cse541
LOGIC for Computer Science

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LECTURE 8a

Chapter 8
Classical Predicate Semantics and Proof Systems

PART 2: **Classical Semantics**

Classical Semantics

The notion of **predicate tautology** is much more **complicated** then that of the propositional

Predicate tautologies are also called **valid** formulas, or **laws of quantifiers** to distinguish them from the propositional case

The formulas of a predicate language $\mathcal{L}$ have meaning only when an **interpretation** is given for all its symbols

Classical Semantics

We define an **interpretation** I by interpreting

predicate and functional symbols as a concrete **relation** and **function** defined in a certain set $U \neq \emptyset$

Constants symbols are interpreted as **elements** of the set U

The set U is called the **universe** of the interpretation I

These two items specify a **structure**

$$\mathbf{M} = (U, I) \quad \text{for the language} \quad \mathcal{L}_{CON}(\mathbf{P}, \mathbf{F}, \mathbf{C})$$

Classical Semantics

The **semantics** for a first order (predicate) language $\mathcal{L}$ in general, and for the first order classical logic in particular, is **defined**, after Tarski (1936), in terms of

the **structure** $\mathbf{M} = [U, I]$

an **assignment** s of $\mathcal{L}$

a **satisfaction relation** $(\mathbf{M}, s) \models A$ between structures, assignments and formulas of $\mathcal{L}$

The definition of the structure $\mathbf{M} = [U, I]$ and the assignment s of $\mathcal{L}$ is **common** for different predicate languages and for different semantics and we define them as follows.

Structure Definition

Definition

Given a predicate language

$$\mathcal{L} = \mathcal{L}_{CON}(\mathbf{P}, \mathbf{F}, \mathbf{C})$$

A **structure** for $\mathcal{L}$ is a pair

$$\mathbf{M} = [U, I]$$

where U is a non empty set called a **universe**

I is an assignment called an **interpretation** of the language $\mathcal{L}(\mathbf{P}, \mathbf{F}, \mathbf{C})$ in the universe U

The structure $\mathbf{M} = [U, I]$ components are defined as follows

Structure Definition

Structure $\mathbf{M} = [U, I]$ **Components**

1. I assigns to any predicate symbol $P \in \mathbf{P}$ a **relation** P_I defined in the universe U, i.e. for any $P \in \mathbf{P}$, if $\#P = n$, then

$$P_I \subseteq U^n$$

2. I assigns to any functional symbol $f \in \mathbf{F}$ a **function** f_I defined in the universe U, i.e. for any $f \in \mathbf{F}$, if $\#f = n$, then

$$f_I : \quad U^n \longrightarrow U$$

3. I assigns to any constant symbol $c \in \mathbf{C}$ an **element** c_I of the universe, i.e for any $c \in \mathbf{C}$,

$$c_I \in U$$

Structure Example

Example

Let $\mathcal{L}$ be a language with one two-place predicate symbol, two functional symbols: one -place and one two-place, and two constants, i.e.

$$\mathcal{L} = \mathcal{L}(\{R\}, \{f, g\}, \{c, d\},)$$

where #R = 2, #f = 1, #g = 2, and $c, d \in$ **C**

We define a **structure** **M** = [U, I] as follows

We take as the universe the set $U = \{1, 3, 5, 6\}$

The **predicate** R is interpreted as $\leq$ what we write as

$$R_I : \leq$$

Structure Example

We interpret f as a **function** $f_I : \{1,3,5,6\} \longrightarrow \{1,3,5,6\}$ such that

$$f_I(x) = 5 \quad \text{for all} \quad x \in \{1,3,5,6\}$$

We put $g_I : \{1,3,5,6\} \times \{1,3,5,6\} \longrightarrow \{1,3,5,6\}$ such that

$$g_I(x,y) = 1 \quad \text{for all} \quad x \in \{1,3,5,6\}$$

The constant c becomes $c_I = 3$, and $d_I = 6$

We write the structure **M** as

$$\mathbf{M} = [\{1,3,5,6\} \quad \leq, \quad f_I, \quad g_I, \quad c_I = 3, \quad d_I = 6]$$

