cse541
LOGIC for Computer Science

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LECTURE 9c
PART 4: Deduction Theorem
Deduction Theorem

In mathematical arguments, one often assumes a statement \( A \) on the assumption (hypothesis) of some other statement \( B \) and then concludes that we have proved the implication "if \( A \), then \( B \)"

This reasoning is justified by the following theorem, called a Deduction Theorem

It was first formulated and proved for a certain Hilbert proof system \( S \) for the classical propositional logic by Herbrand in 1930 in a form stated as follows
Deduction Theorem

Deduction Theorem (Herbrand, 1930)
For any formulas $A, B$ of the language of a **propositional** proof system $S$,

\[
\text{if } A \vdash_S B \text{ then } \vdash_S (A \Rightarrow B)
\]

In chapter 5 we formulated and proved the following, more general version of the Herbrand Theorem for a very simple (two logical axioms and Modus Ponens) **propositional** proof system $H1$
Deduction Theorem

Deduction Theorem
For any subset $\Gamma$ of the set of formulas $\mathcal{F}$ of $H_1$ and for any formulas $A, B \in \mathcal{F}$,

$$\Gamma, A \vdash_{H_1} B \quad \text{if and only if} \quad \Gamma \vdash_{H_1} (A \Rightarrow B)$$

In particular,

$$A \vdash_{H_1} B \quad \text{if and only if} \quad \vdash_{H_1} (A \Rightarrow B)$$

A natural question arises:
does deduction theorem hold for the predicate logic in general and for its proof system $H$ we defined here?.
Deduction Theorem

The Deduction Theorem can not be carried directly to the predicate logic, but it nevertheless holds with some modifications. Here is where the problem lays.

Fact

Given the proof system

\[ H = (\mathcal{L}(P, F, C), \mathcal{F}, LA, \mathcal{R} = \{(MP), (G), (G1), (G2)\}) \]

For any formula \( A(x) \in \mathcal{F} \),

\[ A(x) \vdash \forall x A(x) \]

but it is not always the case that

\[ \vdash (A(x) \Rightarrow \forall x A(x)) \]
Deduction Theorem

Proof
Obviously, $A(x) \vdash \forall x A(x)$ by Generalization rule (G)
Let now $A(x)$ be an atomic formula $P(x)$
By the **H Completeness Theorem**

$$\vdash (P(x) \Rightarrow \forall x P(x)) \text{ if and only if } \models (P(x) \Rightarrow \forall x P(x))$$

Consider a structure

$$M = [M, I]$$

where $M$ contains at least two elements $c$ and $d$
We define $P_I \subseteq M$ as a property that holds only for $c$, i.e.

$$P_I = \{c\}$$
Deduction Theorem

Take any assignment $s : \text{VAR} \rightarrow M$

Then $(M, s) \models P(x)$ only when $s(x) = c$ for all $x \in \text{VAR}$

$M = [M, I]$ is a counter model for $(P(x) \Rightarrow \forall x P(x))$

as we found $s$ such $(M, s) \models P(x)$ and obviously $(M, s) \not\models \forall x P(x)$

We proved that $\not\models (P(x) \Rightarrow \forall x P(x))$

By the \textbf{H Completeness Theorem} this is equivalent to

\[ \not\models (P(x) \Rightarrow \forall x P(x)) \]

and the \textbf{Deduction Theorem} fails as

\[ Px \vdash \forall x P(x) \]
Deduction Theorem

The **Fact** shows that the **problem** is with application of the generalization rule \((G)\) to the formula \(A \in \Gamma\)

To handle this we introduce, after **Mendelson(1987)** the following notion
Deduction Theorem

Definition
Let \( A \) be one of formulas in \( \Gamma \) and let

\[(P) \quad B_1, B_2, ..., B_n\]

be a proof (deduction) of \( B_n \) from \( \Gamma \), together with justification at each step. We say that the formula \( B_i \) depends upon \( A \) in the proof \( B_1, B_2, ..., B_n \) if and only if the following holds

