cse541
LOGIC for Computer Science

Professor Anita Wasilewska

LECTURE 9

Chapter 9
Hilbert Proof Systems
Completeness of Classical Predicate Logic

PART 1: **Reduction Predicate Logic to Propositional Logic**

Proofs of Completeness Theorem

There are several quite distinct **approaches** to the proof of the **completeness** theorem

They correspond to the ways of thinking about **proofs**

Within each of these **approaches** there are endless variations in exact formulation, corresponding to the choice of **methods** we want to use to prove the **completeness** theorem

Different basic approaches are important, though, for they lead to different **applications**

Proofs of Completeness Theorem

We have already presented two of the **approaches** for the propositional logic, namely

Hilbert style formalizations (proof systems) in chapter 5 and

Gentzen style **automated** proof systems in chapter 6

We have also presented, for each of these approaches several **methods** of proving the **completeness** theorem:

two very different proofs for Hilbert style proof systems in chapter 5 and

constructive proofs for several **automated** Gentzen style proof systems in chapter 6

Proofs of Completeness Theorem

There are many proofs of the **completeness** theorem for predicate (first order) logic

We present here in a great detail, a version of **Henkin's** proof as included in a classic

Handbook of Mathematical Logic, North Holland Publishing Company- Amsterdam - Newy York -Oxford (1977)

It contains a **method** for reducing certain problems of **first order** logic back to problems about **propositional** logic

Proofs of Completeness Theorem

We follow **Henkin method** and give independent proof of **compactness** theorem for propositional logic

As the next steps we prove the most important, classical for logic theorems:

Reduction to Propositional Logic Theorem, *Compactness Theorem for first-order logic*, *Löwenheim-Skolem Theorem* and *Gödel Completeness Theorem*

They fall out of the **Henkin method**

Proofs of Completeness Theorem

We choose this particular proof of **completeness** not only for it being one of the oldest and most classical, but also for its **connection** with the propositional logic

Moreover, the proof of the **compactness** theorem is based on **semantical** version of **syntactical** notions and techniques crucial to the second proof of **completeness** theorem for propositional logic covered in chapter 5 and hence is familiar to the reader

Reduction Predicate Logic to Propositional Logic

Reduction Predicate Logic to Propositional Logic

Let $\mathcal{L} = \mathcal{L}(\mathbf{P}, \mathbf{F}, \mathbf{C})$ be a first order language with equality

We assume that the sets **P, F, C** are infinitely enumerable

We also assume that it has a full set of propositional connectives, i.e.

$$\mathcal{L} = \mathcal{L}_{\{\neg, \cap, \cup, \Rightarrow\}}(\mathbf{P}, \mathbf{F}, \mathbf{C})$$

Our **goal** now is to define a **propositional logic** within

$$\mathcal{L} = \mathcal{L}(\mathbf{P}, \mathbf{F}, \mathbf{C})$$

We do it in a sequence of steps

Reduction Predicate Logic to Propositional Logic

First we define a special subset $\mathcal{PF}$ of formulas of $\mathcal{L}$ called a set of all **propositional formulas** of $\mathcal{L}$

Intuitively, these are formulas of $\mathcal{L}$ which are **direct** propositional combination of **simpler formulas**, that are atomic formulas or formulas beginning with quantifiers

These simpler formulas are called **prime formulas** and are formally defined as follows.

Prime Formulas

Definition

Prime formula of $\mathcal{L}$ is any formula from the set

$$\mathcal{P} = A\mathcal{F} \cup \{\forall xB : \quad B \in \mathcal{F}\} \cup \{\exists xB : \quad B \in \mathcal{F}\}$$

where the set $A\mathcal{F}$ is the set of all **atomic** formulas of $\mathcal{L}$

The set

$$\mathcal{P} \subseteq \mathcal{F}$$

is called a set of all **prime formulas** of $\mathcal{L}$

Prime Formulas

Example

The following are **prime** formulas

$R(t_1, t_2), \ \forall x(A(x) \Rightarrow \neg A(x)), \ (c = c), \ \exists x(Q(x, y) \cap \forall y A(y))$

The following **are not** prime formulas.

$$(R(t_1, t_2) \Rightarrow (c = c)), \ (R(t_1, t_2) \cup \forall x(A(x) \Rightarrow \neg A(x))$$

Given a set $\mathcal{P}$ of **prime** formulas we define in a standard way the set $P\mathcal{F}$ of **propositional** formulas of $\mathcal{L}$ as follows

