cse541
LOGIC for Computer Science

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LECTURE 9c

Chapter 9
Hilbert Proof Systems
Completeness of Classical Predicate Logic

PART 4: **Deduction Theorem**

Deduction Theorem

In mathematical arguments, one often assumes a statement A on the assumption (hypothesis) of some other statement B and then concludes that we have proved the implication

"if A, then B"

This reasoning is justified by the following theorem, called a **Deduction Theorem**

It was first formulated and **proved** for a certain Hilbert proof system S for the classical **propositional** logic by Herbrand in 1930 in a form stated as follows

Deduction Theorem

Deduction Theorem (Herbrand,1930)

For any formulas A, B of the language of a **propositional** proof system S,

$$\text{if} \quad A \vdash_S B \quad \text{then} \quad \vdash_S (A \Rightarrow B)$$

In chapter 5 we formulated and proved the following, more genera l version of the Herbrand Theorem for a very simple (two logical axioms and Modus Ponens) propositional proof system $H1$

Deduction Theorem

Deduction Theorem

For any subset Γ of the set of formulas $\mathcal{F}$ of H_1 and for any formulas $A, B \in \mathcal{F}$,

$$\Gamma,\ A \vdash_{H_1} B \quad \text{if and only if} \quad \Gamma \vdash_{H_1} (A \Rightarrow B)$$

In particular,

$$A \vdash_{H_1} B \quad \text{if and only if} \quad \vdash_{H_1} (A \Rightarrow B)$$

A natural question arises:

does **deduction theorem** hold for the **predicate** logic in general and for its proof system **H** we defined here?.

Deduction Theorem

The **Deduction Theorem** can not be carried directly to the **predicate** logic, but it nevertheless **holds** with some modifications. Here is where the problem lays.

Fact

Given the proof system

$$\mathbf{H} = (\mathcal{L}(\mathbf{P}, \mathbf{F}, \mathbf{C}),\ \mathcal{F},\ LA,\ \mathcal{R} = \{(MP),\ (G), (G1),\ (G2)\})$$

For any formula $A(x) \in \mathcal{F}$,

$$A(x)\ \vdash\ \forall x A(x)$$

but it is **not always** the case that

$$\vdash\ (A(x) \Rightarrow \forall x A(x))$$

Deduction Theorem

Proof

Obviously, $A(x) \vdash \forall x A(x)$ by Generalization rule (G)

Let now $A(x)$ be an atomic formula $P(x)$

By the **H Completeness Theorem**

$$\vdash (P(x) \Rightarrow \forall x P(x)) \quad \text{if and only if} \quad \models (P(x) \Rightarrow \forall x P(x))$$

Consider a structure

$$\mathcal{M} = [M, I]$$

where M contains at least two elements c and d

We define $P_I \subseteq M$ as a property that holds only for c, i.e.

$$P_I = \{c\}$$

Deduction Theorem

Take any assignment $s : VAR \longrightarrow M$

Then $(\mathcal{M}, s) \models P(x)$ only when $s(x) = c$ for all $x \in VAR$

$$\mathcal{M} = [M, I] \text{ is a \textbf{counter model} for } (P(x) \Rightarrow \forall x P(x))$$

as we found s such $(\mathcal{M}, s) \models P(x)$ and obviously $(\mathcal{M}, s) \not\models \forall x P(x)$

We proved that $\not\models (P(x) \Rightarrow \forall x P(x))$

By the **H Completeness Theorem** this is equivalent to

$$\nvdash (P(x) \Rightarrow \forall x P(x))$$

and the **Deduction Theorem** fails as

$$Px \vdash \forall x P(x)$$

Deduction Theorem

The **Fact** shows that the problem is with application of the generalization rule (G) to the formula $A \in \Gamma$

To handle this we introduce, after Mendelson(1987) the following notion

Definition

Let A be one of formulas in Γ and let

$$(P) \quad B_1, B_2, ..., B_n$$

be a proof (deduction) of B_n from Γ, together with justification at each step. We say that the formula