1. \( B_i \) is \( A \) and the justification for \( B_i \) is \( B_i \in \Gamma \)
   or
2. \( B_i \) is justified as direct consequence by MP or
   (\( G \)) of some preceding formulas in the proof sequence (\( P \)), where at least one of these preceding formulas depends upon \( A \).
Deduction Theorem

Example
Here is a proof (deduction)

\[ B_1, B_2, \ldots, B_5 \]

showing that

\[ A, (\forall xA \Rightarrow C) \vdash \forall xC \]

\[ B_1 \quad A \]
Hyp
\[ B_1 \text{ depends upon } A \]
\[ B_2 \quad \forall xA \]
\[ B_1, (G) \]
\[ B_2 \text{ depends upon } A \]
\[ B_3 \quad (\forall xA \Rightarrow C) \]
Hyp
\[ B_3 \text{ depends upon } (\forall xA \Rightarrow C) \]
Deduction Theorem

\[ B_3 \quad (\forall x A \Rightarrow C) \]
Hyp
\[ B_3 \text{ depends upon } (\forall x A \Rightarrow C) \]
\[ B_4 \quad C \]
MP on \( B_2, B_3 \)
\[ B_4 \text{ depends upon } A \text{ and } (\forall x A \Rightarrow C) \]
\[ B_5 \quad \forall x C \]
\( (G) \)
\[ B_4 \text{ depends upon } A \text{ and } (\forall x A \Rightarrow C) \]
Observe that the formulas \( A, C \) may, or may not have \( x \) as a \textbf{free} variable
Deduction Theorem

DT Lemma
If $B$ does not depend upon $A$ in a proof (deduction) showing that $\Gamma, A \vdash B$, then $\Gamma \vdash B$

Proof
Let $B_1, B_2, \ldots, B_n = B$

be a proof (deduction) of $B$ from $\Gamma, A$,
in which $B$ does not depend upon $A$
We prove by induction over the length of the proof that

$\Gamma \vdash B$
Deduction Theorem

Assume that **DT Lemma** holds for all proofs of the length less than \( n \).

If \( B \in \Gamma \) or \( B \in LA \), by definition then \( \Gamma \vdash B \).

If \( B \) is a direct **consequence** of two preceding formulas, then, since \( B \) does not depend upon \( A \), neither do theses preceding formulas.

By inductive hypothesis, theses preceding formulas have a proof from \( \Gamma \) alone.

Hence so does \( B \), i.e.

\[ \Gamma \vdash B \]

Now we are ready to **formulate** and prove the **Deduction Theorem** for predicate logic.
Deduction Theorem

For any formulas $A, B$ of the language of proof system $\mathcal{H}$ the following holds:

(1) Assume that in some proof (deduction) showing that

$$\Gamma, A \vdash B$$

no application of the generalization rule $(G)$ to a formula that depends upon $A$ has as its quantified variable a free variable of the formula $A$.

Then we have that

$$\Gamma \vdash (A \Rightarrow B)$$

(2) If $\Gamma \vdash (A \Rightarrow B)$, then $\Gamma, A \vdash B$.
Deduction Theorem

Proof
The proof we present extends the proof of the Deduction Theorem for propositional logic from chapter 5.

We adopt the propositional proof to the system $H$ and add the relevant predicate cases.

For the sake of clarity and independence we write now the whole proof in all details.
Deduction Theorem

(1) Assume that

\[ \Gamma, A \vdash B \]

i.e. that we have a formal proof

\[ B_1, B_2, \ldots, B_n \]

of \( B \) from the set of formulas \( \Gamma \cup \{A\} \)

In order to prove that

\[ \Gamma \vdash (A \Rightarrow B) \]

we will prove the following a stronger statement

(S) \( \Gamma \vdash (A \Rightarrow B_i) \) for all \( B_i \) \((1 \leq i \leq n)\) in the proof of \( B \)
Deduction Theorem

Hence, in particular case, when \( i = n \), we will obtain that also

\[
\Gamma \vdash (A \Rightarrow B)
\]

The proof of the statement \((S)\) is conducted by induction on \(1 \leq i \leq n\)

