cse541
LOGIC for Computer Science

Professor Anita Wasilewska
LECTURE 6b
Chapter 6
Automated Proof Systems
Completeness of Classical Propositional Logic

PART 5: Original Gentzen Systems $\textbf{LK, LI}$
Classical and Intuitionistic Completeness and Hauptzatz Theorem
Original Gentzen Systems LK, LI

The original systems LK and LI were created by Gentzen in 1935 for classical and intuitionistic predicate logics, respectively.

We present now their propositional versions and use the same names LK and LI.

The proof system LI for intuitionistic logic is a particular case of the proof system LK.
Original Gentzen Systems **LK, LI**

Both systems **LK** and **LI** have **two groups** of the inference **rules**

They both have a **special** rule called a **cut rule**

**First group** consists of a set of rules **similar** to the rules of systems **GL** and **G** called **Logical Rules**

**Second group** contains a **new type of rules**
We call them **Structural Rules**
Original Gentzen Systems \textbf{LK, LI}

The \textbf{cut} rule in \textit{Gentzen} sequent systems \textit{corresponds} to the \textit{Modus Ponens} rule in \textit{Hilbert} proof systems.

\textit{Modus Ponens} is a particular \textit{case} of the \textbf{cut} rule.

The \textbf{cut} rule is needed to carry out the original \textit{Gentzen} proof of the \textit{completeness} of the system \textbf{LK} and for proving the \textit{adequacy} of \textbf{LI} system for \textit{intuitionistic} logic.
Original Gentzen Systems LK, LI

Gentzen proof of completeness of LK was not direct

He used the completeness of already known Hilbert proof system H and proved that any formula that is provable in the systems H is also provable in LK

Hence the need of the cut rule
Original Gentzen Systems LK, LI

For the system LI he proved only the adequacy of LI system for intuitionistic logic since the semantics for the intuitionistic logic didn’t yet exist.

He used the acceptance of Heying intuitionistic axiom system as a definition of the intuitionistic logic and proved that any formula provable in the Heyting system is also provable in LI.
Original Gentzen Systems **LK, LI**

*Observe* that by presence of the **cut** rule, Gentzen systems **LK** and **LI** are also Hilbert system.

What **distinguishes** the Gentzen systems from all other known Hilbert proof systems is the **fact** that the **cut rule** could be **eliminated** from them, what is not the case of regular Hilbert proof systems.

This is why Gentzen famous **Hauptzatz Theorem**, is also called **Cut Elimination Theorem**.
Original Gentzen Systems \textbf{LK, LI}

The elimination of the \textit{cut} rule and the structure of other rules makes it possible to define an effective automatic procedures for proof search, what is \textit{impossible} in a case of the Hilbert style systems.

Gentzen in his proof of \textbf{Hauptzatz Theorem} developed a powerful technique of proof \textit{adaptable} to other logics.
Original Gentzen Systems LK, LI

We present here the Gentzen cut elimination technique for the classical propositional case and show how to adapt it to the intuitionistic case.

Gentzen proof is purely syntactical.

The proof defines a constructive method of transformation of any formal proof (derivation) of a sequent $\Gamma \rightarrow \Delta$ that uses the cut rule (and other rules) into its proof without use of the cut rule.

Hence the English name Cut Elimination Theorem.
Gentzen System LK
LK Components

Language

\[ \mathcal{L} = \mathcal{L} \{ \neg, \land, \lor, \Rightarrow \} \quad \text{and} \quad \mathcal{E} = \mathcal{SQ} \]

for

\[ \mathcal{SQ} = \{ \Gamma \rightarrow \Delta : \Gamma, \Delta \in \mathcal{F}^* \} \]

Logical Axioms

There is only one logical axiom, namely

\[ A \rightarrow A \]

where \( A \) is any formula of \( \mathcal{L} \)
LK Components

Rules of Inference

Group one: Structural Rules

Weakening

\[
\frac{\Gamma \rightarrow \Delta}{A, \Gamma \rightarrow \Delta} \quad (\text{weak } \rightarrow)
\]

\[
\frac{\Gamma \rightarrow \Delta}{\Gamma \rightarrow \Delta, A} \quad (\rightarrow \text{ weak})
\]

