Chapter 3 REVIEW
Some Definitions and Problems
SOME DEFINITIONS: Part One

There are some basic **DEFINITIONS** from Chapter 3

You have to **KNOW** them for **Q1** and **MIDTERM**

Knowing all basic **Definitions** is the first step for understanding the material
Definition 1

Given a propositional language $\mathcal{L}_{\text{CON}}$ for the set $\text{CON} = C_1 \cup C_2$, where $C_1, C_2$ are respectively the sets of unary and binary connectives.

Let $V$ be a non-empty set of logical values.

Connectives $\nabla \in C_1$, $\circ \in C_2$ are called extensional iff their semantics is defined by respective functions:

\[
\nabla : V \rightarrow V \quad \text{and} \quad \circ : V \times V \rightarrow V
\]
DEFINITIONS: Propositional Extensional Semantics

Definition 2

Formal definition of a propositional extensional semantics for a given language $L_{\text{CON}}$ consists of providing definitions of the following four main components:

1. Logical Connectives
2. Truth Assignment
3. Satisfaction, Model, Counter-Model
4. Tautology
CLASSICAL PROPOSITIONAL SEMANTICS
DEFINITIONS: Truth Assignment Extension v^*

Definition 3
The Language: \( \mathcal{L} = \mathcal{L}_{\neg, \Rightarrow, \cup, \cap} \)

Given the truth assignment \( v : \text{VAR} \rightarrow \{ T, F \} \) in classical semantics for the language \( \mathcal{L} = \mathcal{L}_{\neg, \Rightarrow, \cup, \cap} \)

We define its extension \( v^* \) to the set \( \mathcal{F} \) of all formulas of \( \mathcal{L} \)

as \( v^* : \mathcal{F} \rightarrow \{ T, F \} \) such that

(i) for any \( a \in \text{VAR} \)

\[ v^*(a) = v(a) \]

(ii) and for any \( A, B \in \mathcal{F} \) we put

\[ v^*(\neg A) = \neg v^*(A); \]

\[ v^*((A \cap B)) = \cap(v^*(A), v^*(B)); \]

\[ v^*((A \cup B)) = \cup(v^*(A), v^*(B)); \]

\[ v^*((A \Rightarrow B)) = \Rightarrow(v^*(A), v^*(B)); \]

\[ v^*((A \Leftrightarrow B)) = \Leftrightarrow(v^*(A), v^*(B)) \]
DEFINITIONS: Truth Assignment Extension $v^*$ Revisited

Notation
For binary connectives (two argument functions) we adopt a convention to write the symbol of the connective (name of the 2 argument function) between its arguments as we do in a case arithmetic operations.

The condition (ii) of the definition of the extension $v^*$ can be hence written as follows:

(ii) and for any $A, B \in \mathcal{F}$ we put

$$v^*(\neg A) = \neg v^*(A);$$
$$v^*((A \cap B)) = v^*(A) \cap v^*(B);$$
$$v^*((A \cup B)) = v^*(A) \cup v^*(B);$$
$$v^*((A \Rightarrow B)) = v^*(A) \Rightarrow v^*(B);$$
$$v^*((A \Leftrightarrow B)) = v^*(A) \Leftrightarrow v^*(B).$$
DEFINITIONS: Satisfaction Relation

Definition 4  Let $v : VAR \rightarrow \{T, F\}$
We say that $v$ satisfies a formula $A \in \mathcal{F}$ iff $v^*(A) = T$

Notation: $v \models A$

We say that $v$ does not satisfy a formula $A \in \mathcal{F}$ iff $v^*(A) \neq T$

Notation: $v \not\models A$
DEFINITIONS: Model, Counter-Model, Classical Tautology

Definition 5
Given a formula \( A \in \mathcal{F} \) and \( v : \text{VAR} \rightarrow \{T, F\} \)
We say that
\( v \) is a \textbf{model} for \( A \) iff \( v \models A \)
\( v \) is a \textbf{counter-model} for \( A \) iff \( v \not\models A \)

Definition 6
\( A \) is a \textbf{tautology} iff for any \( v : \text{VAR} \rightarrow \{T, F\} \) we have that \( v \models A \)

Notation
We write symbolically \( \models A \) to denote that \( A \) is a \textbf{classical tautology}
DEFINITIONS: Restricted Truth Assignments

Notation: for any formula $A$, we denote by $\text{VAR}_A$ a set of all variables that appear in $A$

Definition 7 Given a formula $A \in \mathcal{F}$, any function

$$v_A : \text{VAR}_A \rightarrow \{T, F\}$$

is called a truth assignment restricted to $A$
DEFINITIONS: Restricted Model, Counter Model

Notation: for any formula $A$, we denote by $\text{VAR}_A$ a set of all variables that appear in $A$

Definition 8  Given a formula $A \in \mathcal{F}$
Any function

$$w : \text{VAR}_A \rightarrow \{T, F\}$$
such that $w^*(A) = T$

is called a restricted MODEL for $A$

Any function

$$w : \text{VAR}_A \rightarrow \{T, F\}$$
such that $w^*(A) \neq T$

is called a restricted Counter- MODEL for $A$
DEFINITIONS: Models for Sets of Formulas