B_i **depends upon** A in the proof $B_1, B_2, ..., B_n$

if and only if the following holds

(1) B_i is A and the justification for B_i is $B_i \in \Gamma$

or

(2) B_i is justified as direct consequence by MP

or

(G) of some preceding formulas in the proof sequence (P), where at least one of these preceding formulas **depends upon** A

Deduction Theorem

Example

Here is a proof (deduction)

$$B_1,\ B_2,\ \ldots,\ B_5$$

showing that

$$A,\ (\forall x A \Rightarrow C)\ \vdash\ \forall x C$$

B_1 A

Hyp

B_1 depends upon A

B_2 $\forall x A$

B_1, (G)

B_2 depends upon A

B_3 $(\forall x A \Rightarrow C)$

Hyp

B_3 depends upon $(\forall x A \Rightarrow C)$

Deduction Theorem

B_3 $(\forall x A \Rightarrow C)$

Hyp

B_3 depends upon $(\forall x A \Rightarrow C)$

B_4 C

MP on B_2, B_3

B_4 depends upon A and $(\forall x A \Rightarrow C)$

B_5 $\forall x C$

(G)

B_4 depends upon A and $(\forall x A \Rightarrow C)$

Observe that the formulas A, C may, or may not have x as a free variable

Deduction Theorem

DT Lemma

If B **does not** depend upon A in a proof (deduction) showing that $\Gamma, A \vdash B$, then $\Gamma \vdash B$

Proof

Let

$$B_1, B_2, \ldots, B_n = B$$

be a proof (deduction) of B from Γ, A,

in which B **does not** depend upon A

We prove by induction over the length of the proof that

$$\Gamma \vdash B$$

Deduction Theorem

Assume that **DT Lemma** holds for all proofs of the length less than n

If $B \in \Gamma$ or $B \in LA$, by definition then $\Gamma \vdash B$

If B is a direct **consequence** of two preceding formulas, then, since B **does not** depend upon A, **neither do** theses preceding formulas

By inductive hypothesis, theses preceding formulas have a proof from Γ alone

Hence **so does** B, i.e.

$$\Gamma \vdash B$$

Now we are ready to formulate and prove the **Deduction Theorem** for predicate logic

Deduction Theorem

For any formulas A, B of the language of proof system **H** the following holds

(1) Assume that **in some** proof (deduction) showing that

$$\Gamma, A \vdash B$$

no application of the generalization rule (G) **to** a formula that **depends** upon A has as its quantified variable a free variable of the formula A

Then we have that

$$\Gamma \vdash (A \Rightarrow B)$$

(2) If $\Gamma \vdash (A \Rightarrow B)$, then $\Gamma, A \vdash B$

Deduction Theorem

Proof

The proof we present extends the proof of the **Deduction Theorem** for propositional logic from chapter 5

We **adopt** the propositional proof to the system **H** and add the relevant predicate cases

For the sake of clarity and **independence** we write now the whole proof in all **details**

Deduction Theorem

(1) Assume that

$$\Gamma,\ A\ \vdash B$$

i.e. that we have a formal proof

$$B_1,\ B_2,\ \ldots,B_n$$

of B from the set of formulas $\Gamma \cup \{A\}$

In order to prove that

$$\Gamma \vdash (A \Rightarrow B)$$

we will prove the following a **stronger** statement

(**S**) $\Gamma \vdash (A \Rightarrow B_i)$ for all B_i $(1 \leq i \leq n)$ in the proof of B

Deduction Theorem

Hence, in particular case, when $i = n$, we will obtain that also

$$\Gamma \vdash (A \Rightarrow B)$$

The proof of the statement (**S**) is conducted by induction on $1 \leq i \leq n$

Base Step $i = 1$

When $i = 1$, it means that the formal proof contains only one element B_1

By the definition of the formal proof from $\Gamma \cup \{A\}$, we have that $B_1 \in LA$, or $B_1 \in \Gamma$, or $B_1 = A$, i.e.

