{T}_A$ for the formula

$$A: \quad (\neg\forall x(B(x) \Rightarrow A(x)) \ \Rightarrow \ \exists x(B(x)\cap\neg A(x)))$$

we proceed as follows.

We put the countably infinite set of all **terms** in a one-to one sequence

$$(*) \quad t_1, t_2, ..., t_n, ...$$

and use carefully **Condition 1** ($\forall$) and **Condition 2** ($\exists$) of the decomposition tree definition and obtain the tree below.

$$\mathbf{T}_A$$

$$(\neg\forall x(B(x) \Rightarrow A(x)) \ \Rightarrow \ \exists x(B(x)\cap\neg A(x)))$$

$$\mid (\Rightarrow)$$

$$\neg\neg\forall x(B(x) \Rightarrow A(x)), \ \ \exists x(B(x)\cap\neg A(x))$$

$$\mid (\neg\neg)$$

$$\forall x(B(x) \Rightarrow A(x)), \ \ \exists x(B(x)\cap\neg A(x))$$

$$\mid (\forall)$$

$$(B(x_1) \Rightarrow A(x_1), \ \ \exists x(B(x)\cap\neg A(x))$$

where x_1 is a first free variable in the sequence $(*)$ of all terms, such that x_1 does not appear in $\forall x(B(x) \Rightarrow A(x)), \ \exists x(B(x)\cap\neg A(x))$

$$\mid (\Rightarrow)$$

$$\neg B(x_1), \ A(x_1), \ \ \exists x(B(x)\cap\neg A(x))$$

$$\mid (\exists)$$

$$\neg B(x_1), \ A(x_1), \ (B(x_1)\cap\neg A(x_1)), \ \ \exists x(B(x)\cap\neg A(x))$$

where x_1 is the first term (variables are terms) in the sequence $(*)$ such that $(B(x_1) \cap \neg A(x_1))$ does not appear on a tree above

$$\neg B(x_1),\ A(x_1),\ (B(x_1) \cap \neg A(x_1))\ \exists x(B(x) \cap \neg A(x))$$

$$\bigwedge(\cap)$$

$$\neg B(x_1), A(x_1)), B(x_1),\ \exists x(B(x) \cap \neg A(x)) \qquad \neg B(x_1), A(x_1)), \neg A(x_1), \exists x(B(x) \cap \neg A(x))$$

axiom *axiom*

PROBLEM 3 (5pts)

Find prenex normal form **PNF** of the following formula A.

$$(\exists x\ (Q(x,y) \cap P(z))\ \Rightarrow\ \forall y \exists x\ R(x,y))$$

Reminder

1. We assume that the formula A in **PNF** is always **closed**. If it is not closed you have to form its **closure**.

2. At each step of transformation list Laws of Quantifiers you used

Solution

Equational Laws of Quantifiers

Q1 $\forall x(A(x) \Rightarrow B) \equiv (\exists x A(x) \Rightarrow B)$ **Q2** $\forall x(B \Rightarrow A(x)) \equiv (B \Rightarrow \forall x A(x))$

Q3 $\exists x(B \Rightarrow A(x)) \equiv (B \Rightarrow \exists x A(x))$ **Q4** $\exists x(A(x) \Rightarrow B) \equiv (\forall x A(x) \Rightarrow B)$

where B is a formula such that B *does not contain any free occurrence* of x.

Here are the following steps for finding the **PNF** of

$$(\exists x\ (Q(x,y) \cap P(z))\ \Rightarrow\ \forall y \exists x\ R(x,y))$$

s1. Rename Variables Apart

$$(\exists x\ (Q(x,y) \cap P(z))\ \Rightarrow\ \forall t \exists w\ R(w,t))$$

s2. Pull out $\exists x$ by **Q1** for $B = \forall t \exists w\ R(w,t)$

$$\forall x\ ((Q(x,y) \cap P(z))\ \Rightarrow\ \forall t \exists w\ R(w,t))$$

s3. Pull out $\forall t$ by **Q2** for $B = (Q(x,y) \cap P(z))$

$$\forall x \forall t\ ((Q(x,y) \cap P(z))\ \Rightarrow\ \exists w\ R(w,t))$$

s4. Pull out $\exists w$ by **Q3** for $B = (Q(x,y) \cap P(z))$

$$\forall x \forall t \exists w\ ((Q(x,y) \cap P(z))\ \Rightarrow\ R(w,t))$$

s5. This is PNF with free variables y, z - we form its bf closure and get:

$$\textbf{PNF} \quad \forall y \forall z \forall x \forall t \exists w\, ((Q(x,y) \cap P(z)) \;\Rightarrow\; R(w,t))$$

