cse541
LOGIC for Computer Science

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LECTURE 9a

Chapter 9
Hilbert Proof Systems
Completeness of Classical Predicate Logic

PART 2: **Henkin Method**

Reduction to Propositional Logic Theorem,

Compactness Theorem, Löwenheim-Skolem Theorem

Henkin Method

Propositional tautologies within $\mathcal{L}$ barely scratch the surface of the collection of **predicate** (first -order) tautologies

For **example** the following first-order formulas are **propositional** tautologies

$$(\exists x A(x) \cup \neg \exists x A(x)), \quad (\forall x A(x) \cup \neg \forall x A(x)$$

$$(\neg(\exists x A(x) \cup \forall x A(x)) \Rightarrow (\neg \exists x A(x) \cap \neg \forall x A(x)))$$

but the following are **predicate** (first order) tautologies that are not **propositional** tautologies

$$\forall x (A(x) \cup \neg A(x))$$

$$(\neg \forall x A(x) \Rightarrow \exists x \neg A(x))$$

Henkin Method

To stress the difference between the **propositional** tautologies of a propositional language and **predicate** (first order) tautologies the word **tautology** is used only for the propositional tautologies of a propositional language

The word a **valid formula** is used for the predicate (first order) tautologies in this case

We use here **both** notions, with preference to word **predicate tautology** or **tautology** for short when there is **no room** for misunderstanding

To make sure that there is no misunderstandings we **remind** the following basic definitions from chapter 8

Basic Definitions

Given a first order language $\mathcal{L}$ with the set of variables VAR and the set of formulas $\mathcal{F}$. Let

$$\mathcal{M} = [M, I]$$

be a **structure** for the language $\mathcal{L}$, with the **universe** M and the **interpretation** I and let

$$s : VAR \longrightarrow M$$

be an **assignment** of $\mathcal{L}$ in M

Here are some basic **definitions**

Basic Definitions

D1. A is **satisfied** in $\mathcal{M}$

Given a structure $\mathcal{M} = [M, I]$, we say that a formula A is **satisfied** in $\mathcal{M}$ if **there is** an assignment $s : VAR \longrightarrow M$ such that

$$(\mathcal{M}, s) \models A$$

D2. A is **true** in $\mathcal{M}$

Given a structure $\mathcal{M} = [M, I]$, we say that a formula A is **true** in $\mathcal{M}$ if

$$(\mathcal{M}, s) \models A$$

for **all** assignments $s : VAR \longrightarrow M$

Basic Definitions

D3. Model $\mathcal{M}$

If A is **true** in a structure $\mathcal{M} = [M, I]$, then $\mathcal{M}$ is called a **model** for A

We denote it as

$$\mathcal{M} \models A$$

D4. A is **predicate tautology (valid)**

A formula A is a **predicate** tautology (valid) if it is **true** in all structures $\mathcal{M} = [M, I]$, i.e. if all structures are **models** of A

We use use the term **predicate tautology** and and denote it, when there is no confusion with propositional case as

$$\models A$$

Case: A is a **sentence**

If the formula A is a sentence, then the truth or falsity of the statement $(\mathcal{M}, s) \models A$ is completely independent of s

Thus we write

$$\mathcal{M} \models A$$

and read $\mathcal{M}$ is a **model** of A, if for some (hence every) valuation s

$$(\mathcal{M}, s) \models A$$

D5. Model of a **set** S of formulas

$\mathcal{M}$ is a model of a set S (of sentences) if and only if $\mathcal{M} \models A$ for all $A \in S$. We write it

$$\mathcal{M} \models S$$

Predicate and Propositional Models

Relationship

Given a predicate language $\mathcal{L}$

The **predicate models** for $\mathcal{L}$ are defined in terms of

structures $\mathcal{M} = [M, I]$ and assignments $s : VAR \longrightarrow M$

The **propositional models** for $\mathcal{L}$ are defined in terms of of

truth assignments $v : \mathcal{P} \longrightarrow \{T, F\}$

The **relationship** between the predicate models and propositional models is established by the following **Lemma**

Relationship Lemma

Lemma

Let $\mathcal{M} = [M, I]$ be a structure for the language $\mathcal{L}$ and let $s : VAR \longrightarrow M$ an assignment in $\mathcal{M}$