Consider $\mathcal{L} = \mathcal{L}_{\{\neg, \cup, \cap, \Rightarrow\}}$ and let $S \neq \emptyset$ be any non-empty set of formulas of $\mathcal{L}$, i.e.

$$S \subseteq \mathcal{F}$$

Definition 9

A truth assignment $v : \text{VAR} \rightarrow \{T, F\}$ is a model for the set $S$ of formulas if and only if

$$v \models A \text{ for all formulas } A \in S$$

We write

$$v \models S$$

to denote that $v$ is a model for the set $S$ of formulas.
**Definition 10**

A non-empty set \( G \subseteq \mathcal{F} \) of formulas is called **consistent** if and only if \( G \) has a model, i.e. we have that

\[
G \subseteq \mathcal{F} \text{ is consistent if and only if there is } v \text{ such that } v \models G
\]

Otherwise \( G \) is called **inconsistent**.
DEFINITIONS: Independent Statements

Definition 11
A formula $A$ is called independent from a non-empty set $G \subseteq \mathcal{F}$ if and only if there are truth assignments $v_1, v_2$ such that

$$v_1 \models G \cup \{A\} \quad \text{and} \quad v_2 \models G \cup \{\neg A\}$$

i.e. we say that a formula $A$ is independent if and only if $G \cup \{A\}$ and $G \cup \{\neg A\}$ are consistent.
Many Valued Extensional Semantics
DEFINITIONS: Semantics \( M \)

Definition 11
The extensional semantics \( M \) is defined for a non-empty set of \( V \) of logical values of any cardinality.

We only assume that the set \( V \) of logical values of \( M \) always has a special, distinguished logical value which serves to define a notion of tautology.

We denote this distinguished value as \( T \).

Formal definition of many valued extensional semantics \( M \) for the language \( L_{\text{CON}} \) consists of giving definitions of the following main components:

1. Logical Connectives under semantics \( M \)
2. Truth Assignment for \( M \)
3. Satisfaction Relation, Model, Counter-Model under semantics \( M \)
4. Tautology under semantics \( M \)
Definition of $M$ - Extensional Connectives

Given a propositional language $L_{CON}$ for the set $CON = C_1 \cup C_2$, where $C_1$ is the set of all unary connectives, and $C_2$ is the set of all binary connectives.

Let $V$ be a non-empty set of logical values adopted by the semantics $M$.

**Definition 12**

Connectives $\nabla \in C_1$, $\circ \in C_2$ are called $M$-extensional iff their semantics $M$ is defined by respective functions

$$\nabla : V \rightarrow V \quad \text{and} \quad \circ : V \times V \rightarrow V$$
DEFINITION: Definability of Connectives under a semantics $M$

Given a propositional language $\mathcal{L}_{CON}$ and its extensional semantics $M$

We adopt the following definition

Definition 13

A connective $\circ \in CON$ is definable in terms of some connectives $\circ_1, \circ_2, ... \circ_n \in CON$ for $n \geq 1$ under the semantics $M$ if and only if the connective $\circ$ is a certain function composition of functions $\circ_1, \circ_2, ... \circ_n$ as they are defined by the semantics $M$
DEFINITION: M Truth Assignment Extension \( v^* \) to \( \mathcal{F} \)

**Definition 14**

Given the \( \textbf{M} \) truth assignment \( v : \text{VAR} \rightarrow \text{V} \)

We define its \( \textbf{M} \) extension \( v^* \) to the set \( \mathcal{F} \) of all formulas of \( \mathcal{L} \) as any function \( v^* : \mathcal{F} \rightarrow \text{V} \), such that the following conditions are satisfied

(i) for any \( a \in \text{VAR} \)

\[ v^*(a) = v(a); \]

(ii) For any connectives \( \nabla \in \mathcal{C}_1 \), \( \circ \in \mathcal{C}_2 \) and for any formulas \( A, B \in \mathcal{F} \) we put

\[ v^*(\nabla A) = \nabla v^*(A) \]

\[ v^*((A \circ B)) = \circ(v^*(A), v^*(B)) \]
DEFINITION: **M** Satisfaction, Model, Counter Model, Tautology

**Definition 15** Let $v : \text{VAR} \rightarrow V$

Let $T \in V$ be the distinguished logical value

We say that $v$ **M** satisfies a formula $A \in \mathcal{F}$ ($v \models_M A$) iff $v^*(A) = T$

**Definition 16**

Given a formula $A \in \mathcal{F}$ and $v : \text{VAR} \rightarrow V$

Any $v$ such that $v \models_M A$ is called a **M** model for $A$

Any $v$ such that $v \not\models_M A$ is called a **M** counter model for $A$

$A$ is a **M** tautology ($\models_M A$) iff $v \models_M A$, for all $v : \text{VAR} \rightarrow V$
CHAPTER 3: Some Questions
Chapter 3: Question 1