$$B_1 \in LA \cup \Gamma \cup \{A\}$$

Here we have two cases

Deduction Theorem

Case 1 $B_1 \in LA \cup \Gamma$

Observe that the formula

$$(B_1 \Rightarrow (A \Rightarrow B_1))$$

is a particular case of the axiom A2 of **H**

By assumption $B_1 \in LA \cup \Gamma$, hence we get the required proof of $(A \Rightarrow B_1)$ from Γ by the following application of the MP rule

$$(MP)\ \frac{B_1 \ ; \ (B_1 \Rightarrow (A \Rightarrow B_1))}{(A \Rightarrow B_1)}$$

Deduction Theorem

Case 2 $B_1 = A$

When $B_1 = A$, then to prove

$$\Gamma \vdash (A \Rightarrow B)$$

means to prove $\Gamma \vdash (A \Rightarrow A)$

But $(A \Rightarrow A) \in LA$ (axiom A1) of **H**, i.e. $\vdash (A \Rightarrow A)$. By the monotonicity of the consequence we have that

$$\Gamma \vdash (A \Rightarrow A)$$

The above cases conclude the proof of the Base Case $i = 1$

Deduction Theorem

Inductive Step

Assume that

$$\Gamma \vdash (A \Rightarrow B_k)$$

for all $k < i$, we will show that using this fact we can conclude that also

$$\Gamma \vdash (A \Rightarrow B_i)$$

Consider a formula B_i in the proof sequence

By the definition, $B_i \in LA \cup \Gamma \cup \{A\}$

or B_i follows by MP from certain B_j, B_m such that $j < m < i$

We have to consider again two cases

Case 1

$B_i \in LA \cup \Gamma \cup \{A\}$

The proof of $(A \Rightarrow B_i)$ from Γ in this case is obtained from the proof of the Base Step for $i = 1$ by replacement B_1 by B_i and will be omitted here as a straightforward repetition

Case 2

B_i is a conclusion of MP

If B_i is a conclusion of MP , then we must have two formulas B_j, B_m in the proof sequence, such that $j < i, m < i, \; j \neq m$ and

$$(MP)\ \frac{B_j \ ; \ B_m}{B_i}$$

item[[] By the inductive assumption, the formulas B_j, B_m are such that

$$\Gamma \vdash (A \Rightarrow B_j) \quad \text{and} \quad \Gamma \vdash (A \Rightarrow B_m)$$

Deduction Theorem

Moreover, by the definition of the Modus Ponens rule, the formula B_m has to have a form $(B_j \Rightarrow B_i)$, i.e.

$$B_m = (B_j \Rightarrow B_i)$$

and the the inductive assumption can be re-written as

$$(*)\quad \Gamma \vdash (A \Rightarrow B_j) \quad \text{and} \quad \Gamma \vdash (A \Rightarrow (B_j \Rightarrow B_i)) \quad \text{for } j < i$$

Observe now that the formula

$$((A \Rightarrow (B_j \Rightarrow B_i)) \Rightarrow ((A \Rightarrow B_j) \Rightarrow (A \Rightarrow B_i)))$$

is a substitution of the axiom A3 of **H** and hence

$$\vdash \ ((A \Rightarrow (B_j \Rightarrow B_i)) \Rightarrow ((A \Rightarrow B_j) \Rightarrow (A \Rightarrow B_i)))$$

Deduction Theorem

By the monotonicity,

$$(**)\quad \Gamma \vdash ((A \Rightarrow (B_j \Rightarrow B_i)) \Rightarrow ((A \Rightarrow B_j) \Rightarrow (A \Rightarrow B_i)))$$

Applying the rule MP to formulas $(*)$ and $(**)$ i.e. performing the following

$$(MP)\ \frac{(A \Rightarrow (B_j \Rightarrow B_i));\ ((A \Rightarrow (B_j \Rightarrow B_i)) \Rightarrow ((A \Rightarrow B_j) \Rightarrow (A \Rightarrow B_i)))}{((A \Rightarrow B_j) \Rightarrow (A \Rightarrow B_i))}$$