There is a truth assignment

$$v : \mathcal{P} \longrightarrow \{T, F\}$$

such that for all formulas A of $\mathcal{L}$,

$$(\mathcal{M}, s) \models A \quad \text{if and only if} \quad v^*(A) = T$$

In particular, for any set S of sentences of $\mathcal{L}$,

if $\mathcal{M} \models S$ then S is **consistent** in the propositional sense

Relationship Lemma Proof

Proof

For any prime formula $A \in P$ we define

$$v(A) = \begin{cases} T & \text{if } (\mathcal{M}, s) \models A \\ F & \text{otherwise.} \end{cases}$$

Since every formula in $\mathcal{L}$ is either prime or is built up from prime formulas by means of propositional connectives, the conclusion is obvious

Relationship Lemma

Observe, that the converse of the **Lemma** implication:

if $\mathcal{M} \models S$ then S is **consistent** in the propositional sense

is **far** from **true**

Consider a set

$$S = \{\forall x(A(x) \Rightarrow B(x)), \forall x A(x), \exists x \neg B(x)\}$$

All formulas of S are different prime formulas

So S has and obvious **model** and hence is consistent in the propositional sense

Obviously S has **no** predicate (first-order)**model**

Definition (Language with Equality)

Let $\mathcal{L}$ be a **predicate** (first order) language with **equality**

$$\mathcal{L} = \mathcal{L}_{\{\neg,\cap,\cup,\Rightarrow\}}(\mathbf{P}, \mathbf{F}, \mathbf{C})$$

Equality Axioms

For any free variable or constant of $\mathcal{L}$, i.e for any $u, w, u_i, w_i \in (VAR \cup \mathbf{C})$,

E1 $u = u$

E2 $(u = w \Rightarrow w = u)$

E3 $((u_1 = u_2 \cap u_2 = u_3) \Rightarrow u_1 = u_3)$

E4

$((u_1 = w_1 \cap ... \cap u_n = w_n) \Rightarrow (R(u_1, ..., u_n) \Rightarrow R(w_1, ..., w_n)))$

E5

$((u_1 = w_1 \cap ... \cap u_n = w_n) \Rightarrow (t(u_1, ..., u_n) \Rightarrow t(w_1, ..., w_n)))$

where $R \in \mathbf{P}$ and $t \in \mathbf{T}$, i.e. R is an arbitrary n-ary relation symbol of $\mathcal{L}$ and $t \in \mathbf{T}$ is an arbitrary n-ary term of $\mathcal{L}$

Language with Equality

Observe that given any structure $\mathcal{M} = [M, I]$

We have by simple verification that

for all $s : VAR \longrightarrow M$, and

for all $A \in \{E1, E2, E3, E4, E5\}$,

$$(\mathcal{M}, s) \models A$$

This proves the following

Fact

All **equality axioms** are predicate tautologies of $\mathcal{L}$

This is why we call logic with **equality axioms** added to it, still just a **logic**

Henkin's Witnessing Expansion of $\mathcal{L}$

Henkin's Witnessing Expansion

Now we are going to **define** notions that are **fundamental** to the Henkin's technique for **reducing** predicate logic to propositional logic

The **first** one is that of witnessing expansion of $\mathcal{L}$

We construct an **expansion** of the language $\mathcal{L}$ by **adding** a set of new constants to it

It means the we **add** a specially constructed the set C to the set $\mathbf{C}$ of constants of $\mathcal{L}$ such that

$$C \cap \mathbf{C} = \emptyset$$

The language such **constructed** is called witnessing expansion of the language $\mathcal{L}$

The construction of the **expansion** is described as follows

Henkin's Witnessing Expansion

Definition
For any predicate language

$$\mathcal{L} = \mathcal{L}(\mathbf{P}, \mathbf{F}, \mathbf{C})$$

the language

$$\mathcal{L}(C) = \mathcal{L}(\mathbf{P}, \mathbf{F}, \mathbf{C} \cup C))$$

is called a **witnessing expansion** of $\mathcal{L}$

The set C of **new** constants and the language $\mathcal{L}(C)$ defined by the **construction** described below