Question 1
Find a restricted model for formula $A$, where

$$A = (\neg a \Rightarrow (\neg b \cup (b \Rightarrow \neg c)))$$

You can’t use short-hand notation
Show each step of solution

Solution
For any formula $A$, we denote by $\text{VAR}_A$ a set of all variables that appear in $A$
In our case we have $\text{VAR}_A = \{a, b, c\}$
Any function $\nu_A : \text{VAR}_A \rightarrow \{T, F\}$ is called a truth assignment restricted to $A$
Let \( v : \text{VAR} \rightarrow \{ T, F \} \) be any truth assignment such that
\[
v(a) = v_A(a) = T, \quad v(b) = v_A(b) = T, \quad v(c) = v_A(c) = F
\]

We evaluate the value of the extension \( v^* \) of \( v \) on the formula \( A \) as follows
\[
\begin{align*}
v^*(A) &= v^*((\neg a \Rightarrow (\neg b \cup (b \Rightarrow \neg c)))) \\
&= v^*(\neg a) \Rightarrow v^*((\neg b \cup (b \Rightarrow \neg c))) \\
&= \neg v^*(a) \Rightarrow (v^*(\neg b) \cup v^*((b \Rightarrow \neg c))) \\
&= \neg v(a) \Rightarrow (\neg v(b) \cup (v(b) \Rightarrow \neg v(c))) \\
&= \neg v_A(a) \Rightarrow (\neg v_A(b) \cup (v_A(b) \Rightarrow \neg v_A(c))) \\
\end{align*}
\]
\[
(\neg T \Rightarrow (\neg T \cup (T \Rightarrow \neg F))) = F \Rightarrow (F \cup T) = F \Rightarrow T = T, \text{ i.e.}
\]
\[
v_A \models A \quad \text{and} \quad v \models A
\]
Chapter 3: Question 2

Question 2
Find a restricted model and a restricted counter-model for $A$, where

$$A = (\neg a \Rightarrow (\neg b \cup (b \Rightarrow \neg c)))$$

You can use short-hand notation. Show work.

Solution

**Notation:** for any formula $A$, we denote by $\text{VAR}_A$ a set of all variables that appear in $A$.

In our case we have $\text{VAR}_A = \{a, b, c\}$.

Any function $\nu_A : \text{VAR}_A \rightarrow \{T, F\}$ is called a truth assignment restricted to $A$.

We define now $\nu_A(a) = T, \nu_A(b) = T, \nu_A(c) = F$, in shorthand: $a = T, b = T, c = F$ and evaluate

$$(\neg T \Rightarrow (\neg T \cup (T \Rightarrow \neg F))) = F \Rightarrow (F \cup T) = F \Rightarrow T = T$$

i.e.

$$\nu_A \models A$$
Chapter 3: Question 2

Observe that

\[(\neg a \Rightarrow (\neg b \cup (b \Rightarrow \neg c))) = T\] when \(a = T\) and \(b, c\) any truth values as by definition of implication we have that \(F \Rightarrow \text{anything} = T\)

Hence \(a = T\) gives us 4 models as we have \(2^2\) possible values on \(b\) and \(c\)
Chapter 3: Question 2

We take as a restricted counter-model: \( a = F, \ b = T \) and \( c = T \)

**Evaluation:** observe that

\[
(\neg a \Rightarrow (\neg b \cup (b \Rightarrow \neg c))) = F \quad \text{if and only if}
\]

\[
\neg a = T \quad \text{and} \quad (\neg b \cup (b \Rightarrow \neg c)) = F \quad \text{if and only if}
\]

\[
a = F, \ \neg b = F \quad \text{and} \quad (b \Rightarrow \neg c) = F \quad \text{if and only if}
\]

\[
a = F, \ b = T \quad \text{and} \quad (T \Rightarrow \neg c) = F \quad \text{if and only if}
\]

\[
a = F, \ b = T \quad \text{and} \quad \neg c = F \quad \text{if and only if}
\]

\[
a = F, \ b = T \quad \text{and} \quad c = T
\]

The above proves also that \( a = F, \ b = T \) and \( c = T \) is the only restricted counter-model for \( A \)
Chapter 3: Question 3

**Question 3** Justify whether the following statements true or false

**S1** There are more then 3 possible restricted counter-models for $A$

**S2** There are more then 2 possible restricted models of $A$

**Solution**

**S1** Statement: There are more then 3 possible restricted counter-models for $A$ is **false**

We have just proved that there is only one possible restricted counter-model for $A$

**S2** Statement: There are more then 2 possible restricted models of $A$ is **true**

There are 7 possible restricted models for $A$

**Justification:** $2^3 - 1 = 7$
Chapter 3: Question 4

Question 4
1. List 3 models for $A$ from Question 2, i.e. for formula

$$A = (\neg a \Rightarrow (\neg b \cup (b \Rightarrow \neg c)))$$

that are extensions to the set $VAR$ of all variables of one of the restricted models that you have found in Questions 1,

2. List 2 counter models for $A$ that are extensions of one of the restricted counter models that you have found in the Questions 1, 2
Chapter 3: Question 4

Solution

1. One of the **restricted models** is, for example a function

   \[ v_A : \{a, b, c\} \rightarrow \{T, F\} \] such that

   \[ v_A(a) = T, \ v_A(b) = T, \ v_A(c) = F \]