we get that also

$$\Gamma \vdash ((A \Rightarrow B_j) \Rightarrow (A \Rightarrow B_i))$$

Deduction Theorem

Applying again the rule MP to formulas (∗) and the above

$$\Gamma \vdash ((A \Rightarrow B_j) \Rightarrow (A \Rightarrow B_i))$$

i.e. performing the following

$$(MP)\ \frac{(A \Rightarrow B_j)\ ;\ ((A \Rightarrow B_j) \Rightarrow (A \Rightarrow B_i))}{(A \Rightarrow B_i)}$$

we get that

$$\Gamma \vdash (A \Rightarrow B_i)$$

Deduction Theorem

Finally, suppose that there is some $j < i$ such that

$$B_i \text{ is } \forall x B_j$$

By inductive assumption

$$\Gamma \vdash (A \Rightarrow B_j)$$

and either

(i) B_j **does not** depend upon A or

(ii) x is **not free** variable in A

We want to prove

$$\Gamma \vdash B_i$$

We have theses two cases **(i)** and **(ii)** to consider.

Deduction Theorem

Case **(i)**

$$\Gamma \vdash (A \Rightarrow B_j)$$

and B_j **does not** depend upon A

Then by **DT Lemma** we have that $\Gamma \vdash B_j$

and, consequently, by the generalization rule (G)

$$\Gamma \vdash \forall x B_j$$

Thus we proved

$$\Gamma \vdash B_i$$

Deduction Theorem

Now, from just proved

$$\Gamma \vdash B_i$$

and axiom A2 of **H**

$$\vdash (B_i \Rightarrow (A \Rightarrow B_i))$$

and monotonicity

$$\Gamma \vdash (B_i \Rightarrow (A \Rightarrow B_i))$$

and MP applied to them we get

$$\Gamma \vdash (A \Rightarrow B_i)$$

Deduction Theorem

Case **(ii)**

$\Gamma \vdash (A \Rightarrow B_j)$ and x **is not** free variable in A

We know that $\models (\forall x(A \Rightarrow B_j) \Rightarrow (A \Rightarrow \forall x B_j))$

hence the **Completeness Theorem** we get

$\vdash (\forall x(A \Rightarrow B_j) \Rightarrow (A \Rightarrow \forall x B_j))$

Since $\Gamma \vdash (A \Rightarrow B_j)$ by inductive assumption, we get by the generalization rule (G) and nmonotonicity

$$\Gamma \vdash \forall x(A \Rightarrow B_j)$$

By MP applied to the above

$$\Gamma \vdash (A \Rightarrow \forall x B_j)$$

That is we got

$$\Gamma \vdash A \Rightarrow B_i)$$

Deduction Theorem

Since $\Gamma \vdash (A \Rightarrow B_j)$ by inductive assumption, we get by the generalization rule (G),

$$\Gamma \vdash \forall x(A \Rightarrow B_j)$$

and so, by MP

$$\Gamma \vdash A \Rightarrow \forall x B_j)$$

That is we proved

$$\Gamma \vdash (A \Rightarrow B_i)$$

This completes the induction and the **proves** part (1) of the **Deduction Theorem**

Deduction Theorem part (2)

The **proof** of the implication

$$\text{if } \Gamma \vdash (A \Rightarrow B) \text{ then } \Gamma, A \vdash B$$

is straightforward

Assume $\Gamma \vdash (A \Rightarrow B)$. By monotonicity we have also that

$$\Gamma, A \vdash (A \Rightarrow B)$$

Obviously, $\Gamma, A \vdash A$. Applying MP to the above, we get the proof of B from $\{\Gamma, A\}$ i.e. we have proved that

$$\Gamma, A \vdash B$$

This **ends** the proof of the **Deduction Theorem** for **H**

PART 5: Some other Axiomatizations

Hilbert and Ackermann (1928)

We present here some of most known, and historically important axiomatizations of classical **predicate** logic, i.e. the following Hilbert style proof systems

1. Hilbert and Ackermann (1928)

This formalization is based on D. Hilbert and W. Ackermann book *Grundzügen der Theoretischen Logik* (Principles of Theoretical Logic), Springer - Verlag, 1928

The book grew from the **courses** on logic and foundations of mathematics Hilbert gave in years 1917-1922