We denote $\mathcal{L}(C)$ as

$$\mathcal{L}(C) = \mathcal{L} \cup C$$

Construction of the witnessing expansion of $\mathcal{L}$

We **define** the set C of **new** constants by constructing (by induction) an infinite sequence

$$C_0, C_1, ..., C_n, \ldots$$

of **sets of constants** together with an infinite sequence

$$\mathcal{L}_0, \mathcal{L}_1, ..., \mathcal{L}_n, \ldots$$

of **languages** as follows

$$C_0 = \emptyset \quad \text{and} \quad \mathcal{L}_0 = \mathcal{L} \cup C_0 = \mathcal{L}$$

We denote by

$$A[x]$$

the fact that the formula A has **exactly one** free variable

Henkin's Witnessing Expansion

For each such a formula $A[x]$ we introduce a distinct **new constant** denoted by

$$c_{A[x]}$$

We **define**

$$C_1 = \{c_{A[x]} : \ A[x] \in \mathcal{L}_0 \} \quad \text{and} \quad \mathcal{L}_1 = \mathcal{L} \cup C_1$$

Assume that we have already defined the set C_n of constants and the language $\mathcal{L}_n$

To each formula $A[x]$ of $\mathcal{L}_n$ which **is not** already a formula of $\mathcal{L}_{n-1}$ we assign distinct **new** constant symbol

$$c_{A[x]}$$

Henkin's Witnessing Expansion

We write it informally as $A[x] \in (\mathcal{L}_n - \mathcal{L}_{n-1})$ to denote that $A[x]$ of $\mathcal{L}_n$ which **is not** already a formula of $\mathcal{L}_{n-1}$

We define

$$C_{n+1} = C_n \cup \{c_{A[x]} : \ A[x] \in (\mathcal{L}_n - \mathcal{L}_{n-1})\}$$

$$\mathcal{L}_{n+1} = \mathcal{L} \cup C_{n+1}$$

We put

$$(*) \quad C = \bigcup C_n \quad \text{and} \quad \mathcal{L}(C) = \mathcal{L} \cup C$$

For any formula $A(x)$, a constant $c_{A[x]} \in C$ as defined by $(*)$ is called a **witnessing constant**

Reduction to Propositional Logic Theorem

Definition(Henkin Axioms)

The following **sentences**

H1 $(\exists x A(x) \Rightarrow A(c_{A[x]}))$

H2 $(A(c_{\neg A[x]}) \Rightarrow \forall x A(x))$

are called **Henkin axioms**

The informal idea behind the Henkin axioms is the following

The axiom **H1** says:

If $\exists x A(x)$ *is* **true** *in a structure, choose an element a satisfying* $A(x)$ *and give it a* **new name** $c_{A[x]}$

The axiom **H2** says:

If $\forall x A(x)$ *is* **false***, choose a counter example* b *and call it by a* **new name** $c_{\neg A[x]}$

Quantifiers Axioms

Definition (Quantifiers Axioms)

The following **sentences**

Q1 $(\forall x A(x) \Rightarrow A(t))$

where t is a closed term of $\mathcal{L}(C)$

Q2 $(A(t) \Rightarrow \exists x A(x))$

where t is a closed term of $\mathcal{L}(C)$

re called **quantifiers axioms**

Observe that the quantifiers axioms **Q1, Q2** obviously are predicate tautologies

Henkin Set

Henkin Set

Any set of **sentences** of $\mathcal{L}(C)$ which are either Henkin axioms or quantifiers axioms is called the **Henkin set** and denoted by

$$S_{Henkin}$$

The sentences of S_{Henkin} are obviously **not true** in every $\mathcal{L}(C)$-structure

But we are going to show now that

every $\mathcal{L}$-structure can be transformed into an $\mathcal{L}(C)$-structure which is a **model** of S_{Henkin}

Before we do so we need to introduce two new notions

Reduct and Expansion

Given two languages $\mathcal{L}$ and $\mathcal{L}'$ such that

$$\mathcal{L} \subseteq \mathcal{L}'$$

Let $\mathcal{M}' = [M, I']$ be a structure for $\mathcal{L}'$. The structure

$$\mathcal{M} = [M, I' \mid \mathcal{L}]$$

is called the **reduct** of $\mathcal{M}'$ to the language $\mathcal{L}$ and $\mathcal{M}'$ is called the **expansion** of $\mathcal{M}$ to the language $\mathcal{L}'$