   We **extend** \( v_A \) to the set of all propositional variables \( VAR \) to obtain a (non restricted) **models** as follows
Chapter 3: Question 4

Model $w_1$ is a function

\[ w_1 : \text{VAR} \rightarrow \{ T, F \} \] such that

\[ w_1(a) = v_A(a) = T, \quad w_1(b) = v_A(b) = T, \]
\[ w_1(c) = v_A(c) = F, \quad \text{and} \quad w_1(x) = T, \quad \text{for all} \]
\[ x \in \text{VAR} - \{ a, b, c \} \]

Model $w_2$ is defined by a formula

\[ w_2(a) = v_A(a) = T, \quad w_2(b) = v_A(b) = T, \]
\[ w_2(c) = v_A(c) = F, \quad \text{and} \quad w_2(x) = F, \quad \text{for all} \]
\[ x \in \text{VAR} - \{ a, b, c \} \]
Chapter 3: Question 4

Model $w_3$ is defined by a formula

$w_3(a) = v_A(a) = T$, $w_3(b) = v_A(b) = T$, $w_3(c) = v(c) = F$, $w_3(d) = F$ and $w_3(x) = T$ for all $x \in VAR - \{a, b, c, d\}$

There is as many of such models, as extensions of $v_A$ to the set $VAR$, i.e. as many as real numbers
Chapter 3: Question 4

2. A counter-model for a formula
   \[ A = (\neg a \Rightarrow (\neg b \cup (b \Rightarrow \neg c))) \] is, by definition any function
   \[ v : \text{VAR} \rightarrow \{ T, F \} \]
   such that \[ v^*(A) = F \]

A restricted counter-model for the formula \( A \), the only one, as already proved in is a function

\[ v_A : \{ a, b \} \rightarrow \{ T, F \} \]

such that

\[ v_A(a) = F, \quad v_A(b) = T, \quad v_A(c) = T \]
Chapter 3: Question 4

We extend $v_A$ to the set of all propositional variables $VAR$ to obtain (non restricted) some counter-models.

Here are two of such extensions

Counter-model $w_1$:  
\[ w_1(a) = v_A(a) = F, \quad w_1(b) = v_A(b) = T, \]
\[ w_1(c) = v(c) = T, \quad \text{and} \quad w_1(x) = F, \quad \text{for all} \quad x \in VAR - \{a, b, c\} \]

Counter-model $w_2$:  
\[ w_2(a) = v_A(a) = T, \quad w_2(b) = v_A(b) = T, \]
\[ w_2(c) = v(c) = T, \quad \text{and} \quad w_2(x) = T \quad \text{for all} \quad x \in VAR - \{a, b, c\} \]

There is as many of such counter-models, as extensions of $v_A$ to the set $VAR$, i.e. as many as real numbers
Chapter 3: Models for Sets of Formulas

Definition
A truth assignment $v$ is a model for a set $G \subseteq F$ if and only if
$v \models B$ for all $B \in G$

We denote it by $v \models G$

Observe that the set $G \subseteq F$ can be finite or infinite
Definition
A set $G \subseteq \mathcal{F}$ of formulas is called \textbf{consistent} if and only if $G$ has a model, i.e. we have that

$G \subseteq \mathcal{F}$ is \textbf{consistent} if and only if

\text{there is} $v$ such that $v \models G$

Otherwise $G$ is called \textbf{inconsistent}
Chapter 3: Independent Statements

Definition

A formula $A$ is called **independent** from a set $G \subseteq \mathcal{F}$ if and only if there are truth assignments $v_1, v_2$ such that

$$v_1 \models G \cup \{A\} \text{ and } v_2 \models G \cup \{\neg A\}$$

i.e. we say that a formula $A$ is **independent** if and only if

$$G \cup \{A\} \text{ and } G \cup \{\neg A\} \text{ are consistent}$$
Chapter 3: Question 5

Question 5
Given a set

\[ G = \{ ((a \cap b) \Rightarrow b), (a \cup b), \neg a \} \]

Show that \( G \) is consistent

Solution
We have to find \( v : VAR \rightarrow \{ T, F \} \) such that

\[ v \models G \]

It means that we need to find \( v \) such that

\[ v^*((a \cap b) \Rightarrow b) = T, \quad v^*(a \cup b) = T, \quad v^*(\neg a) = T \]
Chapter 3: Question 5

Observe that \( \models ((a \cap b) \Rightarrow b) \), hence we have that

1. \( v^*((a \cap b) \Rightarrow b) = T \) for any \( v \)

\( v^*(-a) = \neg v^*(a) = \neg v(a) = T \)

only when \( v(a) = F \) hence

2. \( v(a) = F \)

\( v^*(a \cup b) = v^*(a) \cup v^*(b) = v(a) \cup v(b) = F \cup v(b) = T \)

only when \( v(b) = T \) so we get

3. \( v(b) = T \)