Propositional Formulas of $\mathcal{L}$

Definition (Propositional Formulas)

Let $\mathcal{F}$, $\mathcal{P}$ be sets of all formulas and prime formulas of $\mathcal{L}$, respectively

The **smallest** set $P\mathcal{F} \subseteq \mathcal{F}$, such that

(i) $\mathcal{P} \subseteq P\mathcal{F}$

(ii) If $A, B \in P\mathcal{F}$, then $(A \Rightarrow B), (A \cup B), (A \cap B)$ and $\neg A \in P\mathcal{F}$

is called a set of all **propositional formulas** of the predicate language $\mathcal{L}$

The set $\mathcal{P}$ is called the set of all **atomic propositional** formulas of $\mathcal{L}$

Propositional Semantics for $\mathcal{L}$

We define propositional semantics for propositional formulas in $P\mathcal{F}$ as follows

Definition (Truth assignment)

Let $\mathcal{P}$ be a set of **atomic propositional** formulas of $\mathcal{L}$ and $\{T, F\}$ be the set of logical values "true" and "false"

Any function

$$v : \mathcal{P} \longrightarrow \{T,\ F\}$$

is called a **truth assignment** in $\mathcal{L}$

Propositional Semantics for $\mathcal{L}$

We **extend** v to the set $P\mathcal{F}$ of all **propositional** formulas by defining the mapping

$$v^* : P\mathcal{F} \longrightarrow \{T, F\}$$

as follows

$v^*(A) = v(A)$ for $A \in \mathcal{P}$

and for any $A, B \in P\mathcal{F}$

$v^*(A \Rightarrow B) = v^*(A) \Rightarrow v^*(B)$

$v^*(A \cup B) = v^*(A) \cup v^*(B)$

$v^*(A \cap B) = v^*(A) \cap v^*(B)$

$v^*(\neg A) = \neg v^*(A)$

Propositional Model, Tautology

Definition

A truth assignment $v : \mathcal{P} \longrightarrow \{T, F\}$ is called a **propositional model** for a formula $A \in P\mathcal{F}$ if and only if $v^*(A) = T$

Definition

For any formula $A \in P\mathcal{F}$

$A \in P\mathcal{F}$ is a **propositional tautology** of $\mathcal{L}$ if and only if $v^*(A) = T$ for all $v : \mathcal{P} \longrightarrow \{T, F\}$

For the sake of simplicity we will often say model, tautology instead propositional model, propositional tautology when there is no confusion

Consistent Inconsistent Sets

Definition

Given a set S of propositional formulas

We say that v is a **model** for the set S if and only if

v is a model for all formulas $A \in S$

Definition (Consistent Set)

A set $S \subseteq \mathcal{PF}$ of propositional formulas of $\mathcal{L}$ is consistent if it has a (propositional) model

Definition (Inconsistent Set)

A set $S \subseteq \mathcal{PF}$ of propositional formulas of $\mathcal{L}$ is inconsistent if it **does not** have a (propositional) model

Compactness Theorem

Compactness Theorem for Propositional Logic of $\mathcal{L}$

A set $S \subseteq P\mathcal{F}$ of propositional formulas of $\mathcal{L}$ is **consistent** if and only if every finite subset of S is **consistent**

Proof

Assume that S is a **consistent** set. By definition, it has a **model**. Its model is also a model for **all** its subsets, including all finite subsets

Hence **all** its finite subsets are **consistent**

Compactness Theorem

To prove the converse implication, i.e. the **nontrivial** half of the **Compactness Theorem** we write it in a slightly modified form. To do so, we introduce the following definition

Definition

Any set S such that **all** its finite subsets are consistent is called **finitely consistent**

We re-write the **Compactness Theorem** as follows.