Thus the reduct of $\mathcal{M}'$ and the expansion of $\mathcal{M}$ are the same except that $\mathcal{M}'$ **assigns** meanings to the symbols in $\mathcal{L} - \mathcal{L}'$

Reduct and Expansion Lemma

Lemma

Let $\mathcal{M} = [M, I]$ be any structure for the language $\mathcal{L}$ and let $\mathcal{L}(C)$ be the **witnessing expansion** of $\mathcal{L}$

There is an **expansion** $\mathcal{M}' = [M, I']$ of $\mathcal{M} = [M, I]$ such that

$$\mathcal{M}' \models S_{Henkin}$$

Proof

In order to define the **expansion** of $\mathcal{M}$ to $\mathcal{M}'$ we have to **define** the interpretation I' for the symbols of the language $\mathcal{L}(C) = \mathcal{L} \cup C$, such that I' **restricted** to $\mathcal{L}$ is the interpretation I, i.e. such that

$$I' \mid \mathcal{L} = I$$

Lemma Proof

This means that we have to define $c_{I'}$ for all $c \in C$

By the definition, $c_{I'} \in M$, so this also means that we have to **assign** the elements of M to all constants $c \in C$ in such a way that the resulting expansion is a **model** for all sentences from S_{Henkin}

The **quantifier axioms** are predicate tautologies so they are going to be **true** regardless

so we have to worry only about the **Henkin axioms**

Lemma Proof

Observe now that if the **Lemma** holds for the Henkin axiom

H1 $(\exists x A(x) \Rightarrow A(c_{A[x]}))$

then it must hold for the axiom **H2**

Namely, let's consider the axiom **H2**:

$$(A(c_{\neg A[x]}) \Rightarrow \forall x A(x))$$

Assume that $A(c_{\neg A[x]})$ is **true** in the expansion $\mathcal{M}'$, i.e. that

$$\mathcal{M}' \models A(c_{\neg A[x]}) \quad \text{and that} \quad \mathcal{M}' \not\models \forall x A(x)$$

This means that

$$\mathcal{M}' \models \neg \forall x A(x)$$

and by the De Morgan Laws

$$\mathcal{M}' \models \exists x \neg A(x)$$

Lemma Proof

But we have assumed that $\mathcal{M}'$ is a **model** for **H1**

In particular

$$\mathcal{M}' \models (\exists x \neg A(x) \Rightarrow \neg A(c_{\neg A[x]}))$$

and hence as $\mathcal{M}' \models \exists x \neg A(x)$ we have that

$$\mathcal{M}' \models \neg A(c_{\neg A[x]})$$

This **contradicts** the assumption that

$$\mathcal{M}' \models A(c_{\neg A[x]})$$

Thus we **proved** that

if $\mathcal{M}'$ is a **model** for all axioms of the type **H1**, it is also a **model** for all axioms of the type **H2**

Lemma Proof

We **define** now $c_{I'}$ for all $c \in C$, where

$$C = \bigcup C_n$$

We do so by **induction** on n

Base case: $n = 1$ and $c_{A[x]} \in C_1$

By definition,

$$C_1 = \{c_{A[x]} : \; A[x] \in \mathcal{L}\}$$

In this case we have that $\exists x A(x) \in \mathcal{L}$ and hence the notion

$$\mathcal{M} \models \exists x A(x)$$

is well defined, as $\mathcal{M} = [M, I]$ is the structure for the language $\mathcal{L}$

Lemma Proof

As we consider arbitrary structure $\mathcal{M}$, there are two possibilities:

$$\mathcal{M} \models \exists x A(x) \quad \text{or} \quad \mathcal{M} \not\models \exists x A(x)$$

We **define** $c_{I'}$, for all $c \in C_1$ as follows

If $\mathcal{M} \models \exists x A(x)$, then $(\mathcal{M}, v') \models A(x)$ for certain $v'(x) = a \in M$. We set

$$(c_{A[x]})_{I'} = a$$

If $\mathcal{M} \not\models \exists x A(x)$, we set

$$(c_{A[x]})_{I'} \text{ arbitrarily}$$

Lemma Proof

This makes all the positive **H1** type Henkin axioms about the $c_{A[x]} \in C_1$ **true**, i.e.

$$\mathcal{M} = (M, I) \models (\exists x A(x) \Rightarrow A(c_{A[x]}))$$

But once $c_{A[x]} \in C_1$ are all interpreted in M, then the notion

$$\mathcal{M}' \models A$$

is defined for all formulas $A \in \mathcal{L} \cup C_1$

We carry the same argument and **define** $c_{I'}$, for all $c \in C_2$ and so on ...