This means that for any \( v : VAR \rightarrow \{T, F\} \) such that

\( v(a) = F, \ v(b) = T, \ v \models \mathcal{G} \)

and we proved that \( \mathcal{G} \) is consistent
Chapter 3: Question 6

Question 6
Show that a formula $A = (\neg a \cap b)$ is not independent of

$G = \{((a \cap b) \Rightarrow b), (a \cup b), \neg a\}$

Solution
We have to show that it is impossible to construct $v_1, v_2$ such that

$v_1 \models G \cup \{A\}$ and $v_2 \models G \cup \{\neg A\}$

Observe that we have just proved that any $v$ such that $v(a) = F$, and $v(b) = T$ is the only model restricted to the set of variables $\{a, b\}$ for $G$ so we have to check now if it is possible that $v \models A$ and $v \models \neg A$
We have to evaluate $v^*(A)$ and $v^*(\neg A)$ for

$v(a) = F$, and $v(b) = T$

$v^*(A) = v^*((\neg a \cap b) = \neg v(a) \cap v(b) = \neg F \cap T = T \cap T = T$

and so $v \models A$

$v^*(\neg A) = \neg v^*(A) = \neg T = F$

and so $v \not\models \neg A$

This ends the proof that $A$ is not independent of $G$
Chapter 3: Question 7

Question 7
Find an infinite number of formulas that are independent of
\[ G = \{((a \cap b) \Rightarrow b), (a \cup b), \neg a\} \]

This my solution - there are many others, but this one seemed to me to be the simplest

Solution
We just proved that any \( v \) such that \( v(a) = F, \ v(b) = T \) is the only model restricted to the set of variables \( \{a, b\} \) and so all other possible models for \( G \) must be extensions of \( v \).
We define a countably infinite set of formulas (and their negations) and corresponding extensions of $v$ (restricted to to the set of variables $\{a, b\}$) such that $v \models G$ as follows.

Observe that all extensions of $v$ restricted to to the set of variables $\{a, b\}$ have as domain the infinitely countable set

$$VAR - \{a, b\} = \{a_1, a_2, \ldots, a_n, \ldots\}$$

We take as a set of formulas (to be proved to be independent) the set of atomic formulas

$$\mathcal{F}_0 = VAR - \{a, b\} = \{a_1, a_2, \ldots, a_n, \ldots\}$$
Chapter 3: Question 7

**proof** of independence of any formula of $\mathcal{F}_0$

Let $c \in \mathcal{F}_0$

We define truth assignments $v_1, v_2 : \text{VAR} \rightarrow \{T, F\}$ such that

$$v_1 \models \mathcal{G} \cup \{c\} \quad \text{and} \quad v_2 \models \mathcal{G} \cup \{-c\}$$

as follows

$v_1(a) = v(a) = F, \quad v_1(b) = v(b) = T \quad \text{and} \quad v_1(c) = T$

for all $c \in \mathcal{F}_0$

$v_2(a) = v(a) = F, \quad v_2(b) = v(b) = T \quad \text{and} \quad v_2(c) = F$

for all $c \in \mathcal{F}_0$
CHAPTER 3
Some Extensional Many Valued Semantics
Chapter 3: Question 8

Question 8
We define a 4 valued $H_4$ logic semantics as follows

The language is $\mathcal{L} = \mathcal{L}_{\neg, \Rightarrow, \cup, \cap}$

The logical connectives $\neg$, $\Rightarrow$, $\cup$, $\cap$ of $H_4$ are operations in the set $\{F, \bot_1, \bot_2, T\}$, where $\{F < \bot_1 < \bot_2 < T\}$ and are defined as follows

**Conjunction** $\cap$ is a function

$\cap : \{F, \bot_1, \bot_2, T\} \times \{F, \bot_1, \bot_2, T\} \rightarrow \{F, \bot_1, \bot_2, T\}$, such that for any $x, y \in \{F, \bot_1, \bot_2, T\}$

$$x \cap y = \text{min}\{x, y\}$$
Chapter 3: Question 8

Disjunction $\cup$ is a function
$\cup: \{F, \bot_1, \bot_2, T\} \times \{F, \bot_1, \bot_2, T\} \longrightarrow \{F, \bot_1, \bot_2, T\}$, such that for any $x, y \in \{F, \bot_1, \bot_2, T\}$

$$x \cup y = \max\{x, y\}$$

Implication $\Rightarrow$ is a function
$\Rightarrow: \{F, \bot_1, \bot_2, T\} \times \{F, \bot_1, \bot_2, T\} \longrightarrow \{F, \bot_1, \bot_2, T\}$, such that for any $x, y \in \{F, \bot_1, \bot_2, T\}$,

$$x \Rightarrow y = \begin{cases} T & \text{if } x \leq y \\ y & \text{otherwise} \end{cases}$$