**Base Step \( i = 1 \)**

When \( i = 1 \), it means that the formal proof contains only one element \( B_1 \)

By the definition of the formal proof from \( \Gamma \cup \{A\} \), we have that \( B_1 \in LA \), or \( B_1 \in \Gamma \), or \( B_1 = A \), i.e.

\[
B_1 \in LA \cup \Gamma \cup \{A\}
\]

Here we have two cases
Case 1 \( B_1 \in LA \cup \Gamma \)

Observe that the formula

\[
(B_1 \Rightarrow (A \Rightarrow B_1))
\]

is a particular case of the axiom A2 of H

By assumption \( B_1 \in LA \cup \Gamma \), hence we get the required proof of \( (A \Rightarrow B_1) \) from \( \Gamma \) by the following application of the MP rule

\[
\begin{array}{c}
B_1 \quad (B_1 \Rightarrow (A \Rightarrow B_1)) \\
\hline
(A \Rightarrow B_1)
\end{array}
\]

\( (MP) \)
Deduction Theorem

Case 2 \( B_1 = A \)

When \( B_1 = A \), then to prove

\[ \Gamma \vdash (A \Rightarrow B) \]

means to prove \( \Gamma \vdash (A \Rightarrow A) \)

But \((A \Rightarrow A) \in LA \) (axiom A1 ) of H, i.e. \( \vdash (A \Rightarrow A) \). By the monotonicity of the consequence we have that

\[ \Gamma \vdash (A \Rightarrow A) \]

The above cases conclude the proof of the Base Case \( i = 1 \)
Deduction Theorem

Inductive Step
Assume that
\[ \Gamma \vdash (A \Rightarrow B_k) \]
for all \( k < i \), we will show that using this fact we can conclude that also
\[ \Gamma \vdash (A \Rightarrow B_i) \]
Consider a formula \( B_i \) in the proof sequence
By the definition, \( B_i \in LA \cup \Gamma \cup \{A\} \)
or \( B_i \) follows by MP from certain \( B_j, B_m \) such that \( j < m < i \)
We have to consider again two cases
Deduction Theorem

Case 1

\( B_i \in LA \cup \Gamma \cup \{A\} \)

The proof of \( (A \Rightarrow B_i) \) from \( \Gamma \) in this case is obtained from the proof of the Base Step for \( i = 1 \) by replacement \( B_1 \) by \( B_i \) and will be omitted here as a straightforward repetition.

Case 2

\( B_i \) is a conclusion of MP

If \( B_i \) is a conclusion of MP, then we must have two formulas \( B_j, B_m \) in the proof sequence, such that \( j < i, m < i, j \neq m \) and

\[
\frac{B_j \mid B_m}{B_i} \quad \text{(MP)}
\]

item[[ ]] By the inductive assumption, the formulas \( B_j, B_m \) are such that

\[ \Gamma \vdash (A \Rightarrow B_j) \quad \text{and} \quad \Gamma \vdash (A \Rightarrow B_m) \]
Deduction Theorem

Moreover, by the definition of the Modus Ponens rule, the formula $B_m$ has to have a form $(B_j \Rightarrow B_i)$, i.e.

$$B_m = (B_j \Rightarrow B_i)$$

and the inductive assumption can be re-written as

$$(* ) \quad \Gamma \vdash (A \Rightarrow B_j) \quad \text{and} \quad \Gamma \vdash (A \Rightarrow (B_j \Rightarrow B_i)) \quad \text{for } j < i$$

Observe now that the formula

$$(((A \Rightarrow (B_j \Rightarrow B_i))) \Rightarrow ((A \Rightarrow B_j) \Rightarrow (A \Rightarrow B_i)))$$

is a substitution of the axiom $A_3$ of $H$ and hence

$$\vdash ((A \Rightarrow (B_j \Rightarrow B_i))) \Rightarrow ((A \Rightarrow B_j) \Rightarrow (A \Rightarrow B_i)))$$
Deduction Theorem