Observe that in a similar way we can also get the following another form of PNF formula

$$\textbf{PNF}' \quad \forall y \forall z \forall t \exists w \forall x\, ((Q(x,y) \cap P(z)) \;\Rightarrow\; R(w,t))$$

PROBLEM 4 (5pts)

Find the **clausal form** of a formula

$$\forall x \exists y \forall z \exists w\, ((Q(x,y) \cup \neg R(x)) \Rightarrow\; \neg Q(z,w))$$

Solution

Part 1. As our formula is a **closed PNF** formula we can perform Skolem Procedure of Elimination of Quantifiers to obtain its **Skolem form** A^* as follows

s1 We eliminate $\forall x$ in the formula

$$\forall x \exists y \forall z \exists w\, ((Q(x,y) \cup \neg R(x)) \Rightarrow\; \neg Q(z,w))$$

and get a formula A_1

$$\exists y \forall z \exists w\, ((Q(x,y) \cup \neg R(x)) \Rightarrow\; \neg Q(z,w))$$

s2 We eliminate $\exists y$ by replacing y by $f(x)$ where f is a **new** one argument functional symbol **added** to the original language $\mathcal{L}$

We get a formula A_2

$$\forall z \exists w\, ((Q(x,f(x)) \cup \neg R(x)) \Rightarrow\; \neg Q(z,w))$$

s3 We eliminate $\forall z$ and get a formula A_3

$$\exists w\, ((Q(x,f(x)) \cup \neg R(x)) \Rightarrow\; \neg Q(z,w))$$

s4 We eliminate $\exists w$ by replacing w by $g(x,z)$ where g is a **new** two argument functional symbol **added** to the original language $\mathcal{L}$

We get a formula A_4 that is our resulting **open** formula A^*, called the **Skolem form** of A

$$((Q(x,f(x)) \cup \neg R(x)) \Rightarrow\; \neg Q(z,g(x,z)))$$

Part 2. We use the proof system $\mathbf{QRS}^*$ and construct the decomposition tree $\mathbf{T}_A$ of the **Skolem form** of A

$$\mathbf{T}_A$$

$$((Q(x,f(x)) \cup \neg R(x)) \Rightarrow\; \neg Q(z,g(x,z)))$$

$$\mid (\Rightarrow)$$

$$\neg(Q(x,f(x)) \cup \neg R(x)),\; \neg Q(z,g(x,z))$$

$$\bigwedge(\cap)$$

$$\neg Q(x, f(x)),\ \neg Q(z, g(x, z)) \qquad\qquad \neg\neg R(x),\ \neg Q(z, g(x, z))$$

$$| (\neg\neg)$$

$$R(x),\ \neg Q(z, g(x, z))$$

Part 3. We use the leaves of $\mathbf{T}_A$ to obtain the **clausal form** of the formula A.

The leaves of $\mathbf{T}_A$ are

$$L_1 = \neg Q(x, f(x)),\ \neg Q(z, g(x, z)) \quad \text{and} \quad L_2 = R(x),\ \neg Q(z, g(x, z))$$

The corresponding clauses are

$$C_1 = \{\neg Q(x, f(x)),\ \neg Q(z, g(x, z))\} \quad \text{and} \quad C_2 = \{R(x),\ \neg Q(z, g(x, z))\}$$

The **clausal form** of the formula A is

$$\mathbf{C}_A = \{\{\neg Q(x, f(x)),\ \neg Q(z, g(x, z))\},\ \ \{R(x),\ \neg Q(z, g(x, z))\}$$