A set S of propositional formulas of $\mathcal{L}$ is **consistent** if and only if S is **finitely consistent**

Compactness Theorem

The nontrivial half of the **Compactness Theorem** still to be proved is now stated now as follows

Every **finitely consistent** set of propositional formulas of $\mathcal{L}$ is **consistent**

The **proof** consists of the following four steps

S1 We introduce the notion of a maximal finitely consistent set

S2 We show that every maximal finitely consistent set is consistent by constructing its **model**

Compactness Theorem

S3 We show that every **finitely consistent** set S can be extended to a **maximal** finitely consistent set S^*, we show that for every finitely consistent set S there is a set S^*, such that $S \subseteq S^*$ and S^* is **maximal** finitely consistent

S4 We use steps **S2** and **S3** to justify the following **reasoning**

Given a finitely consistent set S. We bf extend it, via construction to be defined in the step **S3** to a **maximal** finitely consistent set S^*

By the **S2**, the set S^* is **consistent** and so is the set S

This ends the proof of the **Compactness Theorem**

Proof of Step **S1**

Here are the details and proofs needed for completion of steps **S1** - **S4**

Step **S1** We introduce the following definition

Definition of Maximal Finitely Consistent Set (MFC)

Any set

$$S \subseteq P\mathcal{F}$$

is maximal finitely consistent if it is finitely consistent and for every formula A,

$$\text{either} \quad A \in S \quad \text{or} \quad \neg A \in S$$

We use notation MFC for maximal finitely consistent set, and FC for the finitely consistent set

Proof of Step **S2**

Step **S2** consists of proving the following Lemma

MFC Lemma

Any MFC set is consistent

Proof

Given a MFC set denoted by S^*

We prove consistency of S^* by constructing **model** for it

It means we are going to **construct** a truth assignment

$$v : \mathcal{P} \longrightarrow \{T, F\}$$

such that for **all** $A \in S^*$

$$v^*(A) = T$$

Proof of Step **S2**

Observe that directly from the definition we have the following property of the the MFC sets.

Property

For any MFC set S^* and for every $A \in P\mathcal{F}$,

exactly one of the formulas $A, \neg A$ belongs to S^*

In particular, for any **atomic** formula $P \in \mathcal{P}$, we have that exactly **one** of formulas $P, \neg P$ belongs to S^*

This justifies the correctness of the following definition

Proof of Step **S2**

Definition

For any MFC set S^*, a mapping

$$v : \mathcal{P} \longrightarrow \{T, F\}$$

such that

$$v(P) = \begin{cases} T & \text{if } P \in S^* \\ F & \text{if } P \notin S^* \end{cases}$$

is called a truth assignment **defined** by S^*

Proof of Step **S2**

We extend v to

$$v^* : P\mathcal{F} \longrightarrow \{T, F\}$$

in a usual, standard way and we prove that the truth assignment v is a **model** for S^*

It means we show for any $A \in P\mathcal{F}$,

$$v^*(A) = \begin{cases} T & \text{if } A \in S^* \\ F & \text{if } A \notin S^* \end{cases}$$

We **prove** it by induction on the degree of the formula A as follows.

Proof of Step **S2**

The **base case** of **atomic** formula $P \in \mathcal{P}$ follows immediately from the definition of v

Inductive Case: $A = \neg C$

1. Assume that $A \in S^*$

This means $\neg C \in S^*$ and by the MFC Property we have that $C \notin S^*$. So by the inductive assumption $v^*(C) = F$ and we get

$$v^*(A) = v^*(\neg C) = \neg v^*(C) = \neg F = T$$

2. Assume now that $A \notin S^*$.

By MFC Property we have that $C \in S^*$

By the inductive assumption $v^*(C) = T$ and

$$v^*(A) = v^*(\neg C) = \neg v^*(T) = \neg T = F$$

Proof of Step **S2**

We proved that for any formula $A \in \mathcal{PF}$,

$$v^*(\neg A) = \begin{cases} T & \text{if } \neg A \in S^* \\ F & \text{if } \neg A \notin S^* \end{cases}$$

Inductive Case: $A = (B \cup C)$

1. Assume that $A \in S^*$. i.e. $(B \cup C) \in S^*$

It is enough to prove that in this case $B \in S^*$ or $C \in S^*$, because then from the inductive assumption $v^*(B) = T$ and

$v^*(B \cup C) = v^*(B) \cup v^*(C) = T \cup v^*(C) = T$ for any C

The case $C \in S^*$ is similar

Proof of Step **S2**

Assume that $(B \cup C) \in S^*$, $B \notin S^*$ and $C \notin S^*$

Then by MFC Property we have that $\neg B \in S^*$, $\neg C \in S^*$ and consequently the set

$$\{(B \cup C), \neg B, \neg C\}$$

is a finite inconsistent subset of S^*, what **contradicts** the fact that S^* is finitely consistent

2. Assume now that $(B \cup C) \notin S^*$

By MFC Property, $\neg(B \cup C) \in S^*$ and by already proven **case** of $A = \neg C$ we have that $v^*(\neg(B \cup C)) = T$