He received **help** in writeup from Barnays and the material was **put into** the book by Ackermann and Hilbert

Hilbert and Ackermann

The Hilbert and Ackermann book was conceived as an introduction to mathematical logic and was followed by another two volumes book written by D. Hilbert and P. Bernays, *Grundzügen der Mathematik I, II* , Springer -Verlag, 1934, 1939

Hilbert and Ackermann **formulated** and **asked** a question of the **completeness** for their deductive (proof) system

It was **answered** affirmatively by Kurt Gödel in 1929 with proof of his **Completeness Theorem**

Hilbert and Ackermann

We define the Hilbert and Ackermann proof system **HA** following a pattern established for the **H** system

The original **language** used by Hilbert and Ackermann contained **only** negation $\neg$ and disjunction $\cup$ and so do we

We **define**

$$\mathbf{HA} = (\mathcal{L}_{\{\neg,\cup\}}(\mathbf{P},\mathbf{F},\mathbf{C}),\ \mathcal{F},\ LA,\ \mathcal{R})$$

where

$$\mathcal{R} = \{(MP),\ (SB),\ (G1),\ (G2)\}$$

The set LA of logical axioms is as follows

Hilbert and Ackermann (1928)

Propositional Axioms

A1 $(\neg(A \cup A) \cup A)$

A2 $(\neg A \cup (A \cup B))$

A3 $(\neg(A \cup B) \cup (B \cup A))$

A4 $(\neg(\neg B \cup C) \cup (\neg(A \cup B) \cup (A \cup C)))$

for any $A, B, C, \in \mathcal{F}$

Quantifiers Axioms

Q1 $(\neg \forall x A(x) \cup A(x))$

Q2 $(\neg A(x) \cup \exists x A(x))$

Q3 $(\neg A(x) \cup \exists x A(x))$,

for any $A(x) \in \mathcal{F}$

Rules of Inference $\mathcal{R}$

(MP) is the Modus Ponens rule. It has, in the language $\mathcal{L}_{\{\neg, \cup\}}$, a form

$$(MP)\ \frac{A\ ;\ \ (\neg A \cup B)}{B}$$

(SB) is a **substitution rule**

$$(SB)\ \frac{A(x_1, x_2, \ \dots \ x_n)}{A(t_1, t_2, \ \dots \ t_n)}$$

where $A(x_1, x_2, \ \dots \ x_n) \in \mathcal{F}$ and $t_1, t_2, \ \dots \ t_n \in \mathbf{T}$

.

Hilbert and Ackermann

$(G1), (G2)$ are **quantifiers generalization rules**

$$(G1) \quad \frac{(\neg B \cup A(x))}{(\neg B \cup \forall x A(x))}$$

$$(G2) \quad \frac{(\neg A(x) \cup B)}{(\neg \exists x A(x) \cup B)}$$

where $A(x), B \in \mathcal{F}$ and B is such that x is **not free** in B

Hilbert and Ackermann

The **HA** system is usually written now with the use of implication, i.e. is based on a language

$$\mathcal{L} = \mathcal{L}_{\{\neg, \Rightarrow\}}(\mathbf{P}, \mathbf{F}, \mathbf{C})$$

We define

$$\mathbf{HAI} = (\mathcal{L}_{\{\neg, \Rightarrow\}}(\mathbf{P}, \mathbf{F}, \mathbf{C}), \mathcal{F}, LA, \mathcal{R})$$

