cse541
LOGIC for Computer Science

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LECTURE 9b

Chapter 9
Hilbert Proof Systems
Completeness of Classical Predicate Logic

PART 3: **Proof of the Completeness Theorem**

Completeness Theorem

The proof of Gödel's **completeness theorem** given by Kurt Gdel in his doctoral dissertation of 1929 and published as an article in 1930 is **not easy** to read today

It uses concepts and formalism that are no longer used and terminology that is often obscure

Gödel's proof was then simplified in 1947, when Leon Henkin observed in his Ph.D. thesis that the hard part of the proof can be presented as the Model Existence Theorem (published in 1949)

Henkin's proof was simplified by Gisbert Hasenjaeger in 1953

Completeness Theorem

Other now classical **proofs** have been published by Rasiowa and Sikorski in 1951, 1952 using Boolean algebraic methods and by Beth in 1953, using topological methods

Still **other proofs** may be found in Hintikka (1955) and in Beth (1959)

We follow a modern version of of Henkin proof

Hilbert-style Proof System **H**

We define now a Hilbert style proof system **H** we are going to prove the **completeness theorem** for

Language $\mathcal{L}$

The language $\mathcal{L}$ of the proof system **H** is a predicate (first order) language with equality

We assume that the sets **P, F, C** are infinitely enumerable

We also assume that $\mathcal{L}$ has a full set of propositional connectives, i.e.

$$\mathcal{L} = \mathcal{L}_{\{\neg, \cap, \cup, \Rightarrow\}}(\mathbf{P}, \mathbf{F}, \mathbf{C})$$

Hilbert-style Proof System **H**

Logical Axioms *LA*

The set *LA* of **logical axioms** consists of three groups of axioms:

propositional axioms *PA*, equality axioms *EA*, and

quantifiers axioms *QA*

We write it symbolically as

$$LA = \{PA, \ EA, \ QA\}$$

For the set *PA* of **propositional axioms** we choose any **complete** set of axioms for propositional logic with a full set $\{\neg, \cap, \cap, \Rightarrow\}$ of propositional connectives

Hilbert-style Proof System **H**

In some formalizations, including the one in the *Handbook of Mathematical Logic, Barwise, ed.* (1977) we **base** our proof system **H** on, the authors just say for this group *PA* of **propositional axioms**: "all tautologies"

They of course mean all **predicate** formulas of $\mathcal{L}$ that are substitutions of propositional tautologies

This is done for the **need** of being able to use freely these **predicate** substitutions of propositional tautologies in the proof of **completeness theorem** for the proof system they formalize this way.

Hilbert-style Proof System **H**

In this case these **tautologies** are listed as **axioms** of the system and hence are provable in it

This is a convenient approach, but also the one that makes such a proof system **not** to be finitely axiomatizable

We **avoid** the infinite axiomatization by choosing a proper finite set of predicate language version of propositional **axioms** that is known (proved already for propositional case) to be **complete**, i.e. the one in which all propositional tautologies are **provable**

We choose, for name of the proof system **H** for Hilbert

Moreover, historical sake, we adopt Hilbert (1928) set of **axioms** from chapter 5

Hilbert-style Proof System **H**

For the set *EA* of **equational axioms** we choose the same set as in before because they were used in the proof of Reduction to Propositional Logic Theorem

We want to be able to carry this proof **within** the system **H**

For the set *QA* of **quantifiers axioms** we choose the **axioms** such that the Henkin set S_{Henkin} axioms **Q1, Q2** are their particular cases

This again is needed, so the proof of the **Reduction Theorem** can be carried within **H**

Rules of inference $\mathcal{R}$

There are four inference rules:

Modus Ponens (MP) and three quantifiers rules $(G), (G1), (G2)$, called **Generalization Rules**

We **define** the proof system **H** as follows

$$\mathbf{H} = (\mathcal{L}_{\{\neg,\cap,\cup,\Rightarrow\}}(\mathbf{P},\mathbf{F},\mathbf{C}),\ \mathcal{F},\ LA,\ \mathcal{R} = \{(MP),(G),(G1),(G2)\})$$

where $\mathcal{L} = \mathcal{L}_{\{\neg,\cap,\cup,\Rightarrow\}}(\mathbf{P},\mathbf{F},\mathbf{C})$ is **predicate** (first order) language with **equality**