Contraction

\[
\frac{A, A, \Gamma \rightarrow \Delta}{A, \Gamma \rightarrow \Delta} \quad (\text{contr } \rightarrow)
\]

\[
\frac{\Gamma \rightarrow \Delta, A, A}{\Gamma \rightarrow \Delta, A} \quad (\rightarrow \text{ contr})
\]
LK Components

Exchange

\[(\text{exch } \rightarrow)\]
\[
\frac{\Gamma_1, A, B, \Gamma_2 \rightarrow \Delta}{\Gamma_1, B, A, \Gamma_2 \rightarrow \Delta}
\]

\[(\rightarrow \text{exch})\]
\[
\frac{\Delta \rightarrow \Gamma_1, A, B, \Gamma_2}{\Delta \rightarrow \Gamma_1, B, A, \Gamma_2}
\]

Cut Rule

\[(\text{cut})\]
\[
\frac{\Gamma \rightarrow \Delta, A ; A, \Sigma \rightarrow \Theta}{\Gamma, \Sigma \rightarrow \Delta, \Theta}
\]

A is called a cut formula
LK Components

Group Two: Logical Rules

Conjunction rules

\[
\frac{A, \Gamma \rightarrow \Delta}{(A \cap B), \Gamma \rightarrow \Delta}
\]

\[
\frac{B, \Gamma \rightarrow \Delta}{(A \cap B), \Gamma \rightarrow \Delta}
\]

\[
\frac{\Gamma \rightarrow \Delta, A}{\Gamma \rightarrow \Delta, (A \cap B)}
\]

\[
\frac{\Gamma \rightarrow \Delta, B, \Delta}{\Gamma \rightarrow \Delta, (A \cap B)}
\]
Disjunction rules

\[(\to \cup)_1\]
\[
\begin{align*}
\Gamma & \to \Delta, A \\
\Gamma & \to \Delta, (A \cup B)
\end{align*}
\]

\[(\to \cup)_2\]
\[
\begin{align*}
\Gamma & \to \Delta, B \\
\Gamma & \to \Delta, (A \cup B)
\end{align*}
\]

\[(\cup \to)\]
\[
\begin{align*}
A, \Gamma & \to \Delta \\
B, \Gamma & \to \Delta \\
(A \cup B), \Gamma & \to \Delta
\end{align*}
\]
Lk Components

Implication rules

\[(\rightarrow\Rightarrow)\]

\[\frac{A, \Gamma \rightarrow \Delta, B}{\Gamma \rightarrow \Delta, (A \Rightarrow B)}\]

\[(\Rightarrow\rightarrow)\]

\[\frac{\Gamma \rightarrow \Delta, A ; B, \Gamma \rightarrow \Delta}{(A \Rightarrow B), \Gamma \rightarrow \Delta}\]

Negation rules

\[(\neg\rightarrow)\]

\[\frac{\Gamma \rightarrow \Delta, A}{\neg A, \Gamma \rightarrow \Delta}\]

\[(\rightarrow \neg)\]

\[\frac{A, \Gamma \rightarrow \Delta}{\Gamma \rightarrow \Delta, \neg A}\]
**LK Definition**

**Classical System LK**

We define the classical Gentzen system **LK** as

\[ \text{LK} = (\mathcal{L}, \text{SQ}, \text{LA}, \mathcal{R}) \]

where

\[ \mathcal{R} = \{ \text{Structural Rules, Cut Rule, Logical Rules} \} \]

as defined by the components definitions
LI Definition

Intuitionistic System \( LI \)

We define the intuitionistic Gentzen system \( LI \) as

\[
LI = (\mathcal{L}, ISQ, AL, R)
\]

\( R = \{ \text{I - Structural Rules, I - Cut Rule, I - Logical Rules} \} \)

where \( R \) are the LK rules restricted to the set \( ISQ \) of the intuitionistic sequents defined as follows

\[
ISQ = \{ \Gamma \rightarrow \Delta : \Delta \text{ consists of at most one formula} \}
\]

We will study the intuitionistic system \( LI \) in Chapter 7
Classical System LK

We say that a formula $A \in \mathcal{F}$ has a proof in LK and denote it by

$$\vdash_{LK} A$$

if the sequent $\rightarrow A$ has a proof in LK, i.e. we write

$$\vdash_{LK} A \text{ if and only if } \vdash_{LK} \rightarrow A$$
LK Proof Trees