Assignment - Interpretation of Variables

Definition

Given a first order language

$$\mathcal{L} = \mathcal{L}(\mathbf{P}, \mathbf{F}, \mathbf{C})$$

with the set VAR of variables

Let $\mathbf{M} = [U, I]$ be a structure for $\mathcal{L}$ with the universe $U \neq \emptyset$

An **assignment of** $\mathcal{L}$ **in** $\mathbf{M} = [U, I]$ is any function

$$s : \quad VAR \longrightarrow U$$

The **assignment** s is also called an **interpretation of variables** VAR of $\mathcal{L}$ in the structure $\mathbf{M} = [U, I]$

Assignment - Interpretation

Let $\mathbf{M} = [U, I]$ be a structure for $\mathcal{L}$ and

$$s : VAR \longrightarrow U$$

be an **assignment** of variables VAR of $\mathcal{L}$ in the structure $\mathbf{M}$

Let $\mathbf{T}$ be the set of all **terms** of $\mathcal{L}$

By definition of terns

$$VAR \subseteq \mathbf{T}$$

We use the interpretation I of the structure $\mathbf{M} = [U, I]$ to **extend** the **assignment** s to the set the set $\mathbf{T}$ of all **terms** of the language $\mathcal{L}$

Interpretation of Terms

Notation

We denote the **extension** of the assignment s o the set the set $\mathbf{T}$ by s_I rather then by s^* as we did before

s_I associates with each term $t \in \mathbf{T}$ an element $s_I(t) \in U$ of the universe of the structure $\mathbf{M} = [U, I]$

We **define** the extension s_I of s by the induction of the length of the term $t \in \mathbf{T}$ and call it an **interpretation of terms** of $\mathcal{L}$ in a structure $\mathbf{M} = [U, I]$

Interpretation of Terms

Definition

Given a language $\mathcal{L} = \mathcal{L}(\mathbf{P}, \mathbf{F}, \mathbf{C})$ and a structure $\mathbf{M} = [U, I]$

Let a function

$$s : VAR \longrightarrow U$$

be any assignment of variables VAR of $\mathcal{L}$ in $\mathbf{M}$

We **extend** s to a function

$$s_I : \mathbf{T} \longrightarrow U$$

called an **interpretation of terms** of $\mathcal{L}$ in $\mathbf{M}$

Interpretation of Terms

We define the function s_I by induction on the complexity of terms as follows

1. For any v $x \in VAR$,

$$s_I(x) = s(x)$$

2. for any $c \in \mathbf{C}$,

$$s_I(c) = c_I;$$

3. for any $t_1, \ t_2, \ \dots, t_n \in \mathbf{T}, \ \ n \geq 1, \ \ f \in \mathbf{F}$, such that $\#f = n$

$$s_I(f(t_1, \ t_2, \ \dots, t_n)) = f_I(s_I(t_1), \ s_I(t_2), \ \dots, \ s_I(t_n))$$

Interpretation of Terms Example

Example

Consider a language

$$\mathcal{L} = \mathcal{L}(\{P,\ R\},\ \{f, h\},\ \emptyset$$

for # P = # R = 2, #f = 1, # h= 2

Let $\mathbf{M} = [Z, I\,]$, where Z is the set on integers and

the **interpretation** I for elements of **F** and **C** is as follows

$f_I : Z \longrightarrow Z$ is given by formula $f(m) = m + 1$ for all $m \in Z$

$h_I : Z \times Z \longrightarrow Z$ is given by formula $f(m, n) = m + n$ for all