We assume that the sets **P, F, C** are infinitely enumerable

$\mathcal{F}$ is the set of all well formed **formulas** of $\mathcal{L}$

Hilbert-style Proof System **H**

LA is the set of **logical axioms**

$$LA = \{PA, EA, QA\}$$

for PA, EA, QA defined as follows

PA is the set of **propositional axioms** (Hilbert, 1928)

A1 $(A \Rightarrow A)$

A2 $(A \Rightarrow (B \Rightarrow A))$

A3 $((A \Rightarrow B) \Rightarrow ((B \Rightarrow C) \Rightarrow (A \Rightarrow C)))$

A4 $((A \Rightarrow (A \Rightarrow B)) \Rightarrow (A \Rightarrow B))$

A5 $((A \Rightarrow (B \Rightarrow C)) \Rightarrow (B \Rightarrow (A \Rightarrow C)))$

A6 $((A \Rightarrow B) \Rightarrow ((B \Rightarrow C) \Rightarrow (A \Rightarrow C)))$

Hilbert-style Proof System **H**

A7 $((A \cap B) \Rightarrow A)$

A8 $((A \cap B) \Rightarrow B)$

A9 $((A \Rightarrow B) \Rightarrow ((A \Rightarrow C) \Rightarrow (A \Rightarrow (B \cap C)))$

A10 $(A \Rightarrow (A \cup B))$

A11 $(B \Rightarrow (A \cup B))$

A12 $((A \Rightarrow C) \Rightarrow ((B \Rightarrow C) \Rightarrow ((A \cup B) \Rightarrow C)))$

A13 $((A \Rightarrow B) \Rightarrow ((A \Rightarrow \neg B) \Rightarrow \neg A))$

A14 $(\neg A \Rightarrow (A \Rightarrow B))$

A15 $(A \cup \neg A)$

for any $A, B, C \in \mathcal{F}$

Hilbert-style Proof System **H**

EA is the set of **equality axioms**

E1 $u = u$

E2 $(u = w \Rightarrow w = u)$

E3 $((u_1 = u_2 \cap u_2 = u_3) \Rightarrow u_1 = u_3)$

E4
$((u_1 = w_1 \cap ... \cap u_n = w_n) \Rightarrow (R(u_1, ..., u_n) \Rightarrow R(w_1, ..., w_n)))$

E5
$((u_1 = w_1 \cap ... \cap u_n = w_n) \Rightarrow (t(u_1, ..., u_n) \Rightarrow t(w_1, ..., w_n)))$

for any free variable or **constant** of $\mathcal{L}$, $R \in \mathbf{P}$, and $t \in \mathbf{T}$
where R is an arbitrary n-ary **relation** symbol of $\mathcal{L}$ and $t \in \mathbf{T}$ is an arbitrary n-ary **term** of $\mathcal{L}$

Hilbert-style Proof System **H**

QA is the set of **quantifiers axioms**.

Q1 $(\forall x A(x) \Rightarrow A(t))$

Q2 $(A(t) \Rightarrow \exists x A(x))$

where where t is a **term**

A(t) is a result of **substitution** of t for all free occurrences of x in $A(x)$ and

t is **free for** x in A(x), i.e. **no** occurrence of a variable in t becomes a **bound** occurrence in A(t)

Hilbert-style Proof System **H**

$\mathcal{R}$ is the set of **rules of inference**

$$\mathcal{R} = \{(MP),\ (G),\ (G1),\ (G2)\}$$

(MP) is **Modus Ponens** rule

$$(MP)\ \frac{A\ ;\ (A \Rightarrow B)}{B}$$

for any formulas $A, B \in \mathcal{F}$

(G) is a quantifier **generalization** rule

$$(G)\ \frac{A}{\forall x A}$$

where $A \in \mathcal{F}$ and in particular we write

$$(G)\ \frac{A(x)}{\forall x A(x)}$$

for $A(x) \in \mathcal{F}$ and $x \in VAR$

Hilbert-style Proof System **H**

(G1) is a quantifier **generalization** rule

$$(G1) \quad \frac{(B \Rightarrow A(x))}{(B \Rightarrow \forall x A(x))}$$

where for $A(x), B \in \mathcal{F}$, $x \in VAR$, and B is such that x is **not free** in B