for

$$\mathcal{R} = \{(MP), (SB), (G1), (G2)\}$$

and the set LA of logical axioms as follows

Hilbert and Ackermann

Propositional Axioms

A1 $((A \cup A) \Rightarrow A)$

A2 $(A \Rightarrow (A \cup B))$

A3 $((A \cup B) \Rightarrow (B \cup A))$

A4 $((\neg B \cup C) \Rightarrow ((A \cup B) \Rightarrow (A \cup C)))$

for any

$A, B, C, \in \mathcal{F}$

Quantifiers Axioms

Q1 $(\forall x A(x) \Rightarrow A(x))$

Q2 $(A(x) \Rightarrow \exists x A(x))$

for any $A(x) \in \mathcal{F}$

Hilbert and Ackermann

Rules of Inference $\mathcal{R}$

(MP) is Modus Ponens rule

$$(MP)\quad \frac{A \; ; \; (A \Rightarrow B)}{B}$$

for any formulas $A, B \in \mathcal{F}$

(SB) is a **substitution rule**

$$(SB)\quad \frac{A(x_1, x_2, \ \ldots \ x_n)}{A(t_1, t_2, \ \ldots \ t_n)}$$

where $A(x_1, x_2, \ \ldots \ x_n) \in \mathcal{F}$ and $t_1, t_2, \ \ldots \ t_n \in \mathbf{T}$

Hilbert and Ackermann

$(G1), (G2)$ are **quantifiers generalization rules**.

$$(G1) \quad \frac{(B \Rightarrow A(x))}{(B \Rightarrow \forall x A(x))}$$

$$(G2) \quad \frac{(A(x) \Rightarrow B)}{(\exists x A(x) \Rightarrow B)}$$

where $A(x), B \in \mathcal{F}$ and B is such that x is **not free** in B

The form of the **quantifiers** axioms Q1, Q2, and **quantifiers generalization** rule (G2) is due to Bernays

Mendelson (1987)

Here is the **first order** logic proof system as introduced in Elliott Mendelson's book *Introduction to Mathematical Logic* (1987). Hence the name **HM**

HM is a generalization to the **predicate** language of the proof system H_2 for **propositional** logic defined after Mendelson's book and studied in Chapter 5

$$\mathbf{HM} = (\mathcal{L}_{\{\neg,\cup\}}(\mathbf{P},\mathbf{F},\mathbf{C}),\ \mathcal{F},\ LA,\ \mathcal{R} = \{(MP),\ (G)\})$$

The **HM** components are as follows

Mendelson (1987)

Propositional Axioms

A1 $(A \Rightarrow (B \Rightarrow A))$

A2 $((A \Rightarrow (B \Rightarrow C)) \Rightarrow ((A \Rightarrow B) \Rightarrow (A \Rightarrow C)))$

A3 $((\neg B \Rightarrow \neg A) \Rightarrow ((\neg B \Rightarrow A) \Rightarrow B)))$

for any $A, B, C, \in \mathcal{F}$

Mendelson

Quantifiers Axioms

Q1 $(\forall x A(x) \Rightarrow A(t))$

where t is a term, A(t) is a result of **substitution** of t for all **free** occurrences of x in $A(x)$ and t is **free for** x in A(x), i.e. **no** occurrence of a variable in t becomes a **bound** occurrence in A(t)

Q2 $(\forall x(B \Rightarrow A(x)) \Rightarrow (B \Rightarrow \forall x A(x)))$

where $A(x), B \in \mathcal{F}$ and B is such that x is **not free** in B

Mendelson

Rules of Inference $\mathcal{R}$

(MP) is the Modus Ponens rule

$$(MP) \quad \frac{A \; ; \; (A \Rightarrow B)}{B}$$

for any formulas $A, B \in \mathcal{F}$

(G) is the **generalization rule**

$$(G) \quad \frac{A(x)}{\forall x A(x)}$$

where $A(x) \in \mathcal{F}$ and $x \in VAR$

Rasiowa, Sikorski (1950)

Helena Rasiowa and Roman Sikorski are the authors of the first **algebraic proof** of the Gödel **completeness theorem** ever given in 1950

Other **algebraic** proofs were later given by Rieger, Beth, Łos in 1951 , and Scott in 1954

Rasiowa and Sikorski (1950)

Here is Rasiowa- Sikorski original formalization

$$RS = (\mathcal{L}_{\{\neg,\cap,\cup,\Rightarrow\}}(\mathbf{P}, \mathbf{F}, \mathbf{C}),\ \mathcal{F},\ LA,\ \mathcal{R})$$

for

$$\mathcal{R} = \{(MP), (SB), (Q1), (Q2), (Q3), (Q4)\}$$

The logical axioms LA are as follows

Propositional Axioms

A1 $((A \Rightarrow B) \Rightarrow ((B \Rightarrow C) \Rightarrow (A \Rightarrow C)))$