b $m, n \in Z$

Interpretation of Terms Example

Let s be any assignment $s : VAR \longrightarrow Z$ such that

$s(x) = -5, \quad s(y) = 2$ and $t_1, t_2 \in \mathbf{T}$

Let $t_1 = h(y, f(x))$ and $t_2 = h(f(x), h(x, f(y))$

We **evaluate**

$$s_I(t_1) = s_I(h(y, f(x)) = h_I(s_I(y), f_I(s_I(x))) =$$

$$+(2, f_I(-5)) = 2 - 4 = -2$$

and

$$s_I(t_2) = s_I(h(f(x), h(x, f(y))) =$$

$$+(f_I(-5), +(-5, 3)) = -4 + (-5 + 3) = -6$$

Observation

Given $t \in \mathbf{T}$

Let $x_1, x_2, \ldots, x_n \in VAR$ be all variables appearing in t

We write it as

$$t(x_1, x_2, \ldots, x_n)$$

Observation

For any term $t(x_1, x_2, \ldots, x_n) \in \mathbf{T}$, any structure $\mathbf{M} = [U, I]$ and any assignments s, s' of $\mathcal{L}$ in $\mathbf{M}$, the following holds

If $s(x) = s'(x)$ for all $x \in \{x_1, x_2, \ldots, x_n\}$, i.e

if the assignments s, s' **agree** on all variables appearing in t,

then

$$s_I(t) = s'_I(t)$$

Notation

Thus for any $t \in \mathbf{T}$, the function $s_I : \mathbf{T} \longrightarrow U$ **depends** on only a **finite** number of values of $s(x)$ for $x \in VAR$

Notation

Given a structure $\mathbf{M} = [U, I]$ and an assignment $s : VAR \longrightarrow U$ We write

$$s(\tfrac{a}{x})$$

to **denote** any assignment

$$s' : VAR \longrightarrow U$$

such that s, s' **agree** on all variables **except** on x and such that

$$s'(x) = a \quad \text{for certain } a \in U$$

Classical Satisfaction

We introduce now a notion of a **satisfaction relation** $(\mathbf{M}, s) \models A$ that acts between structures, assignments and formulas of $\mathcal{L}$

It is the **satisfaction relation** that allows us to distinguish **one** semantics for a given $\mathcal{L}$ from the **other**,

and consequently one logic from the other

We define now only a classical satisfaction and the notion of classical predicate **tautology**

Classical Satisfaction

Definition

Given a predicate (first order) language $\mathcal{L} = \mathcal{L}(\mathbf{P}, \mathbf{F}, \mathbf{C})$

Let $\mathbf{M} = [U, I]$ be a structure for $\mathcal{L}$ and

$s : VAR \longrightarrow U$ be any assignment of $\mathcal{L}$ in $\mathbf{M}$

Let $A \in \mathcal{F}$ be any formula of $\mathcal{L}$

We define a **satisfaction relation**

$$(\mathbf{M}, s) \models A$$

that reads: " the assignment s **satisfies** the formula A in $\mathbf{M}$"

by **induction** on the complexity of A as follows

Classical Satisfaction

(i) A is **atomic formula**

$(\mathbf{M}, s) \models P(t_1, \ldots, t_n)$ if and only if $(s_I(t_1), \ldots, s_I(t_n)) \in P_I$

(ii) A **is not** atomic formula and has one of connectives of $\mathcal{L}$ as the main connective

$(\mathbf{M}, s) \models \neg A$ if and only if $(\mathbf{M}, s) \not\models A$

$(\mathbf{M}, s) \models (A \cap B)$ if and only if $(\mathbf{M}, s) \models A$ and $(\mathbf{M}, s) \models B$

$(\mathbf{M}, s) \models (A \cup B)$ if and only if $(\mathbf{M}, s) \models A$ or $(\mathbf{M}, s) \models B$ or both

$(\mathbf{M}, s) \models (A \Rightarrow B)$ if and only if ether $(\mathbf{M}, s) \not\models A$ or else $(\mathbf{M}, s) \models B$ or both