(G2) is a quantifier **generalization** rule

$$(G2) \quad \frac{(A(x) \Rightarrow B)}{(\exists x A(x) \Rightarrow B)}$$

where for $A(x), B \in \mathcal{F}$, $x \in VAR$, and B is such that x is **not free** in B

Hilbert-style Proof System **H**

We define now, as we do for any proof system, a notion of a **formal proof** of a formula A **from** a set S of formulas in **H** as a finite **sequence**

$$B_1, B_2, \ \ldots \ B_n$$

of formulas with each of which is **either** a logical axiom of **H**, a member of S , **or** else follows from earlier formulas in the sequence by one of the inference rules from $\mathcal{R}$

and is such that

$$B_n = A$$

We write it formally as follows.

Formal Proof in **H**

Definition

Let $\Gamma \subseteq \mathcal{F}$ be any set of formulas of $\mathcal{L}$

A **proof** in **H** of a formula $A \in \mathcal{F}$ **from** a set Γ of formulas is a sequence

$$B_1, B_2, \ \ldots \ B_n$$

of formulas, such that

$$B_1 \in LA \cup \Gamma, \qquad B_n = A$$

and for each $1 < i \leq n$, **either** $B_i \in LA \cup \Gamma$ **or** B_i is a **conclusion** of some of the preceding expressions in the sequence $B_1, B_2, \ \ldots \ B_n$ by virtue of one of the rules of inference from $\mathcal{R}$

Formal Proof in $\mathbf{H}$

We write

$$\Gamma \vdash_{\mathbf{H}} A$$

to denote that the formula A has a **proof** from Γ in $\mathbf{H}$

The case when $\Gamma = \emptyset$ is a special one

By the definition, $\emptyset \vdash_{\mathbf{H}} A$ means that in the proof of A **only** logical axioms LA are used. We hence write

$$\vdash_{\mathbf{H}} A$$

to denote that a formula A has a proof in $\mathbf{H}$

Formal Proof in **H**

As we work now with a **fixed** (and only one) proof system **H**, we use the notation

$$\Gamma \vdash A \quad \text{and} \quad \vdash A$$

to denote the **proof** of a formula A **from** a set Γ in **H** and the proof of a formula A in **H**, respectively

Completeness Theorem

Any proof of the **completeness theorem** for a given proof system consists always of **two parts**

First we have show that

all formulas that have a proof in the system are tautologies

This is called a **soundness theorem** or soundness part of the completeness theorem

Completeness Theorem

The **second** implication says:

if a formula is a tautology then it has a proof in the proof system

This alone is sometimes called a **completeness theorem** (on assumption that the proof system is **sound**)

Traditionally it is called a completeness part of the **completeness theorem**

Soundness Theorem

We know that all **axioms** of **H** are predicate tautologies (proved in chapter 8)

All **rules** of inference from $\mathcal{R}$ are **sound** as the corresponding formulas were also proved in chapter 8 to be predicate tautologies and so the system **H** is **sound** i.e. the following holds for **H**

Soundness Theorem

For every formula $A \in \mathcal{F}$ of the language $\mathcal{L}$ of the proof system **H**,

$$\text{if} \quad \vdash A \quad \text{then} \quad \models A$$

Completeness Theorem

The **soundness theorem** proves that the proofs in the system **H** "produce" only tautologies

We show here, as the next step that our proof system **H** "produces" not only tautologies, but that all tautologies are **provable** in it

This is called a **completeness theorem** for classical predicate (first order logic, as it all is proven with respect to **classical** semantics

This is why it is called a **completeness** of classical predicate logic

Completeness Theorem

The goal is now to prove the **completeness part** of the following original theorem Gödel' s theorem

Theorem (completeness of predicate logic)