But $v^*(\neg(B \cup C)) = \neg v^*((B \cup C)) = T$

This means that $v^*((B \cup C)) = F$, what **ends** the proof of this case

Step **S3**

The remaining cases of $A = (B \cap C)$ and $A = (B \Rightarrow C)$ are similar to the above and are left to the as an exercise

This **ends** the proof of MFC **Lemma** and completes the step **S2**

S3: Maximal finitely consistent (MFC) extension S^*

Given a finitely consistent set S

We **construct** the MFC **extension** S^* of the set S as follows

Proof of Step **S3**

The set of all formulas of $\mathcal{L}$ is infinitely countable and so is the set $P\mathcal{F}$. We assume that the set $P\mathcal{F}$ of all **propositional** formulas form a one-to-one sequence

$$(*) \quad A_1, A_2, \ldots, A_n, \ldots,$$

We **define** a chain

$$(**) \quad S_0 \subseteq S_1 \subseteq S_2, \ldots, \subseteq S_n \subseteq, \ldots$$

of **extensions** of the set S as follows

$$S_0 = S$$

$$S_{n+1} = \begin{cases} S_n \cup \{A_n\} & \text{if } S_n \cup \{A_n\} \text{ is finitely consistent} \\ S_n \cup \{\neg A_n\} & \text{otherwise.} \end{cases}$$

Proof of Step **S3**

We take

$$S^* = \bigcup_{n \in N} S_n$$

Obviously $S \subseteq S^*$ also is MFC as clearly and for every A, either $A \in S^*$ or $\neg A \in S^*$

To complete the **proof** that S^* is MFC set we have to show that it is **finitely** consistent

First, let observe that **if** all sets S_n are **finitely** consistent, **so is** the set $S^* = \bigcup_{n \in N} S_n$. Namely, let

$$S_F = \{B_1, ..., B_k\}$$

be a **finite** subset of S^*

Proof of Step **S3**

This means that there are sets $S_{i_1}, ...S_{i_k}$ in the chain $(**)$ such that

$$B_m \in S_{i_m} \quad \text{for} \quad m = 1, \dots k$$

Let $M = max(i_1, ...i_k)$. Obviously

$$S_F \subseteq S_M$$

and the set S_M is **finitely** consistent as an element of the chain $(**)$. This **proves** that **if** all sets S_n are finitely consistent, so is S^*

Now we have to **prove only** that **all** sets S_n **are** FC (finitely consistent) We carry the proof by induction over the length of the chain

Proof of Step **S3**

Base Case

$S_0 = S$, so it is FC (finitely consistent) by assumption of the Compactness Theorem

Inductive Step

Assume now that S_n is FC (finitely consistent)

We prove that S_{n+1} is FC

We have two cases to consider

Case 1 $S_{n+1} = S_n \cup \{A_n\}$

Then S_{n+1} is FC by the definition of the chain

Case 2 $S_{n+1} = S_n \cup \{\neg A_n\}$

Observe that this can happen only if $S_n \cup \{A_n\}$ is **not** FC, i.e. there is a finite subset $S_n' \subseteq S_n$, such that $S_n' \cup \{A_n\}$ is **not** consistent

Proof of Step **S3**

Suppose now that S_{n+1} is **not** FC

This means that there is a finite subset $S_n'' \subseteq S_n$, such that $S_n'' \cup \{\neg A_n\}$ is **not** consistent

Take $S_n' \cup S_n''$. It is a finite subset of S_n so it is consistent by the inductive assumption

Let v be a **model** of $S_n' \cup S_n''$

Then **one of** $v^*(A), v^*(\neg A)$ **must** be T

This **contradicts** the inconsistency of both

$$S_n' \cup \{A_n\} \quad \text{and} \quad S_n' \cup \{\neg A_n\}$$

Thus, in ether case, S_{n+1} is FC

We hence proved that **all** sets S_n **are** FC (finitely consistent)

Compactness Theorem

This **completes** the proof of the step **S3**

We complete the proof of the **Compactness Theorem** for propositional logic of $\mathcal{L}$ via the following argument as presented in the step **S4**

Given a finitely consistent set S

We **extend** it, via construction defined in the step **S3** to a **maximal** finitely consistent set S^*

By the **S2**, the set S^* is **consistent** and so is the set S

This ends the proof of the **Compactness Theorem**