By the monotonicity,

\[(**) \quad \Gamma \vdash ((A \Rightarrow (B_j \Rightarrow B_i)) \Rightarrow ((A \Rightarrow B_j) \Rightarrow (A \Rightarrow B_i)))\]

Applying the rule MP to formulas (*) and (**) i.e. performing the following

\[(\text{MP}) \quad \frac{(A \Rightarrow (B_j \Rightarrow B_i)); ((A \Rightarrow (B_j \Rightarrow B_i)) \Rightarrow ((A \Rightarrow B_j) \Rightarrow (A \Rightarrow B_i)))}{((A \Rightarrow B_j) \Rightarrow (A \Rightarrow B_i))}\]

we get that also

\[\Gamma \vdash ((A \Rightarrow B_j) \Rightarrow (A \Rightarrow B_i))\]
Deduction Theorem

Applying again the rule $\text{MP}$ to formulas (\*) and the above

\[ \Gamma \vdash ((A \Rightarrow B_j) \Rightarrow (A \Rightarrow B_i)) \]

i.e. performing the following

\[
\begin{align*}
(MP) \quad & (A \Rightarrow B_j) ; ((A \Rightarrow B_j) \Rightarrow (A \Rightarrow B_i)) \\
& \quad \overline{\quad (A \Rightarrow B_i)}
\end{align*}
\]

we get that

\[ \Gamma \vdash (A \Rightarrow B_i) \]
Deduction Theorem

Finally, suppose that there is some $j < i$ such that

$$B_i \text{ is } \forall x B_j$$

By inductive assumption

$$\Gamma \vdash (A \Rightarrow B_j)$$

and either

(i) $B_j$ does not depend upon $A$ or
(ii) $x$ is not free variable in $A$

We want to prove

$$\Gamma \vdash B_i$$

We have these two cases (i) and (ii) to consider.
Deduction Theorem

Case (i)

\[ \Gamma \vdash (A \Rightarrow B_j) \]

and \( B_j \) does not depend upon \( A \)

Then by DT Lemma we have that \( \Gamma \vdash B_j \)

and, consequently, by the generalization rule \((G)\)

\[ \Gamma \vdash \forall x B_j \]

Thus we proved

\[ \Gamma \vdash B_i \]
Deduction Theorem

Now, from just proved

$$\Gamma \vdash B_i$$

and axiom $A2$ of $H$

$$\vdash (B_i \Rightarrow (A \Rightarrow B_i))$$

and monotonicity

$$\Gamma \vdash (B_i \Rightarrow (A \Rightarrow B_i))$$

and $MP$ applied to them we get

$$\Gamma \vdash (A \Rightarrow B_i)$$
Deduction Theorem

Case (ii)

\( \Gamma \vdash (A \Rightarrow B_j) \) and \( x \) is not free variable in \( A \)

We know that \( \models (\forall x (A \Rightarrow B_j) \Rightarrow (A \Rightarrow \forall x B_j)) \)

hence the Completeness Theorem we get

\( \vdash (\forall x (A \Rightarrow B_j) \Rightarrow (A \Rightarrow \forall x B_j)) \)

Since \( \Gamma \vdash (A \Rightarrow B_j) \) by inductive assumption, we get by the generalization rule \((G)\) and nmonotonicity

\[ \Gamma \vdash \forall x (A \Rightarrow B_j) \]

By \( \text{MP} \) applied to the above

\[ \Gamma \vdash (A \Rightarrow \forall x B_j) \]

That is we got

\[ \Gamma \vdash A \Rightarrow B_i \)
**Deduction Theorem**

Since $\Gamma \vdash (A \Rightarrow B_j)$ by inductive assumption, we get by the generalization rule $(G)$,

$$\Gamma \vdash \forall x(A \Rightarrow B_j)$$

and so, by **MP**

$$\Gamma \vdash A \Rightarrow \forall xB_j$$

That is we proved

$$\Gamma \vdash (A \Rightarrow B_i)$$

This completes the induction and the **proves** part (1) of the **Deduction Theorem**
Deduction Theorem part (2)