For any formula A of the language $\mathcal{L}$ of the proof system **H**,

A is **provable** in **H** if and only if

A is a predicate **tautology (valid)**

We write it symbolically as

$$\vdash A \quad \text{if and only if} \quad \models A$$

Completeness Theorem

We are going to prove the above **Theorem** (completeness of predicate logic) as a **particular case** of the **Gödel Completeness Theorem** that follows

This theorem is its more general, and more modern version

Its formulation, as well as the method of proving it, was first introduced by Henkin in 1947

It uses a notion of a **logical implication**, and some other notions that we introduce now below

Completeness Theorem

Sentence, Closure

Any formula of $\mathcal{L}$ **without** free variables is called a **sentence**

For any formula $A(x_1, \dots x_n)$, a sentence

$$\forall x_1 \forall x_2 \dots \forall x_n \, A(x_1, \dots x_n)$$

is called a **closure** of $A(x_1, \dots x_n)$

Directly from the above definition have that the following hold

Closure Fact

For any formula $A(x_1, \dots x_n)$,

$$\models A(x_1, \dots x_n) \quad \text{if and only if} \quad \models \forall x_1 \forall x_2 \dots \forall x_n \, A(x_1, \dots x_n)$$

Completeness Theorem

Logical Implication

For any set $\Gamma \subseteq \mathcal{F}$ of formulas of $\mathcal{L}$ and any $A \in \mathcal{F}$, we say that the set Γ **logically implies** the formula A and write it as

$$\Gamma \models A$$

if and only if all **models** of Γ are **models** of A

Observe, that in order to **prove** that $\Gamma \models B$ we have to show that the implication

$$\text{if } \mathcal{M} \models \Gamma \text{ then } \mathcal{M} \models B$$

holds for all structures $\mathcal{M} = [U, I]$ for $\mathcal{L}$

Completeness Theorem

Directly from the **Closure Lemma** we get the following

Lemma

Let Γ be a set of sentences of $\mathcal{L}$

For any formula $A(x_1, \ldots x_n)$ that **is not** a sentence,

$\Gamma \vdash A(x_1, \ldots x_n)$ if and only if $\Gamma \models \forall x_1 \forall x_2 \ldots \forall x_n \, A(x_1, \ldots x_n)$

Completeness Theorem

The above **Lemma** and **Closure Lemma** show that we need to consider only **sentences** (closed formulas) of $\mathcal{L}$ since they prove two properties:

(1) a formula of $\mathcal{L}$ is a **tautology** if and only if its closure is a **tautology**

(2) a formula of $\mathcal{L}$ is **provable** from Γ if and only if its closure is **provable** from Γ

This justifies the following **generalization** of the original Gödel' s completeness of predicate logic theorem

Gödel Completeness Theorem

Let Γ be any set of sentences and A any sentence of a language $\mathcal{L}$ of Hilbert proof system **H**

A sentence A is **provable** from Γ in **H** if and only if the set Γ **logically implies** A

We write it in symbols,

$$\Gamma \vdash A \quad \text{if and only if} \quad \Gamma \models A.$$

Remark

We want to remind that the Section: Reduction Predicate Logic to Propositional Logic is an integral and the first part of the proof the **Gödel Completeness Theorem**

We presented it separately for two reasons

R1. The reduction method and theorems and their proofs are purely **semantical** in their nature and hence are independent of the proof system **H**

R2. Because of the reason **R1.** the reduction method can be **used/adapted** to a proof of **completeness theorem** of any other proof system one needs to prove the classical **completeness theorem** for

Consistency

There are two definitions of consistency: semantical and syntactical

The **semantical** definition uses the notion of a model and says, in plain English:

a set of formulas is consistent if it has a model

The **syntactical** one uses the notion of provability and says:

a set of formulas is consistent if one can't prove a contradiction from it

We have used, in the Proof Two of the **Completeness Theorem** for propositional logic (chapter 5) the syntactical definition of consistency

We use now the following **semantical** definition

Consistency

Definition (Consistent/Inconsistent)