Classical Satisfaction

(iii) A **is not** atomic formula and A begins with one of the **quantifiers**

$(\mathbf{M}, s) \models \exists x A$ if and only if **there is** s' such that s, s' **agree** on all variables except on x, and

$$(\mathbf{M}, s') \models A$$

$(\mathbf{M}, s) \models \forall x A$ if and only if **for all** s' such that s, s' **agree** on all variables except on x, and

$$(\mathbf{M}, s') \models A$$

Classical Satisfaction

Observe that that the **truth** or **falsity** of $(\mathbf{M}, s) \models A$ depends only on the values of $s(x)$ for variables x which are actually **free** in the formula A. This is why we often write the condition **(iii)** as follows

(iii)' $A(x)$ (with a free variable x) **is not** atomic formula and A begins with one of the **quantifiers**

$(\mathbf{M}, s) \models \exists x A(x)$ if and only if **there is** s' such that $s(y) = s'(y)$ such that for all $y \in VAR - \{x\}$,

$$(\mathbf{M}, s') \models A(x)$$

$(\mathbf{M}, s) \models \forall x A$ if and only if **for all** s' such that $s(y) = s'(y)$ for all $y \in VAR - \{x\}$,

$$(\mathbf{M}, s') \models A(x)$$

Satisfaction Relation Exercise

Exercise

For the structures $\mathbf{M}_i$, find assignments $s_i,\ s'_i$ for $1 \leq i \leq 2$ such that

$$(\mathbf{M}_i, s_i) \models Q(x, c), \quad \text{and} \quad (\mathbf{M}_i, s'_i) \not\models Q(x, c)$$

where $Q \in \mathbf{P},\ c \in \mathbf{C}$

The structures $\mathbf{M}_i$ are defined as follows (the interpretation I for each of them is specified only for symbols in the **atomic** formula $Q(x, c)$, and N denotes the set of natural numbers

$$\mathbf{M}_1 = [\{1\},\ Q_I := ,\ c_I : 1] \text{ and } \mathbf{M}_2 = [\{1, 2\},\ Q_I :\leq,\ c_I : 1]$$

Satisfaction Relation Exercise

Solution

Given Q(x,c). Consider

$$\mathbf{M}_1 = [\{1\},\ Q_I :=,\ c_I : 1]$$

Observe that **all** assignments

$$s : VAR \longrightarrow \{1\}$$

must be defined by a formula $s(x) = 1$ for all $x \in VAR$

We evaluate $s_I(x) = 1,\ s_I(c) = c_I = 1$

By definition

$$(\mathbf{M}_1, s) \models Q(x,c) \quad \text{if and only if} \quad (s_I(x),\ s_I(c)) \in Q_I$$

This means that $(1,1) \in =$ what is **true** as $1 = 1$

We have proved

$$(\mathbf{M}_1, s) \models Q(x,c) \text{ for all assignments } s : VAR \longrightarrow \{1\}$$

Satisfaction Relation Exercise

Given Q(x,c). Consider

$$\mathbf{M}_2 = [\{1,2\},\ Q_I :\leq,\ c_I : 1]$$

Let $s : VAR \longrightarrow \{1,2\}$ be any assignment, such that

$$s(x) = 1$$

We evaluate $s_I(x) = 1,\ s_I(c) = 1$ and **verify** whether $(s_I(x),\ s_I(c)) \in Q_I$ i.e. whether $(1,1) \in \leq$

This is **true** as $1 \leq 1$

We have found s such that

$$(\mathbf{M}_2, s) \models Q(x,c)$$

In fact, have found uncountably many such assignments s

Satisfaction Relation Exercise

Given Q(x,c) and the structure

$$\mathbf{M}_2 = [\{1,2\},\ Q_I :\leq,\ c_I : 1]$$

Let now s' we be any assignment

$$s' : VAR \longrightarrow \{1,2\} \text{ such that } s'(x) = 2$$

We evaluate $s'_I(x) = 1, \quad s'_I(c) = 1$

We verify whether $s'_I(x),\ s'_I(c)) \in Q_I$, i.e. whether $(2,1) \in \leq$

This is **not true** as $2 \not\leq 1$

We have **found** $s' \neq s$ such that

$$(\mathbf{M}_2, s') \not\models Q(x,c)$$

In fact, have found uncountably many such assignments s'