The proof of the implication

\[ \text{if } \Gamma \vdash (A \Rightarrow B) \text{ then } \Gamma, A \vdash B \]

is straightforward

Assume \( \Gamma \vdash (A \Rightarrow B) \). By monotonicity we have also that

\[ \Gamma, A \vdash (A \Rightarrow B) \]

Obviously, \( \Gamma, A \vdash A \). Applying MP to the above, we get the proof of \( B \) from \( \{\Gamma, A\} \) i.e. we have proved that

\[ \Gamma, A \vdash B \]

This ends the proof of the Deduction Theorem for \( H \)
PART 5: Some other Axiomatizations
Hilbert and Ackermann (1928)

We present here some of most known, and historically important axiomatizations of classical predicate logic, i.e. the following Hilbert style proof systems

1. Hilbert and Ackermann (1928)
This formalization is based on D. Hilbert and W. Ackermann book *Grundzügen der Theoretischen Logik* (Principles of Theoretical Logic), Springer - Verlag, 1928

The book grew from the courses on logic and foundations of mathematics Hilbert gave in years 1917-1922
He received help in writeup from Barnays and the material was put into the book by Ackermann and Hilbert
Hilbert and Ackermann

The Hilbert and Ackermann book was conceived as an introduction to mathematical logic and was followed by another two volumes book written by D. Hilbert and P. Bernays, *Grundzüge der Mathematik I, II*, Springer-Verlag, 1934, 1939.

Hilbert and Ackermann formulated and asked a question of the completeness for their deductive (proof) system.

It was answered affirmatively by Kurt Gödel in 1929 with proof of his Completeness Theorem.
Hilbert and Ackermann

We define the Hilbert and Ackermann proof system $\text{HA}$ following a pattern established for the $\text{H}$ system. The original language used by Hilbert and Ackermann contained only negation $\neg$ and disjunction $\cup$ and so do we. We define

$$\text{HA} = (\mathcal{L}_{\neg,\cup}(P,F,C), F, LA, \mathcal{R})$$

where

$$\mathcal{R} = \{(\text{MP}), (\text{SB}), (G1), (G2)\}$$

The set $\text{LA}$ of logical axioms is as follows.
Hilbert and Ackermann (1928)

Propositional Axioms

A1 \((\neg(A \cup A) \cup A)\)
A2 \((\neg A \cup (A \cup B))\)
A3 \((\neg(A \cup B) \cup (B \cup A))\)
A4 \((\neg(\neg B \cup C) \cup (\neg (A \cup B) \cup (A \cup C)))\)

for any \(A, B, C, \in \mathcal{F}\)

Quantifiers Axioms

Q1 \((\neg \forall xA(x) \cup A(x))\)
Q2 \((\neg A(x) \cup \exists xA(x))\)
Q3 \((\neg A(x) \cup \exists xA(x))\),

for any \(A(x) \in \mathcal{F}\)
Rules of Inference $\mathcal{R}$

(MP) is the Modus Ponens rule. It has, in the language $\mathcal{L}_{\{\neg, \cup\}}$, a form

$$
(\text{MP}) \quad \frac{A ; (\neg A \cup B)}{B}
$$

(SB) is a substitution rule

$$
(\text{SB}) \quad \frac{A(x_1, x_2, \ldots x_n)}{A(t_1, t_2, \ldots t_n)}
$$

where $A(x_1, x_2, \ldots x_n) \in \mathcal{F}$ and $t_1, t_2, \ldots t_n \in \mathcal{T}$.
Hilbert and Ackermann

\((G1), (G2)\) are quantifiers generalization rules

\[(G1) \quad \frac{(\neg B \cup A(x))}{(\neg B \cup \forall xA(x))}\]

\[(G2) \quad \frac{(\neg A(x) \cup B)}{(\neg \exists xA(x) \cup B)}\]

where \(A(x), B \in \mathcal{F}\) and \(B\) is such that \(x\) is not free in \(B\)
Hilbert and Ackermann

The **HA** system is usually written now with the use of implication, i.e. is based on a language

\[ \mathcal{L} = \mathcal{L}_{\{\neg, \Rightarrow\}}(P, F, C) \]