We write **formal proofs** in **LK**, as we did for other Gentzen style proof systems in a form of the **proof trees** defined as follows

**Definition**

By a **proof tree** of a sequent $\Gamma \rightarrow \Delta$ in **LK** we understand a tree $D_{\Gamma \rightarrow \Delta}$ satisfying the following conditions:

1. The topmost sequent, i.e the **root** of $D_{\Gamma \rightarrow \Delta}$ is $\Gamma \rightarrow \Delta$
2. All **leaves** are **axioms**
3. The **nodes** are sequents such that each sequent on the tree **follows** from the ones **immediately preceding** it by one of the rules
Derivations in LK

Proofs are often called derivations. In particular, Gentzen, in his work used the term derivation for the proof and we will use this notion as well.

This is why we denote the proof trees by $D$, for the derivation.

Finding derivations $D$ in LK is a complex process. LK logical rules are different, then in GL and G. Consequently, proofs rely strongly on use of the structural rules.
Derivations in \textbf{LK}

For \textit{example}, a \textit{derivation} of Excluded Middle \((A \cup \neg A)\) formula in \textbf{LK} is as follows

\begin{align*}
D &\rightarrow (A \cup \neg A) \\
| \ (\rightarrow \text{contr}) &\rightarrow (A \cup \neg A), (A \cup \neg A) \\
| \ (\rightarrow \cup)_1 &\rightarrow (A \cup \neg A), A \\
| \ (\rightarrow \text{exch}) &\rightarrow A, (A \cup \neg A) \\
| \ (\rightarrow \cup)_1 &\rightarrow A, \neg A \\
| \ (\rightarrow \neg) &\rightarrow A, A \\
\text{axiom}
\end{align*}
Derivations in LK

Here is as yet another example a cut free derivation in LK

\[ D \]

\[ \rightarrow (\neg (A \cap B) \Rightarrow (\neg A \cup \neg B)) \]

\[ \vdash \rightarrow \neg \]

\[ (\neg (A \cap B) \rightarrow (\neg A \cup \neg B)) \]

\[ \vdash \rightarrow \neg \]

\[ \rightarrow (\neg A \cup \neg B), (A \cap B) \]

\[ \wedge (\Rightarrow \rightarrow) \]

\[ \rightarrow (\neg A \cup \neg B), A \]

\[ \vdash \rightarrow \neg \]

\[ A, (\neg A \cup \neg B) \]

\[ \vdash (\rightarrow \cup)_{1} \]

\[ \rightarrow A, \neg A \]

\[ \vdash (\rightarrow \neg) \]

\[ A \rightarrow A \]

axiom

\[ \vdash \rightarrow \neg \]

\[ B \rightarrow B \]

B → B

axiom
LK Soundness
LK Soundness

Observe that the Logical Rules of LK are similar in their structure to the rules of the system G.

Hence LK Logical Rules admit similar proof of their soundness.

The sound rules

\((\rightarrow \cap)_1, (\rightarrow \cap)_2\) and \((\rightarrow \cup)_1, (\rightarrow \cup)_2\)

are not strongly sound because

\(A \neq (A \cap B),\ B \neq (A \cap B)\) and \(A \neq (A \cup B),\ B \neq (A \cup B)\)

All other Logical Rules are strongly sound.
**LK Soundness**

The **Contraction** and **Exchange** structural rules are **strongly sound** as for any formulas $A, B \in \mathcal{F}$,

$$A \equiv (A \cap A), \quad A \equiv (A \cup A) \quad \text{and} \quad (A \cap B) \equiv (B \cap A), \quad (A \cap B) \equiv (B \cap A)$$

The **Weakeninng** rule is **sound** because (we use shorthand notation)

$$\text{if } (\Gamma \Rightarrow \Delta) = T \text{ then } ((A \cap \Gamma) \Rightarrow \Delta) = T$$

for any logical value of the formula $A$

Obviously

$$(\Gamma \Rightarrow \Delta) \neq ((A \cap \Gamma) \Rightarrow \Delta))$$

i.e. the **Weakening** rule is **not** strongly sound
**LK Soundness**

The **Cut rule** is **sound** as the fact that

\[(\Gamma \Rightarrow (\Delta \cup A)) = T \quad \text{and} \quad ((A \cap \Sigma) \Rightarrow \Lambda) = T\]

implies that

\[((\Gamma \cap \Sigma) \Rightarrow (\Delta \cup \Lambda)) = T\]