Negation: for any $x, y \in \{F, \bot_1, \bot_2, T\}$

$$\neg x = x \Rightarrow F$$
Chapter 3: Question 8

Part 1  Write Truth Tables for IMPLICATION and NEGATION in $H_4$

Solution

$H_4$ Implication

<table>
<thead>
<tr>
<th>$\Rightarrow$</th>
<th>$F$</th>
<th>$\perp_1$</th>
<th>$\perp_2$</th>
<th>$T$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$F$</td>
<td>$T$</td>
<td>$T$</td>
<td>$T$</td>
<td>$T$</td>
</tr>
<tr>
<td>$\perp_1$</td>
<td>$F$</td>
<td>$T$</td>
<td>$T$</td>
<td>$T$</td>
</tr>
<tr>
<td>$\perp_2$</td>
<td>$F$</td>
<td>$\perp_1$</td>
<td>$T$</td>
<td>$T$</td>
</tr>
<tr>
<td>$T$</td>
<td>$F$</td>
<td>$\perp_1$</td>
<td>$\perp_2$</td>
<td>$T$</td>
</tr>
</tbody>
</table>

$H_4$ Negation

<table>
<thead>
<tr>
<th>$\neg$</th>
<th>$F$</th>
<th>$\perp_1$</th>
<th>$\perp_2$</th>
<th>$T$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$T$</td>
<td>$F$</td>
<td>$F$</td>
<td>$F$</td>
<td>$F$</td>
</tr>
</tbody>
</table>
Chapter 3: Question 7

Part 2 Verify whether

\[ \models_{H_4} ((a \Rightarrow b) \Rightarrow (\neg a \cup b)) \]

Solution

Take any \( v \) such that
\( v(a) = \bot_1 \quad v(b) = \bot_2 \)

Evaluate
\[ v \star ((a \Rightarrow b) \Rightarrow (\neg a \cup b)) = (\bot_1 \Rightarrow \bot_2) \Rightarrow (\neg \bot_1 \cup \bot_2) = T \Rightarrow (F \cup \bot_2)) = T \Rightarrow \bot_2 = \bot_2 \]

This proves that our \( v \) is a counter-model and hence

\[ \not\models_{H_4} ((a \Rightarrow b) \Rightarrow (\neg a \cup b)) \]
Chapter 3: Question 9

Question 9
Show that (can’t use TTables!)

\[ \models ((\neg a \cup b) \Rightarrow (((c \cap d) \Rightarrow \neg d) \Rightarrow (\neg a \cup b))) \]

Solution
Denote \( A = (\neg a \cup b) \), and \( B = ((c \cap d) \Rightarrow \neg d) \)

Our formula becomes a substitution of a basic tautology

\[ (A \Rightarrow (B \Rightarrow A)) \]

and hence is a tautology
Chapter 3: Challenge Exercise

1. Define your own propositional language $L_{CON}$ that contains also different connectives that the standard connectives $\neg$, $\cup$, $\cap$, $\Rightarrow$

Your language $L_{CON}$ does not need to include all (if any!) of the standard connectives $\neg$, $\cup$, $\cap$, $\Rightarrow$

2. Describe intuitive meaning of the new connectives of your language

3. Give some motivation for your own semantic

4. Define formally your own extensional semantics $M$ for your language $L_{CON}$ - it means write carefully all Steps 1- 4 of the definition of your $M$
Chapter 3: Question 10

Question 10
Definition

Let $S_3$ be a 3-valued semantics for $L\{\neg, \cup, \Rightarrow\}$ defined as follows:

$V = \{F, U, T\}$ is the set of logical values with the distinguished value $T$

$x \Rightarrow y = \neg x \cup y$ for any $x, y \in \{F, U, T\}$

$\neg F = T$, $\neg U = F$, $\neg T = U$

and

\[
\begin{array}{c|ccc}
\cup & F & U & T \\
\hline
F & F & U & T \\
U & U & U & U \\
T & T & U & T \\
\end{array}
\]
Question 10

Part 1
Consider the following classical tautologies:

\[ A_1 = (a \cup \neg a), \quad A_2 = (a \Rightarrow (b \Rightarrow a)) \]

Find \( S_3 \) counter-models for \( A_1, A_2 \), if exist

You can’t use shorthand notation

Solution

Any \( v \) such that \( v(a) = v(b) = U \) is a counter-model for both \( A_1 \) and \( A_2 \), as

\[ v^*(a \cup \neg a) = v^*(a) \cup \neg v^*(b) = U \cup \neg U = U \cup F = U \neq T \]

\[ v^*(a \Rightarrow (b \Rightarrow a)) = v^*(a) \Rightarrow (v^*(b) \Rightarrow v^*(a)) = U \Rightarrow (U \Rightarrow U) = U \Rightarrow U = \neg U \cup U = F \cup U = U \neq T \]
Consider the following classical tautologies:

\[ A_1 = (a \cup \neg a), \quad A_2 = (a \Rightarrow (b \Rightarrow a)) \]

Define your own 2-valued semantics \( S_2 \) for \( \mathcal{L} \), such that none of \( A_1, A_2 \) is a \( S_2 \) tautology.

Verify your results. You can use shorthand notation.

Solution

This is not the only solution, but it is the simplest and most obvious I could think of! Here it is.