We define

\[ \text{HAI} = (\mathcal{L}_{\{\neg, \Rightarrow\}}(P, F, C), F, LA, R) \]

for

\[ R = \{(MP), (SB), (G1), (G2)\} \]

and the set **LA** of logical axioms as follows
Hilbert and Ackermann

Propositional Axioms

A1 \(((A \cup A) \Rightarrow A)\)
A2 \((A \Rightarrow (A \cup B))\)
A3 \(((A \cup B) \Rightarrow (B \cup A))\)
A4 \(((\neg B \cup C) \Rightarrow ((A \cup B) \Rightarrow (A \cup C)))\)

for any \(A, B, C, \in F\)

Quantifiers Axioms

Q1 \((\forall x A(x) \Rightarrow A(x))\)
Q2 \((A(x) \Rightarrow \exists x A(x))\)

for any \(A(x) \in F\)
Hilbert and Ackermann

Rules of Inference $\mathcal{R}$

(MP) is Modus Ponens rule

\[
\text{(MP)} \quad \frac{A; (A \Rightarrow B)}{B}
\]

for any formulas $A, B \in \mathcal{F}$

(SB) is a substitution rule

\[
\text{(SB)} \quad \frac{A(x_1, x_2, \ldots x_n)}{A(t_1, t_2, \ldots t_n)}
\]

where $A(x_1, x_2, \ldots x_n) \in \mathcal{F}$ and $t_1, t_2, \ldots t_n \in \mathcal{T}$
Hilbert and Ackermann

(G1), (G2) are quantifiers generalization rules.

\[
\begin{align*}
(G1) & \quad \frac{(B \Rightarrow A(x))}{(B \Rightarrow \forall xA(x))} \\
(G2) & \quad \frac{(A(x) \Rightarrow B)}{(\exists xA(x) \Rightarrow B)}
\end{align*}
\]

where \(A(x), B \in \mathcal{F}\) and \(B\) is such that \(x\) is not free in \(B\).

The form of the quantifiers axioms Q1, Q2, and quantifiers generalization rule (G2) is due to Bernays.
Mendelson (1987)

Here is the **first order** logic proof system as introduced in Elliott Mendelson’s book *Introduction to Mathematical Logic* (1987). Hence the name **HM**

**HM** is a generalization to the **predicate** language of the proof system $H_2$ for **propositional** logic defined after Mendelson’s book and studied in Chapter 5

$$HM = (\mathcal{L}_{\neg, \cup}(P, F, C), \mathcal{F}, LA, R = \{(MP), (G)\})$$

The **HM** components are as follows
Propositional Axioms

A1 \((A \Rightarrow (B \Rightarrow A))\)

A2 \(((A \Rightarrow (B \Rightarrow C)) \Rightarrow ((A \Rightarrow B) \Rightarrow (A \Rightarrow C)))\)

A3 \(((\neg B \Rightarrow \neg A) \Rightarrow ((\neg B \Rightarrow A) \Rightarrow B)))\)

for any \(A, B, C, \in \mathcal{F}\)
Quantifiers Axioms

Q1  \((\forall x A(x) \Rightarrow A(t))\)

where \(t\) is a term, \(A(t)\) is a result of substitution of \(t\) for all free occurrences of \(x\) in \(A(x)\) and \(t\) is free for \(x\) in \(A(x)\), i.e. no occurrence of a variable in \(t\) becomes a bound occurrence in \(A(t)\)

Q2  \((\forall x (B \Rightarrow A(x))) \Rightarrow (B \Rightarrow \forall x A(x)))\)

where \(A(x), B \in \mathcal{F}\) and \(B\) is such that \(x\) is not free in \(B\)
Rules of Inference $\mathcal{R}$

$(MP)$ is the Modus Ponens rule

\[
\frac{A ; (A \Rightarrow B)}{B}
\]

for any formulas $A, B \in \mathcal{F}$

$(G)$ is the generalization rule

\[
\frac{A(x)}{\forall x A(x)}
\]

where $A(x) \in \mathcal{F}$ and $x \in \text{VAR}$
Rasiowa and Sikorski (1950)

Rasiowa, Sikorski (1950)

Helena Rasiowa and Roman Sikorski are the authors of the first algebraic proof of the Gödel completeness theorem ever given in 1950.