A2 $(A \Rightarrow (A \cup B))$

A3 $(B \Rightarrow (A \cup B))$

Rasiowa and Sikorski

A4 $((A \Rightarrow C) \Rightarrow ((B \Rightarrow C) \Rightarrow ((A \cup B) \Rightarrow C)))$

A5 $((A \cap B) \Rightarrow A)$

A6 $((A \cap B) \Rightarrow B)$

A7 $((C \Rightarrow A) \Rightarrow ((C \Rightarrow B) \Rightarrow (C \Rightarrow (A \cap B)))$

A8 $((A \Rightarrow (B \Rightarrow C)) \Rightarrow ((A \cap B) \Rightarrow C))$

A9 $(((A \cap B) \Rightarrow C) \Rightarrow (A \Rightarrow (B \Rightarrow C))$

A10 $(A \cap \neg A) \Rightarrow B)$

A11 $((A \Rightarrow (A \cap \neg A)) \Rightarrow \neg A)$

A12 $(A \cup \neg A)$

for any $A, B, C \in \mathcal{F}$

Rules of Inference $\mathcal{R}$

(MP) is **Modus Ponens** rule

$$(MP) \quad \frac{A \; ; \; (A \Rightarrow B)}{B}$$

for any formulas $A, B \in \mathcal{F}$

(SB) is a **substitution rule**

$$(SB) \quad \frac{A(x_1, x_2, \; \ldots \; x_n)}{A(t_1, t_2, \; \ldots \; t_n)}$$

where $A(x_1, x_2, \; \ldots \; x_n) \in \mathcal{F}$ and $t_1, t_2, \; \ldots \; t_n \in \mathbf{T}$

Rasiowa and Sikorski

$(G1), (G2)$ are the following **quantifiers introduction rules**

$$(G1)\quad \frac{(B \Rightarrow A(x))}{(B \Rightarrow \forall x A(x))}$$

$$(G2)\quad \frac{(A(x) \Rightarrow B)}{(\exists x A(x) \Rightarrow B)}$$

where $A(x), B \in \mathcal{F}$ and B is such that x is **not free** in B

Rasiowa and Sikorski

$(G3), (G3)$ are the following **quantifiers elimination rules**.

$$(G3) \quad \frac{(B \Rightarrow \forall x A(x))}{(B \Rightarrow A(x))}$$

$$(G4) \quad \frac{\exists x (A(x) \Rightarrow B)}{(A(x) \Rightarrow B)}$$

where $A(x), B \in \mathcal{F}$ and B is such that x is **not free** in B

Rasiowa and Sikorski

The **algebraic logic** starts from purely logical considerations, abstracts from them, places them into a general algebraic context, and makes use of **other branches** of mathematics such as topology, set theory, and functional analysis

For **example**, Rasiowa and Sikorski **algebraic generalization** of the **completeness theorem** for classical predicate logic is the following

Rasiowa and Sikorski

Algebraic Completeness Theorem (Rasiowa, Sikorski 1950)

For every formula A of the classical predicate calculus RS the following conditions are equivalent

i A is derivable in RS;

ii A is valid in every realization of $\mathcal{L}$;

iii A is valid in every realization of $\mathcal{L}$ in any complete Boolean algebra;

iv A is valid in every realization of $\mathcal{L}$ in the field $B(X)$ of all subsets of any set $X \neq \emptyset$;

Rasiowa and Sikorski

v A is valid in every semantic realization of $\mathcal{L}$ in any enumerable set;

vi there exists a non-degenerate Boolean algebra $\mathcal{A}$ and an infinite set J such that A is valid in every realization of $\mathcal{L}$ in J and $\mathcal{A}$;

vii $A_R(\mathbf{I}) = V$ for the canonical realization R of $\mathcal{L}$ in the Lindenbaum-Tarski algebra $\mathcal{LT}$ of RS and the identity valuation $\mathbf{I}$;

viii A is a predicate tautology.