Model Definition

Definition

Given a predicate language $\mathcal{L}$, a formula $A \in \mathcal{F}$, and a structure $\mathbf{M} = [U, I]$ for $\mathcal{L}$

$\mathbf{M}$ is a **model** for the formula A if and only if $(\mathbf{M}, s) \models A$ for all $s : VAR \longrightarrow U$

We denote it as

$$\mathbf{M} \models A$$

For any set $\Gamma \subseteq \mathcal{F}$ of formulas of $\mathcal{L}$,

$\mathbf{M}$ is a **model** for Γ if and only if $\mathbf{M} \models A$ for all $A \in \Gamma$

We denote it as

$$\mathbf{M} \models \Gamma$$

Counter Model Definition

Definition

Given a predicate language $\mathcal{L}$, a formula $A \in \mathcal{F}$, and a structure $\mathbf{M} = [U, I]$ for $\mathcal{L}$

$\mathbf{M}$ is a **counter model** for the formula A if and only if **there is** an assignment $s : VAR \longrightarrow U$, such that $(\mathbf{M}, s) \not\models A$

We denote it as

$$\mathbf{M} \not\models A$$

Counter Model Definition

Definition

For any set $\Gamma \subseteq \mathcal{F}$ of formulas of $\mathcal{L}$,

$\mathbf{M}$ is a **counter model** for Γ if and only if

there is $A \in \Gamma$, such that $\mathbf{M} \not\models A$

We denote it as

$$\mathbf{M} \not\models \Gamma$$

Sentence Model

Observe that if a formula A is a **sentence** then the **truth** or **falsity** of satement

$$(\mathbf{M}, s) \models A$$

is completely independent of s

Hence if $(\mathbf{M}, s) \models A$ for some s, it holds for all s and the following holds

Fact

For any formula A of $\mathcal{L}$
If A is a **sentence**, then if there is an s such that

$$(\mathbf{M}, s) \models A$$

then **M** is a model fo A, i.e.

$$\mathbf{M} \models A$$

Formula Closure

We transform any formula A of $\mathcal{L}$ into a certain sentence by **binding** all its free variables. The resulting sentence is called a **closure** of A and is defined as follows

Definition

Given A of $\mathcal{L}$

By the **closure** of A we mean the formula obtained from A by prefixing in **universal** quantifiers all variables the arefree in A

If A **does not** have free variables, i.e. is a sentence, the **closure** of A is defined to be A itself

Obviously, a **closure** of any formula is always a sentence

Formula Closure Example

Example

Let A, B be formulas

$$(P(x_1, x_2) \Rightarrow \neg\exists x_2\, Q(x_1, x_2, x_3))$$

$$(\forall x_1 P(x_1, x_2) \Rightarrow \neg\exists x_2\, Q(x_1, x_2, x_3))$$

Their respective **closures** are

$$\forall x_1 \forall x_2 \forall x_3\, ((P(x_1, x_2) \Rightarrow \neg\exists x_2\, Q(x_1, x_2, x_3)))$$

$$\forall x_1 \forall x_2 \forall x_3\, ((\forall x_1 P(x_1, x_2) \Rightarrow \neg\exists x_2\, Q(x_1, x_2, x_3)))$$

Example

Let $Q \in \mathbf{P}$, $\#Q = 2$ and $c \in \mathbf{C}$

Consider formulas

$$Q(x, c), \quad \exists x Q(x, c), \quad \forall x Q(x, c)$$

and the structures defined as follows.

$$\mathbf{M}_1 = [\{1\},\ Q_I := ,\ c_I : 1\ ] \quad \text{and} \quad \mathbf{M}_2 = [\{1, 2\},\ Q_I : \leq,\ c_I : 1\ ]$$

Directly from definition and above **Fact** we get that:

1. $\mathbf{M}_1 \models Q(x, c)$, $\mathbf{M}_1 \models \forall x Q(x, c)$, $\mathbf{M}_1 \models \exists x Q(x, c)$