A set $\Gamma \subseteq \mathcal{F}$ of formulas of $\mathcal{L}$ is **consistent**

if and only if it has a **model**, otherwise, is **inconsistent**

Directly from the above definition we have the following

Inconsistency Lemma

For any set $\Gamma \subseteq \mathcal{F}$ of formulas of $\mathcal{L}$ and any $A \in \mathcal{F}$,

if $\Gamma \models A$, then the set $\Gamma \cup \{\neg A\}$ is **inconsistent**

Proof

Assume $\Gamma \models A$ and $\Gamma \cup \{\neg A\}$ is **consistent**

It means there is a structure $\mathcal{M} = [U, I]$, such that

$\mathcal{M} \models \Gamma$ and $\mathcal{M} \models \neg A$, i.e. $\mathcal{M} \not\models A$

This is a **contradiction** with $\Gamma \models A$

Now we are going to prove the following **Lemma** that is crucial, to the proof of the Completeness Theorem

Crucial Lemma

Let Γ be any set of **sentences** of a language $\mathcal{L}$ of **H**

The following conditions hold for any formulas $A, B \in \mathcal{F}$ of $\mathcal{L}$

(i) If $\Gamma \vdash (A \Rightarrow B)$ and $\Gamma \vdash (\neg A \Rightarrow B)$, then $\Gamma \vdash B$

(ii) If $\Gamma \vdash ((A \Rightarrow C) \Rightarrow B)$, then $\Gamma \vdash (\neg A \Rightarrow B)$ and $\Gamma \vdash (C \Rightarrow B)$

(iii) If x does not appear in B and if $\Gamma \vdash ((\exists y A(y) \Rightarrow A(x)) \Rightarrow B)$, then $\Gamma \vdash B$

(iv) If x does not appear in B and if $\Gamma \vdash ((A(x) \Rightarrow \forall y A(y)) \Rightarrow B)$, then $\Gamma \vdash B$

Crucial Lemma Proof

Proof

(i) Notice that the formula $((A \Rightarrow B) \Rightarrow ((\neg A \Rightarrow B) \Rightarrow B))$ is a substitution of a propositional tautology, hence by definition of **H**, is **provable** in it

By monotonicity, $\Gamma \vdash ((A \Rightarrow B) \Rightarrow ((\neg A \Rightarrow B) \Rightarrow B))$

By assuption $\Gamma \vdash (A \Rightarrow B)$ and by Modus Ponens we get

$$\Gamma \vdash ((\neg A \Rightarrow B) \Rightarrow B)$$

By assuption $\Gamma \vdash (\neg A \Rightarrow B)$ and Modus Ponens we get

$$\Gamma \vdash B$$

Crucial Lemma Proof

(ii) The formulas

$$(1) \quad (((A \Rightarrow B) \Rightarrow (\neg A \Rightarrow B)))$$

$$(2) \quad (((A \Rightarrow B) \Rightarrow B) \Rightarrow (C \Rightarrow B))$$

are substitution of a propositional tautologies, hence are **provable** in **H**

Assume $\Gamma \vdash ((A \Rightarrow C) \Rightarrow B)$

By monotonicity and (1) we get

$$\Gamma \vdash (\neg A \Rightarrow B)$$

and by (2) we get

$$\vdash (C \Rightarrow B)$$

Crucial Lemma Proof

(iii) Assume

$$\Gamma \vdash ((\exists y A(y) \Rightarrow A(x)) \Rightarrow B)$$

Observe that it is a particular case of assumption

$$\Gamma \vdash ((A \Rightarrow C) \Rightarrow B)$$

in (ii), for $A = \exists y A(y)$, $C = A(x)$ and B= B

Hence by (ii) we have that

$$\Gamma \vdash (\neg \exists y A(y) \Rightarrow B) \text{ and } \Gamma \vdash (A(x) \Rightarrow B)$$

Apply Generalization Rule G2 to

$$\Gamma \vdash (A(x) \Rightarrow B)$$

and we have

$$\Gamma \vdash (\exists y A(y) \Rightarrow B)$$

Crucial Lemma Proof

Then by (i) applied to

$$\Gamma \vdash (\exists y A(y) \Rightarrow B) \quad \text{and} \quad \Gamma \vdash (\neg \exists y A(y) \Rightarrow B)$$

we get

$$\Gamma \vdash B$$

The proof of (iv) is similar to (iii) but uses the Generalization Rule G1

This **ends** the proof of the **Lemma**

Completeness Theorem for **H**

Now we are ready to conduct the proof of the Completeness Theorem for **H** stated as follows