Lemma Proof

The **inductive step** is performed in the exactly the same way as the one above

Observe that we have aleady we **proved** that
if $\mathcal{M}'$ is a **model** for all axioms of the type **H1**, it is also a **model** for all axioms of the type **H2**

Hence this ends the proof of the **Lemma**

Canonical Structure

Definition (Canonical Structure)

Given a structure $\mathcal{M} = [M, I]$ for the language $\mathcal{L}$

The **expansion**

$$\mathcal{M}' = [M, I']$$

of $\mathcal{M} = [M, I]$ is called a **canonical structure** for $\mathcal{L}(C)$ if all $a \in M$ are **denoted** by some $c \in C$. That is

$$M = \{c_{I'} : c \in C\}$$

Now we are ready to state and prove a theorem that provides the essential step in the proof of the **completeness theorem** for predicate logic.

The Reduction to Propositional Logic

Theorem (The Reduction Theorem)

Let $\mathcal{L} = \mathcal{L}(\mathbf{P}, \mathbf{F}, \mathbf{C})$ be a predicate language and let $\mathcal{L}(C) = \mathcal{L}(\mathbf{P}, \mathbf{F}, \mathbf{C} \cup C)$ be a witnessing expansion of $\mathcal{L}$

For any set S of sentences of $\mathcal{L}$ the following conditions are equivalent

(i) S has a **model**, i.e. there is a structure $\mathcal{M} = [M, I]$ for the language $\mathcal{L}$ such that $\mathcal{M} \models A$ for all $A \in S$

(ii) There is a **canonical structure** $\mathcal{M} = [M, I]$ for $\mathcal{L}(C)$ which is a **model** for S, i.e. such that $\mathcal{M} \models A$ for all $A \in S$

(iii) The set $S \cup S_{Henkin} \cup EQ$ is **consistent** in sense of propositional logic, where EQ denotes the equality axioms $E1 - E5$

Reduction Theorem Proof

Proof

We have to prove that the conditions **(i), (ii), (iii)** of the theorem are equivalent

The implication **(ii)** $\rightarrow$ **(i)** is immediate

The implication (**i**) $\rightarrow$ (**iii**) follows from the **Lemma**

We have to prove only the implication **(iii)** $\rightarrow$ **(ii)**

Assume **(iii)** , i.e. that the set $S \cup S_{Henkin} \cup EQ$ is consistent in sense of propositional logic and let v be a truth assignment to the prime sentences of $\mathcal{L}(C)$, such that

$$v^*(A) = T \quad \text{for all} \quad A \in S \cup S_{Henkin} \cup EQ$$

Reduction Theorem Proof

To prove the theorem, we construct a **canonical** $\mathcal{L}(C)$ structure $\mathcal{M} = [M, I]$ such that, for all sentences A of $\mathcal{L}(C)$,

$$\mathcal{M} \models A \quad \text{if and only if} \quad v^*(A) = T$$

By assumption, the truth assignment v is a propositional **model** for the set S_{Henkin}, so v^* satisfies the following conditions:

(i) $v^*(\exists x A(x)) = T$ if and only if $v^*(A(c_{A[x]})) = T$

(ii) $v^*(\forall x A(x)) = T$ if and only if $v^*(A(t)) = T$

for all **closed** terms t of $\mathcal{L}(C)$

Reduction Theorem Proof

The conditions (i) and (ii) allow us to construct the **canonical** $\mathcal{L}(C)$ **model** $\mathcal{M} = [M, I]$ out of the constants in C in the following way

To define $\mathcal{M} = [M, I]$ we must

(1.) specify the **universe** M of $\mathcal{M}$

(2.) define, for each n-ary predicate symbol $R \in \mathbf{P}$, the **interpretation** R_I as an n-argument relation in M