2. $\mathbf{M}_2 \not\models Q(x, c)$, $\mathbf{M}_2 \not\models \forall x Q(x, c)$, $\mathbf{M}_2 \models \exists x Q(x, c)$

Example

Let $Q \in \mathbf{P}$, $\# Q = 2$ and $c \in \mathbf{C}$

Consider formulas

$$Q(x, c), \quad \exists x Q(x, c), \quad \forall x Q(x, c)$$

and the structures defined as follows.

$$\mathbf{M}_3 = [N, \ Q_I : \geq, \ c_I : 0], \quad \text{and} \quad \mathbf{M}_4 = [N, \ Q_I : \geq, \ c_I : 1]$$

Directly from definition and above **Fact** we get that:

3. $\mathbf{M}_3 \models Q(x, c), \quad \mathbf{M}_3 \models \forall x Q(x, c), \quad \mathbf{M}_3 \models \exists x Q(x, c)$

4. $\mathbf{M}_4 \not\models Q(x, c), \quad \mathbf{M}_4 \not\models \forall x Q(x, c), \quad \mathbf{M}_4 \models \exists x Q(x, c)$

True, False in $\mathbf{M}$

Definition

Given a structure $\mathbf{M} = [U, I]$ for $\mathcal{L}$ and a formula A of $\mathcal{L}$

A is **true** in $\mathbf{M}$ and is written as

$$\mathbf{M} \models A$$

if and only if **all** assignments s of $\mathcal{L}$ in $\mathbf{M}$ **satisfy** A, i.e. when $\mathbf{M}$ is a **model** for A

A is **false** in $\mathbf{M}$ and written as

$$\mathbf{M} =\mid A$$

if and only if there **is no** assignment s of $\mathcal{L}$ in $\mathbf{M}$ that **satisfies** A

True, False in **M**

Here are some **properties** of the notions:

1. " A is **true** in **M**" written symbolically as

$$\mathbf{M} \models A$$

2. " A is **false** in **M**" written symbolically as

$$\mathbf{M} =\!| \; A$$

They are obvious under intuitive understanding of the notion of **satisfaction**

Their formal proofs are left as an exercise

Properties

Given a structure $\mathbf{M} = [U, I]$ and any formulas formula A, B of $\mathcal{L}$. The following properties hold

P1. A is **false** in $\mathbf{M}$ if and only if $\neg A$ is **true** in $\mathbf{M}$, i.e.

$$\mathbf{M} \;=\!\!|\; A \text{ if and only if } \mathbf{M} \models \neg A$$

P2. A is **true** in $\mathbf{M}$ if and only if $\neg A$ is **false** in $\mathbf{M}$, i.e.

$$\mathbf{M} \models A \text{ if and only if } \mathbf{M} \;=\!\!|\; \neg A$$

P3. It is **not** the case that **both** $\mathbf{M} \models A$ and $\mathbf{M} \models \neg A$, i.e. there **is no** formula A, such that

$$\mathbf{M} \models A \text{ and } \mathbf{M} \;=\!\!|\; A$$

True, False in $\mathbf{M}$ Properties

Properties

P4. If $\mathbf{M} \models A$ and $\mathbf{M} \models (A \Rightarrow B)$, then $\mathbf{M} \models B$

P5. $(A \Rightarrow B)$ is **false** in $\mathbf{M}$ if and only if
$\mathbf{M} \models A$ and $\mathbf{M} \models \neg B$

$$\mathbf{M} \models\!\!| \; (A \Rightarrow B) \text{ if and only if } \mathbf{M} \models A \text{ and } \mathbf{M} \models \neg B$$

P6. $\mathbf{M} \models A$ if and only if $\mathbf{M} \models \forall x A$

P7. A formula A is **true** in $\mathbf{M}$ if and only if its closure is **true** in $\mathbf{M}$