We define \( S_2 \) connectives as follows:

\[ \neg x = F, \quad x \Rightarrow y = F, \quad x \cup y = F \quad \text{for all} \quad x, y \in \{F, T\} \]

Obviously, for any \( v \),

\[ v^*(a \cup \neg a) = F \quad \text{and} \quad v^*(a \Rightarrow (b \Rightarrow a)) = F \]
Chapter 3: Question 11

Question 11
Prove using proper classical logical equivalences (list them at each step) that for any formulas \( A, B \) of language \( \mathcal{L}_{\{\neg, \cup, \Rightarrow\}} \)

\[
\neg(A \iff B) \equiv ((A \cap \neg B) \cup (\neg A \cap B))
\]

Solution
\[
\neg(A \iff B) \equiv^{\text{def}} \neg((A \Rightarrow B) \cap (B \Rightarrow A))
\equiv^{\text{deMorgan}} (\neg(A \Rightarrow B) \cup \neg(B \Rightarrow A))
\equiv^{\text{negimpl}} ((A \cap \neg B) \cup (B \cap \neg A)) \equiv^{\text{commut}} ((A \cap \neg B) \cup (\neg A \cap B))
\]
Question 12

Question 12
Prove using proper classical logical equivalences (list them at each step) that for any formulas \( A, B \) of language \( L_{\{\neg, \cup, \Rightarrow\}} \)

\[
((B \cap \neg C) \Rightarrow (\neg A \cup B)) \equiv ((B \Rightarrow C) \cup (A \Rightarrow B))
\]

Solution

\[
((B \cap \neg C) \Rightarrow (\neg A \cup B))
\equiv^{impl} (\neg(B \cap \neg C) \cup (\neg A \cup B))
\equiv^{deMorgan} ((\neg B \cup \neg \neg C) \cup (\neg A \cup B))
\equiv^{dneg} ((\neg B \cup C) \cup (\neg A \cup B)) \equiv^{impl} ((B \Rightarrow C) \cup (A \Rightarrow B))
\]
We define Ł connectives for $L\{\neg, \cup, \Rightarrow\}$ as follows:

**Ł Negation** $\neg$ is a function:

$$\neg : \{T, \bot, F\} \rightarrow \{T, \bot, F\}$$

such that $\neg \bot = \bot, \neg T = F, \neg F = T$

**Ł Conjunction** $\cap$ is a function:

$$\cap : \{T, \bot, F\} \times \{T, \bot, F\} \rightarrow \{T, \bot, F\}$$

such that $x \cap y = \min\{x, y\}$ for all $x, y \in \{T, \bot, F\}$

Remember that we assumed: $F < \bot < T$
Question 13

Ł Implication \( \Rightarrow \) is a function:

\[
\Rightarrow : \{ T, \bot, F \} \times \{ T, \bot, F \} \rightarrow \{ T, \bot, F \}
\]
such that

\[
x \Rightarrow y = \begin{cases} 
\neg x \cup y & \text{if } x > y \\
T & \text{otherwise}
\end{cases}
\]

Given a formula \( ((a \cap b) \Rightarrow \neg b) \in \mathcal{F} \) of \( \mathcal{L}_{\{\neg, \cup, \Rightarrow\}} \)

Use the fact that \( v : \text{VAR} \rightarrow \{F, \bot, T\} \) is such that

\[v^*(((a \cap b) \Rightarrow \neg b)) = \bot \] under Ł semantics to evaluate \( v^*(((b \Rightarrow \neg a) \Rightarrow (a \Rightarrow \neg b)) \cup (a \Rightarrow b)) \)

You can use shorthand notation
Solution

The formula \(((a \cap b) \Rightarrow \neg b) = \bot\) in Ł connectives semantics in two cases written is the shorthand notation as

\[ \textbf{C1} \quad (a \cap b) = \bot \quad \text{and} \quad \neg b = F \]

\[ \textbf{C2} \quad (a \cap b) = T \quad \text{and} \quad \neg b = \bot. \]

Consider case \textbf{C1}

\(\neg b = F\), so \(v(b) = T\), and hence \((a \cap T) = v(a) \cap T = \bot\) if and only if \(v(a) = \bot\)

It means that \(v^*((((a \cap b) \Rightarrow \neg b)) = \bot\) for any \(v\), is such that \(v(a) = \bot\) and \(v(b) = T\)
We now evaluate (in shorthand notation)
\[ v^*(((b \Rightarrow \neg a) \Rightarrow (a \Rightarrow \neg b)) \cup (a \Rightarrow b)) \]
\[ = (((T \Rightarrow \neg \bot) \Rightarrow (\bot \Rightarrow \neg T)) \cup (\bot \Rightarrow T)) = ((\bot \Rightarrow \bot) \cup T) = T \]

Consider now Case C2
\[ \neg b = \bot, \ \text{i.e.} \ b = \bot, \ \text{and hence} \ (a \cap \bot) = T \ \text{what is impossible, hence} \ v \ \text{from the Case C1 is the only one} \]
Use the **Definability of Conjunction** in terms of disjunction and negation **Equivalence**

\[(A \land B) \equiv \neg(\neg A \lor \neg B)\]

to transform a formula

\[A = \neg(\neg(\neg a \land \neg b) \land a)\]

of the language \(\mathcal{L}_{\land,\neg}\) into a logically equivalent formula \(B\)
of the language \(\mathcal{L}_{\lor,\neg}\)
Question 14