(3.) define, for each n-ary function symbol $f \in \mathbf{F}$, the **interpretation** $f_I : M^n \to M$, and

(4.) define, for each constant symbol c of $\mathcal{L}(C)$, i.e. $c \in \mathbf{C} \cup C$, its **interpretation** as element $c_I \in M$

Reduction Theorem Proof

The construction of the structure

$$\mathcal{M} = [M, I]$$

must be such that the condition

$$\text{(CM)} \qquad \mathcal{M} \models A \quad \text{if and only if} \quad v^*(A) = T$$

holds for for all sentences A of $\mathcal{L}(C)$

This condition (CM) tells us how to construct the definitions (1.) - (4.) above

Reduction Theorem Proof

Here are the definitions

(1.) **Definition** of the **universe** M of $\mathcal{M}$

In order to define the universe M we **first** define a relation $\approx$ on C as follows

$$c \approx d \quad \text{if and only if} \quad v(c = d)) = T$$

The **equality axioms** EQ guarantee that the relation $\approx$ is **equivalence** relation on C. Here is the proof

Reflexivity of $\approx$

All equality axioms EQ are predicate **tautologies**, so $v(c = d)) = T$ by axiom E1 and we have

$$c \approx c \quad \text{for all} \quad c \in C$$

Reduction Theorem Proof

Symmetry condition

$$\text{if } \; c \approx d, \quad \text{then} \quad d \approx c$$

holds by axiom E2

Assume $c \approx d$, by definition $v(c = d)) = T$

By axiom E2

$$v^*((c = d \Rightarrow d = c)) = v(c = d) \Rightarrow v(d = c) = T$$

i.e. $T \Rightarrow v(d = c) = T$

This is possible **only if** $v(d = c) = T$

This proves that $d \approx c$

Reduction Theorem Proof

We prove **transitivity** in a similar way

Assume now that $c \approx d$ and $d \approx e$

By the **axiom** E3 we have that

$$v^*(((c = d \cap d = e) \Rightarrow c = e)) = T$$

Since $v(c = d)) = T$ and $v(d = e)) = T$ by the assumption $c \approx d$ and $d \approx e$, we evaluate

$v^*((c = d \cap d = e) \Rightarrow c = e) = (T \cap T \Rightarrow c = e) =$

$(T \Rightarrow c = e) = T$ and get that $(c = e) = T$ and hence

$$d \approx e$$

Reduction Theorem Proof

We denote by $[c]$ the equivalence class of c and we define the **universe** M of $\mathcal{M}$ as

$$M = \{[c] : \ c \in C\}$$

(2.) **Definition** of $R_I \subseteq M^n$

Let M be the the **universe** defined above

We define $R_I \subseteq M^n$ as follows

$$([c_1], [c_2], \ \ldots, [c_n]) \in R_I \ \text{ if and only if } \ v(R(c_1, c_2, \ \ldots, c_n)) = T$$

We have to prove now that R_I is *well defined* by the condition above

Reduction Theorem Proof

In order to prove that R_I is **well defined** we must verify:

$$\text{if } [c_1] = [d_1], \ldots, [c_n] = [d_n] \text{ and } ([c_1], [c_2], \ldots, [c_n]) \in R_I$$

$$\text{then } ([d_1], [d_2], \ldots, [d_n]) \in R_I$$

We have by the **axiom** E4 that

$$v^*(((c_1 = d_1 \cap ... c_n = d_n) \Rightarrow (R(c_1, .., c_n) \Rightarrow R(d_1, .., d_n)))) = T$$

By the assumption $[c_1] = [d_1], \ldots, [c_n] = [d_n]$ we have that

$$v(c_1 = d_1) = T, \ldots, v(c_n = d_n) = T$$

Reduction Theorem Proof

By the assumption $([c_1], [c_2], \ldots, [c_n]) \in R_I$, we have that

$$v(R(c_1, ..., c_n)) = T$$

Hence the **axiom** E4 condition becomes

$$(T \Rightarrow (T \Rightarrow v(R(d_1, ..., d_n)))) = T$$

It holds only when $v(R(d_1, ..., d_n)) = T$

and so we **proved** that

$$([d_1], [d_2], \ldots, [d_n]) \in R_I$$

Reduction Theorem Proof

(3.) **Definition** of $f_I : \ M^n \rightarrow M$

Let $c_1, \ c_2, \ \ldots, \ c_n \in C$ and $f \in \mathbf{F}$

We **claim** that **there is** $c \in C$ such that

$$f(c_1, \ c_2, \ \ldots, \ c_n) = c \quad \text{and} \quad v(f(c_1, \ c_2, \ \ldots, \ c_n) = c) = T$$