Valid, Tautology Definition

Definition

A formula A of $\mathcal{L}$ is a **predicate** tautology (is **valid**) if and only if $\mathbf{M} \models A$ for **all** structures $\mathbf{M} = [U, I]$

We also say

A formula A of $\mathcal{L}$ is a **predicate** tautology (is **valid**) if and only if A is **true** in **all** structures $\mathbf{M}$ for $\mathcal{L}$

We write

$$\models A \qquad \text{or} \qquad \models_p A$$

to denote that a formula A is **predicate** tautology (is **valid**)

Valid, Tautology Definition

We write

$$\models_p A$$

when there is a **need** to stress a distinction between **propositional** and **predicate** tautologies

otherwise we write

$$\models A$$

Predicate tautologies are also called **laws of quantifiers**.

Following the notation $\mathbf{T}$ we have established for the **set** of all propositional tautologies we denote by $\mathbf{T}_p$ the **set** of all predicate tautologies

We put

$$\mathbf{T}_p = \{A \text{ of } \mathcal{L} : \ \models_p A\}$$

Not a Tautology, Counter Model

Definition

For any formula A of predicate language $\mathcal{L}$

A **is not** a predicate tautology and denote it by

$$\not\models A$$

if and only if there is a structure $\mathbf{M} = [U, I]$ for $\mathcal{L}$, such that

$$\mathbf{M} \not\models A$$

We call such structure **M** a **counter-model** for A

Counter Model

In order to **prove** that a formula A **is not** a tautology one has to find a **counter-model** for A

It means one has to define the components of a structure $\mathbf{M} = [U, I]$ for $\mathcal{L}$, i.e.

a non-empty set U, called **universe** and

an interpretation I of $\mathcal{L}$ in the universe U

Moreover, one has to define an assignment $s : VAR \longrightarrow U$ and **prove** that

$$(\mathbf{M}, s) \not\models A$$

Contradictions

We introduce now a notion of predicate **contradiction**

Definition

For any formula A of $\mathcal{L}$,

A is a **predicate contradiction** if and only if

A is **false** in **all** structures **M**

We denote it as $=\!| \, A$ and write symbolically

$=\!| \, A$ if and only if $\mathbf{M} =\!| \, A$, for **all** structures **M**

When there is a need to distinguish between propositional and predicate contradictions we also use symbol

$$=\!|_p \, A$$

Contradictions

Following the notation **C** for the set of all propositional **contradictions** we denote by $\mathbf{C}_p$ the set of all predicate contradictions, i.e.

$$\mathbf{C}_p = \{A \text{ of } \mathcal{L}(\mathbf{P}, \mathbf{F}, \mathbf{C}) : \ =\!|_p\, A\}$$

Directly from the contradiction definition we have the following duality property charecteristic for classical logic

Fact

For any formula A of a predicate language $\mathcal{L}$,

$$A \in \mathbf{T}_p \quad \text{if and only if} \quad \neg A \in \mathbf{C}_p$$

$$A \in \mathbf{C}_p \quad \text{if and only if} \quad \neg A \in \mathbf{T}_p$$

Proving Predicate TAutologies

We **prove**, as an example the following **basic** predicate tautology

Fact

For any formula $A(x)$ of $\mathcal{L}$,

$$\models \ (\forall x\, A(x) \Rightarrow \exists x\, A(x))$$

Proof

Assume that $\not\models (\forall x\, A(x) \Rightarrow \exists x\, A(x))$

It means that there is a structure

$$\mathbf{M} = [U, I] \text{ and } s : VAR \longrightarrow U, \text{ such that}$$

$$(\mathbf{M}, s) \not\models (\forall x\, A(x) \Rightarrow \exists x\, A(x))$$

Proving Predicate Tautologies

Observe that $(\mathbf{M}, s) \not\models (\forall x\, A(x) \Rightarrow \exists x\, A(x))$ is equivalent to

$$(\mathbf{M}, s) \not\models \forall x\, A(x) \text{ and } (\mathbf{M}, s) \not\models \exists x\, A(x)$$