Other algebraic proofs were later given by Rieger, Beth, Łos in 1951, and Scott in 1954.
Here is Rasiowa- Sikorski original formalization

\[ RS = (\mathcal{L}_{\neg, \cap, \cup, \Rightarrow}(P, F, C), \mathcal{F}, LA, R) \]

for

\[ R = \{(MP), (SB), (Q1), (Q2), (Q3), (Q4)\} \]

The logical axioms \( LA \) are as follows

**Propositional Axioms**

A1 \( ((A \Rightarrow B) \Rightarrow ((B \Rightarrow C) \Rightarrow (A \Rightarrow C))) \)
A2 \( (A \Rightarrow (A \cup B)) \)
A3 \( (B \Rightarrow (A \cup B)) \)
for any $A, B, C \in \mathcal{F}$
Rasiowa and Sikorski

Rules of Inference $\mathcal{R}$

*(MP)* is Modus Ponens rule

\[
\begin{align*}
(MP) \quad & A ; (A \Rightarrow B) \\
& \quad \rightarrow B
\end{align*}
\]

for any formulas $A, B \in \mathcal{F}$

*(SB)* is a substitution rule

\[
\begin{align*}
(SB) \quad & A(x_1, x_2, \ldots x_n) \\
& \quad \rightarrow A(t_1, t_2, \ldots t_n)
\end{align*}
\]

where $A(x_1, x_2, \ldots x_n) \in \mathcal{F}$ and $t_1, t_2, \ldots t_n \in \mathcal{T}$
(G1), (G2) are the following quantifiers introduction rules

\[
\begin{align*}
(G1) & \quad \frac{B \Rightarrow A(x)}{B \Rightarrow \forall x A(x)} \\
(G2) & \quad \frac{A(x) \Rightarrow B}{\exists x A(x) \Rightarrow B}
\end{align*}
\]

where \( A(x), B \in \mathcal{F} \) and \( B \) is such that \( x \) is not free in \( B \)
(G3), (G3) are the following quantifiers elimination rules.

\[
(G3) \quad \frac{(B \Rightarrow \forall x A(x))}{(B \Rightarrow A(x))}
\]

\[
(G4) \quad \frac{\exists x (A(x) \Rightarrow B)}{(A(x) \Rightarrow B)}
\]

where \( A(x), B \in \mathcal{F} \) and \( B \) is such that \( x \) is not free in \( B \).
The **algebraic logic** starts from purely **logical** considerations, **abstracts** from them, places them into a **general algebraic** context, and makes use of **other branches** of mathematics such as **topology**, **set theory**, and **functional analysis**.

For **example**, Rasiowa and Sikorski **algebraic generalization** of the **completeness theorem** for classical **predicate logic** is the following.
Rasiowa and Sikorski

Algebraic Completeness Theorem (Rasiowa, Sikorski 1950)

For every formula $A$ of the classical predicate calculus $RS$ the following conditions are equivalent

i. $A$ is derivable in $RS$;

ii. $A$ is valid in every realization of $L$;

iii. $A$ is valid in every realization of $L$ in any complete Boolean algebra;

iv. $A$ is valid in every realization of $L$ in the field $B(X)$ of all subsets of any set $X \neq \emptyset$;
v  \( A \) is valid in every semantic realization of \( L \) in any enumerable set;

vi  there exists a non-degenerate Boolean algebra \( A \) and an infinite set \( J \) such that \( A \) is valid in every realization of \( L \) in \( J \) and \( A \);

vii  \( A_R(I) = V \) for the canonical realization \( R \) of \( L \) in the Lindenbaum-Tarski algebra \( LT \) of \( RS \) and the identity valuation \( I \);

viii  \( A \) is a predicate tautology.