For consider the formula

$$A[x] \quad \text{given by} \quad f(c_1, \ c_2, \ \ldots, \ c_n) = x$$

$$\text{If} \quad v^*(\exists x A(x)) = v^*(\exists x \, f(c_1, \ c_2, \ \ldots, \ c_n) = x) = T$$

we want to **prove**

$$v^*(A(c_{A[x]})) = T \quad \text{i.e.} \quad v(f(c_1, \ c_2, \ \ldots, \ c_n) = c_A) = T$$

Reduction Theorem Proof

So suppose that $v(f(c_1,\ c_2,\ \ldots,\ c_n) = c_A) = F$
But one member of he Henkin set S_{Henkin} is the sentence

$$(A(f(c_1,\ c_2,\ \ldots,\ c_n)) \Rightarrow \exists x A(x))$$

so we must have that

$$v^*(A(f(c_1,\ c_2,\ \ldots,\ c_n))) = F$$

But this says that v assigns F to the atomic sentence

$$f(c_1,\ c_2,\ \ldots,\ c_n) = f(c_1,\ c_2,\ \ldots,\ c_n)$$

Reduction Theorem Proof

By the axiom E1 $v(c_i = c_i) = T$ for $i = 1, 2 \dots n$

By axiom E5 we have that

$$(v^*(c_1 = c_1 \cap \dots\ c_n = c_n) \Rightarrow v^*(f(c_1,\ \dots,\ c_n) = f(c_1, \dots, c_n))) = T$$

This means that $T \Rightarrow F = T$ and this **contradiction** proves there is $c \in C$ such that

$$f(c_1,\ c_2,\ \dots,\ c_n) = c \quad \text{and} \quad v(f(c_1,\ c_2,\ \dots,\ c_n) = c) = T$$

We hence **define**

$$f_I(([c_1], \dots, [c_n]) = [c] \text{ for } c \text{ such that } v(f(c_1, \dots,\ c_n) = c) = T$$

The argument similar to the one used in (2.) proves that f_I is **well defined**

Reduction Theorem Proof

(4.) **Definition** of $c_I \in M$

For any $c \in C$ we take

$$c_I = [c]$$

If $d \in \mathbf{C}$, then an argument similar to that used on (3.) shows that **there is** $c \in C$ such that $v(d = c) = T$, i.e. $d \approx c$, so we put

$$d_I = [c]$$

We hence completed the construction of the **canonical structure** $\mathcal{M} = [M, I]$

Reduction Theorem Proof

Observe that directly from the definition of the **canonical structure** $\mathcal{M} = [M, I]$ we have that the property

$$\text{(CM)} \quad \mathcal{M} \models A \quad \text{if and only if} \quad v^*(A) = T$$

holds for **atomic** propositional sentences, i.e. we proved that

$$\mathcal{M} \models B \text{ if and only if } v^*(B) = T \text{ for sentences } B \in \mathcal{P}$$

To complete the proof of the **Reduction Theorem** we prove now that the property (CM) holds for all other sentences

We carry the proof by induction on length of formulas

The Base Case is already proved. The Inductive Case is as follows

Reduction Theorem Proof

Case of propositional connectives is similar to the case of a formula $(A \cap B)$ below

$$\mathcal{M} \models (A \cap B) \quad \text{if and only if} \quad \mathcal{M} \models A \quad \text{and} \quad \mathcal{M} \models B$$

It follows directly from the **satisfaction** definition

$$\mathcal{M} \models A \text{ and } \mathcal{M} \models B \quad \text{if and only if} \quad v^*(A) = T \quad \text{and} \quad v^*(B) = T$$

$$\text{if and only if} \quad v^*(A \cap B) = T$$

It holds by the **induction** hypothesis

We proved

$$\mathcal{M} \models (A \cap B) \text{ if and only if } v^*(A \cap B) = T$$

for all sentences A, B of $\mathcal{L}(C)$

Reduction Theorem Proof

We prove now the case of a sentence B of the form

$$\exists x A(x)$$

We want to show that

$$\mathcal{M} \models \exists x A(x) \text{ if and only if } v^*(\exists x A(x)) = T$$