Solution

\[ \neg (\neg (\neg a \land \neg b) \land a) \equiv \neg (\neg (\neg a \land \neg b) \lor \neg a) \]

\[ \equiv ((\neg a \land \neg b) \lor \neg a) \equiv (\neg (\neg a \lor \neg b) \lor \neg a) \]

\[ \equiv \neg (a \lor b) \lor \neg a \]

The formula $B$ of $\mathcal{L}_{\{\lor, \neg\}}$ equivalent to $A$ is

$B = (\neg (a \lor b) \lor \neg a)$
Equivalence of Languages Definition

Definition
Given two languages: \( L_1 = L_{CON_1} \) and \( L_2 = L_{CON_2} \), for \( CON_1 \neq CON_2 \)
We say that they are logically equivalent, i.e.

\[ L_1 \equiv L_2 \]

if and only if the following conditions \( C1, C2 \) hold.

\textbf{C1:} for any formula \( A \) of \( L_1 \), there is a formula \( B \) of \( L_2 \), such that \( A \equiv B \)

\textbf{C2:} for any formula \( C \) of \( L_2 \), there is a formula \( D \) of \( L_1 \), such that \( C \equiv D \)
Question 14

Prove the logical equivalence of the languages

\[ L\{\neg, \cup\} \equiv L\{\neg, \Rightarrow\} \]

Solution

We need two definability equivalences:

- implication in terms of disjunction and negation

\[ (A \Rightarrow B) \equiv (\neg A \cup B) \]

and disjunction in terms of implication and negation,

\[ (A \cup B) \equiv (\neg A \Rightarrow B) \]

and the Substitution Theorem
Question 15

Prove the logical equivalence of the languages

\[ \mathcal{L}_{\{\neg, \cap, \cup, \Rightarrow\}} \equiv \mathcal{L}_{\{\neg, \cap, \cup\}} \]

Solution

We need only the **definability of implication** in terms of disjunction and negation equivalence

\[(A \Rightarrow B) \equiv (\neg A \cup B)\]

as the **Substitution Theorem** for any formula \(A\) of \(\mathcal{L}_{\{\neg, \cap, \cup, \Rightarrow\}}\) there is a formula \(B\) of \(\mathcal{L}_{\{\neg, \cap, \cup\}}\) such that \(A \equiv B\) and the condition \(C_1\) holds

**Observe** that any formula \(A\) of language \(\mathcal{L}_{\{\neg, \cap, \cup\}}\) is also a formula of the language \(\mathcal{L}_{\{\neg, \cap, \cup, \Rightarrow\}}\) and of course \(A \equiv A\) so the condition \(C_2\) also holds.
Question 16

Prove that

\[ \mathcal{L}_{\neg, \cap} \equiv \mathcal{L}_{\neg, \Rightarrow} \]

Solution

The equivalence of languages holds due to the following two definability of connectives equivalences, respectively

\[ (A \cap B) \equiv \neg (A \Rightarrow \neg B), \quad (A \Rightarrow B) \equiv \neg (A \cap \neg B) \]

and Substitution Theorem
Question 17

Prove that in classical semantics

$$\mathcal{L}\{\neg, \Rightarrow\} \equiv \mathcal{L}\{\neg, \Rightarrow, \cup\}$$

Solution

OBSERVE that the condition \textbf{C1} holds because any formula of \(\mathcal{L}\{\neg, \Rightarrow\}\) is also a formula of \(\mathcal{L}\{\neg, \Rightarrow, \cup\}\).

Condition \textbf{C2} holds due to the following definability of connectives equivalence

\[(A \cup B) \equiv (\neg A \Rightarrow B)\]

and \textbf{Substitution Theorem}
Question 18

Prove that the equivalence defining $\cup$ in terms of negation and implication in classical logic does not hold under $\mathcal{L}$ semantics, i.e. that

$$(A \cup B) \not\equiv_{\mathcal{L}} (\neg A \Rightarrow B)$$

but nevertheless

$$\mathcal{L}_{\{\neg, \Rightarrow\}} \equiv_{\mathcal{L}} \mathcal{L}_{\{\neg, \Rightarrow, \cup\}}$$
Solution

We prove

\[ \mathcal{L}_{\{\neg, \Rightarrow\}} \equiv_{\mathcal{L}} \mathcal{L}_{\{\neg, \Rightarrow, \cup\}} \]

as follows

Condition \textbf{C2} holds because the definability of connectives equivalence

\[(A \cup B) \equiv_{\mathcal{L}} ((A \Rightarrow B) \Rightarrow B)\]

Check it by verification as an exercise

\textbf{C1} holds because any formula of \(\mathcal{L}_{\{\neg, \Rightarrow\}}\) is a formula of \(\mathcal{L}_{\{\neg, \Rightarrow, \cup\}}\)

\textbf{Observe} that the equivalence \((A \cup B) \equiv (A \Rightarrow B) \Rightarrow B)\) provides also an alternative proof of \textbf{C2} in classical case