By definition, $(\mathbf{M}, s) \not\models \forall x\, A(x)$ means that $(\mathbf{M}, s') \models A(x)$ for **all** s' such that s, s' agree on all variables except on x

At the same time $(\mathbf{M}, s) \not\models \exists x\, A(x)$ means that it is **not true** that **there is** s' such that s, s' agree on all variables except on x, and $(\mathbf{M}, s') \models A(x)$. This **contradiction** proves

$$\models (\forall x\, A(x) \Rightarrow \exists x\, A(x))$$

Disapproving Predicate Tautologies

We show now, as an example of a **counter model** construction that the converse implication to

$$\models \; (\forall x\, A(x) \Rightarrow \exists x\, A(x))$$

is not a predicate tautology i.e. the following holds

Fact

There is a formula A of $\mathcal{L}$, such that

$$\not\models \; (\exists x\, A(x) \Rightarrow \forall x\, A(x))$$

Proof

Observe that to prove the **Fact** we have to provide an example of an instance of a formula $A(x)$ and construct a **counter model** $\mathbf{M} = [U, I]$ for it

Proving Predicate Tautologies

Let $A(x)$ be an **atomic** formula

$$P(x, c) \quad \text{for any} \quad P \in \mathbf{P}, \quad \#P = 2$$

The needed instance is a formula

$$(\exists x\, P(x, c) \Rightarrow \forall x\, P(x, c))$$

We take as its **counter model** a structure

$$\mathbf{M} = [\, N, \quad P_I :<, \quad c_I : 3\,]$$

where N is set of natural numbers. We want to show

$$\mathbf{M} \not\models (\exists x\, P(x, c) \Rightarrow \forall x\, P(x, c))$$

It means we have to define an assignment s such that $s : VAR \longrightarrow N$ and

$$(\mathbf{M}, s) \not\models (\exists x\, P(x, c) \Rightarrow \forall x\, P(x, c))$$

Proving Predicate Tautologies

Let s be any assignment $s : VAR \longrightarrow N$

We show now

$$(\mathbf{M}, s) \models \exists x \, P(x, c)$$

Take any s' such that

$$s'(x) = 2 \quad \text{and} \quad s'(y) = s(y) \quad \text{for all} \;\; y \in VAR - \{x\}$$

We have $(2, 3) \in P_I$, as $2 < 3$

Hence we proved that **there exists** s' that agrees with s on all variables except on x and

$$(\mathbf{M}, s') \models P(x, c)$$

Proving Predicate Tautologies

But at the same time

$$(\mathbf{M}, s) \not\models \forall x\, P(x, c)$$

as for example for s' such that

$$s'(x) = 5 \quad \text{and} \quad s'(y) = s(y) \quad \text{for all} \quad y \in VAR - \{x\}$$

We have that $(2, 3) \notin P_I$, as $5 \not< 3$

This proves that the structure

$$\mathbf{M} = [N, P_I :<, \ c_I : 3\,]$$

is a **counter model** for $\forall x\, P(x, c)$

Hence we proved that

$$\not\models \ (\exists x\, A(x) \Rightarrow \forall x\, A(x))$$

Proving Predicate Tautologies

Short Hand Solution of

$$\not\models \ (\exists x \, P(x,c) \Rightarrow \forall x \, P(x,c))$$

We take as its **counter model** a structure

$$\mathbf{M} = [\, N, \quad P_I :<, \quad c_I : 3 \,]$$

where N is set of natural numbers
The formula

$$(\exists x \, P(x,c) \Rightarrow \forall x \, P(x,c))$$

becomes in $\mathbf{M} = (N, P_I :<, \ c_I : 3)$ a mathematical statement (written with logical symbols):

$$\exists n \; n < 3 \ \Rightarrow \ \forall n \; n < 3$$

It is an obviously **false** statement in the set N of natural numbers, as there is $n \in N$, such that $n < 3$, for example $n = 2$, and it is **not true** that all natural numbers are smaller then 3