Let $v^*(\exists x A(x)) = T$

Then there is a c such that $v^*(A(c)) = T$, so by induction hypothesis, $\mathcal{M} \models A(c)$ so by definition

$$\mathcal{M} \models \exists x A(x)$$

Reduction Theorem Proof

On the other hand, if $v^*(\exists x A(x)) = F$, then by $S_{Henking}$ quantifier axiom **Q2** we have that

$$v^*(A(t)) = F$$

for all closed terms t of $\mathcal{L}(C)$. In particular, for every $c \in C$

$$v^*(A(c)) = F$$

By induction hypothesis,

$$\mathcal{M} \models \neg A(c) \quad \text{for all} \quad c \in C$$

Since every element of M is **denoted** by some $c \in C$ we have that

$$\mathcal{M} \models \neg \exists x A(x)$$

The **proof** of the case of a sentence B of the form $\forall x A(x)$ is similar and is left as and exercise

This ends the proof of the **Reduction Theorem**

Compactness Theorem
and
Löwenheim-Skolem Theorem

Compactness and Löwenheim-Skolem Theorems

The Reduction to Propositional Logic Theorem provides a powerful **method** of constructing **models** of theories out of **symbols** in a form of canonical models

It also gives us immediate **proofs** of the two important theorems: Compactness Theorem for the **predicate** logic and the Löwenheim-Skolem Theorem

Compactness theorem

Let S be any set of **predicate** formulas of $\mathcal{L}$

The set S has a **model** if and only if any finite subset S_0 of S has a **model**

Proof

Assume that S is a set of predicate formulas such that every finite subset S_0 of S has a **model**

We need to **show** that S has a **model**

The implication (**iii**) $\rightarrow$ (**i**) of the Reduction Theorem says:

" If The set $S \cup S_{Henkin} \cup EQ$ is **consistent** in sense of propositional logic, then S has a **model**"

So **showing** that S has a **model** this is equivalent to proving that $S \cup S_{Henkin} \cup EQ$ is **consistent** in the sense of propositional logic

Compactness Theorem

By already proved **Compactness Theorem** for propositional logic of $\mathcal{L}$, it suffices to prove that for every finite subset $S_0 \subset S$, the set $S_0 \cup S_{Henkin} \cup EQ$ has a **model**

This follows from the assumption that S is a set such that every finite subset S_0 of S has a **model** and the implication $(i) \rightarrow (iii)$ of the **Reduction Theorem** that says:

" if S_0 has a **model**, then the set $S_0 \cup S_{Henkin} \cup EQ$ is consistent, i.e. has a **model**

Löwenheim-Skolem Theorem

Let κ be an infinite cardinal

Let $\mathcal{L}$ be a **predicate** language with the **alphabet** $\mathcal{A}$ such that $card(\mathcal{A}) \leq \kappa$

Let Γ be a set of at most κ **formulas** of the $\mathcal{L}$

If the set S has a **model**, then there is a **model**

$$\mathcal{M} = [M, I]$$

of S such that

$$cardM \leq \kappa$$

Proof

Let $\mathcal{L}$ be a predicate language with the alphabet $\mathcal{A}$ such that $card(\mathcal{A}) \leq \kappa$

Obviously, $card(\mathcal{F}) \leq \kappa$

By the definition of the witnessing expansion $\mathcal{L}(C)$ of $\mathcal{L}$, $C = \bigcup_n C_n$ and for each n, $card(C_n) \leq \kappa$. So also $cardC \leq \kappa$

Thus any canonical structure for $\mathcal{L}(C)$ has $\leq \kappa$ elements

By the implication $(i) \rightarrow (ii)$ of the **Reduction Theorem** that says: " if there is a model of S , then there is a canonical structure $\mathcal{M} = [M, I]$ for $\mathcal{L}(C)$ which is a **model** for S"

S has a model (canonical structure) with $\leq \kappa$ elements

This ends the proof