H Completeness Theorem

Let Γ be any set of sentences and A any sentence of a language $\mathcal{L}$ of Hilbert proof system **H**

$$\Gamma \vdash A \quad \text{if and only if} \quad \Gamma \models A$$

In particular, for any formula A of $\mathcal{L}$,

$$\vdash A \quad \text{if and only if} \quad \models A$$

Proof of Completeness Theorem for **H**

Proof

We prove the **completeness part** , i.e. we prove the implication

$$\text{if } \Gamma \models A, \text{ then } \Gamma \vdash A$$

Suppose that $\Gamma \models A$

This means that we assume that all $\mathcal{L}$ models of Γ are models of A

By the **Inconsistency Lemma** the set $\Gamma \cup \{\neg A\}$ is inconsistent

Let $\mathcal{M} \models \Gamma$

We **construct**, as a next step, a witnessing expansion language $\mathcal{L}(C)$ of $\mathcal{L}$

Proof of Completeness Theorem for **H**

By the **Reduction Theorem** the set

$$\Gamma \cup S_{Henkin} \cup EQ$$

is **consistent** in a sense of propositional logic in $\mathcal{L}$

The set S_{Henkin} is a Henkin Set and EQ are equality axioms that are also the equality axioms EQ of **H**

By the **Compactness Theorem** for propositional logic of $\mathcal{L}$ there is a finite set

$$S_0 \subseteq \Gamma \cup S_{Henkin} \cup EQ$$

such that $S_0 \cup \{\neg A\}$ is **inconsistent** in the sense of propositional logic in $\mathcal{L}$

Proof of Completeness Theorem for **H**

We list all elements of S_0 in a sequence

$$A_1,\ A_2, \ldots, A_n,\ B_1,\ B_2, \ldots, B_m$$

where the sequence

$$A_1,\ A_2, \ldots, A_n$$

consists of those elements of S_0 which are **either** in $\Gamma \cup EQ$ **or else** are quantifiers axioms that are particular cases of the quantifiers axioms QA of **H**. We list them in any order

The sequence

$$B_1,\ B_2, \ldots, B_m$$

consists of elements of S_0 which are Henkin Axioms but listed **carefully** as to be described as follows

Proof of Completeness Theorem for **H**

Observe that by definition,

$$\mathcal{L}(C) = \bigcup_{n \in N} \mathcal{L}_n \text{ for } \mathcal{L} = \mathcal{L}_0 \subseteq \mathcal{L}_1 \subseteq \ \ldots$$

We **define** the **rank** of $A \in \mathcal{L}(C)$ to be the **least** n, such that $A \in \mathcal{L}_n$

Now we choose for B_1 a Henkin Axiom in S_0 of the maximum **rank**

We choose for B_2 a Henkin Axiom in $S_0 - \{B_1\}$ of the maximum **rank**

We choose for B_3 a Henkin Axiom in $S_0 - \{B_1, B_2\}$ of the maximum **rank**, etc. . . .

Proof of Completeness Theorem for **H**

The point of choosing the formulas B_i in this way is to make sure that the **witnessing constant** about which B_i speaks, **does not** appear in

$$B_{i+1},\ B_{i+2},\ \ldots,\ B_m$$

For **example** , if B_1 is

$$(\exists x A(x) \Rightarrow A(c_{A[x]}))$$

then $A[x]$ **does not** appear in any of the other $B_2, \ \ldots, B_m$, by the **maximality condition** on B_1

Proof of Completeness Theorem for **H**

We know that that $S_0 \cup \{\neg A\}$ is **inconsistent** in the sense of propositional logic, i.e.

$S_0 \cup \{\neg A\}$ **does not** have a (propositional) model

This means that

$$v^*(\neg A) \neq T \text{ for all } v \quad \text{and so} \quad v^*(A) = T \text{ for all } v$$

Hence a sentence

$$(S) \quad (A_1 \Rightarrow (A_2 \Rightarrow \ldots (A_n \Rightarrow (B_1 \Rightarrow \ldots (B_m \Rightarrow A))..)$$

is a **propositional tautology**

Proof of Completeness Theorem for **H**

We now replace in the sentence (S) each witnessing constant by a distinct new variable and write the result as

$$(S') \quad (A_1{}' \Rightarrow (A_2{}' \Rightarrow \ldots (A_n{}' \Rightarrow (B_1{}' \Rightarrow \ldots (B_m{}' \Rightarrow A))..)$$

We have $A' = A$ since A has **no** witnessing constant in it

The result is still a **tautology** and hence is **provable** in **H** from propositional axioms PA and Modus Ponens

By monotonicity

$$S_0 \vdash (A_1{}' \Rightarrow (A_2{}' \Rightarrow \ldots (A_n{}' \Rightarrow (B_1{}' \Rightarrow \ldots (B_m{}' \Rightarrow A))..)$$

Proof of Completeness Theorem for **H**

Each of $A_1', A_2', \ldots, A_n'$ is **either** a quantifiers axiom from QA of **H** **or else** in S_0, so

$$S_0 \vdash A_i' \quad \text{for all} \quad 1 \leq i \leq n$$

We apply Modus Ponens to the above and (S') n times and get

$$S_0 \vdash (B_1' \Rightarrow (B_2' \Rightarrow \ldots (B_m' \Rightarrow A))..)$$

Proof of Completeness Theorem for **H**

For **example**, if $B_1{}'$ is

$$(\exists x C(x) \Rightarrow C(x))$$

we have

$$S_0 \vdash ((\exists x C(x) \Rightarrow C(x)) \Rightarrow B)$$

for $B = (B_2{}' \Rightarrow \ldots (B_m{}' \Rightarrow A))..)$

By the **Crucial Lemma** part (iii) that says:

(iii) If x does not appear in B and if
$\Gamma \vdash ((\exists y A(y) \Rightarrow A(x)) \Rightarrow B)$, then $\Gamma \vdash B$

we get $S_0 \vdash B$, i.e.

$$S_0 \vdash (B_2{}' \Rightarrow \ldots (B_m{}' \Rightarrow A))..)$$

Proof of Completeness Theorem for **H**

If, for **example**, $B_2{}'$ is

$$(D(x) \Rightarrow \forall x D(x))$$

we have

$$S_0 \vdash ((\exists x C(x) \Rightarrow C(x)) \Rightarrow D)$$

for $D = (B_3{}' \Rightarrow \ldots (B_m{}' \Rightarrow A))..)$

By the **Crucial Lemma** part (iv) that says:

(iv) If x does not appear in B and if
$\Gamma \vdash ((A(x) \Rightarrow \forall y A(y)) \Rightarrow B)$, then $\Gamma \vdash B$

we get $S_0 \vdash D$, i.e.

$$S_0 \vdash (B_3{}' \Rightarrow \ldots (B_m{}' \Rightarrow A))..)$$

Proof of Completeness Theorem for **H**

We hence apply parts (iii) and (iv) of the **Crucial Lemma** to successively remove all

$$B_1{}', \ldots, B_m{}'$$

and obtain

$$S_0 \vdash A$$

This ends the proof that

$$\Gamma \vdash A$$

We hence we completed the proof of the completeness part of the first part

$$\Gamma \vdash A \quad \text{if and only if} \quad \Gamma \models A$$

of the **H Completeness Theorem**

Gödel' s Completeness Theorem

The soundness part of the **H Completeness Theorem** i.e. the implication

$$\text{if } \Gamma \vdash A, \text{ then } \Gamma \models A$$

holds for any sentence A of $\mathcal{L}$ directly by **Closure Lemma** and **Soundness Theorem**

The original Gödel' s **Theorem**, is expressed by the second part of the **H** Completeness Theorem:

$$\vdash A \quad \text{if and only if} \quad \models A$$

It follows from **Closure Lemma** and the first part for $\Gamma = \emptyset$