**Cut rule is not strongly sound**

Any truth assignment such that

\[\Gamma = T \quad \text{and} \quad \Delta = \Sigma = \Lambda = A = F\]

proves that

\[(\Gamma \rightarrow \Delta, A) \cap (A, \Sigma \rightarrow \Lambda) \neq (\Gamma, \Sigma \rightarrow \Delta, \Lambda)\]
**LK Soundness**

Obviously, the *Logical Axiom* is a tautology, i.e.

\[ \models A \rightarrow A \]

We have proved that **LK** is **sound** and the following theorem holds

**Soundness Theorem**

For any sequent \( \Gamma \rightarrow \Delta \),

if \( \vdash_{LK} \Gamma \rightarrow \Delta \), then \( \models \Gamma \rightarrow \Delta \)

In particular, for any \( A \in \mathcal{F} \),

if \( \vdash_{LK} A \), then \( \models A \)
LK Completeness
LK Completeness

We follow Gentzen original proof of completeness of LK.

We choose any complete Hilbert proof system for the LK language

\[ \mathcal{L} = \mathcal{L}\{\neg, \wedge, \vee, \Rightarrow\} \]

and prove, after Gentzen, its equivalency with LK.

Gentzen referred to the Hilbert-Ackerman (1920) system (axiomatization) included in chapter 5.

We choose the Rasiowa-Sikorski (1952) formalization \( R \) also included in Chapter 5.
We choose the formalization $R$ for two reasons.

First, it reflects a connection between classical and intuitionistic logics very much in a spirit Gentzen relationship between $\text{LK}$ and $\text{LI}$.

We obtain a complete proof system $I$ from $R$ by just removing the last axiom $A12$.

Second, both sets of axioms reflect the best what set of rovable formulas is needed to conduct algebraic proofs of completeness of $R$ and $I$, as discussed in Chapter 7.
The set of **logical axioms** of the proof system $R$

- **A1** \[ ((A \Rightarrow B) \Rightarrow ((B \Rightarrow C) \Rightarrow (A \Rightarrow C))) \]
- **A2** \[ (A \Rightarrow (A \cup B)) \]
- **A3** \[ (B \Rightarrow (A \cup B)) \]
- **A4** \[ ((A \Rightarrow C) \Rightarrow ((B \Rightarrow C) \Rightarrow ((A \cup B) \Rightarrow C))) \]
- **A5** \[ ((A \cap B) \Rightarrow A) \]
- **A6** \[ ((A \cap B) \Rightarrow B) \]
- **A7** \[ ((C \Rightarrow A) \Rightarrow ((C \Rightarrow B) \Rightarrow (C \Rightarrow (A \cap B)))) \]
- **A8** \[ ((A \Rightarrow (B \Rightarrow C))) \Rightarrow ((A \cap B) \Rightarrow C)) \]
- **A9** \[ (((A \cap B) \Rightarrow C) \Rightarrow (A \Rightarrow (B \Rightarrow C)) \]
- **A10** \[ (A \cap \neg A) \Rightarrow B) \]
- **A11** \[ ((A \Rightarrow (A \cap \neg A)) \Rightarrow \neg A) \]
Hilbert System R

\[ \text{A12} \quad (A \cup \neg A) \]

where \( A, B, C \in \mathcal{F} \) are any formulas

We adopt a Modus Ponens

\[
\text{(MP)} \quad \frac{A \; (A \implies B)}{B}
\]

as the only inference rule

We define the proof system \( R \) as

\[ R = (\mathcal{L}_{\neg, \land, \lor, \implies}, \mathcal{F}, \{A_1 - A_{12}\}, (\text{MP})) \]

where \( A_1 - A_{12} \) are logical axioms defined above
Hilbert System R

The system $R$ is **complete**, i.e. we have the following

**$R$ Completeness Theorem**

For any formula $A \in \mathcal{F}$,

$$\vdash_R A \text{ if and only if } \models A$$

We leave it as an **exercise** to show that all axioms $A1 - A12$ of the system $R$ are **provable** in $LK$

Moreover, the **Modus Ponens** rule of $R$ is a **particular case** of the **Cut rule**, namely

$$\begin{array}{c}
(MP) \quad \to A ; \quad A \to B \\
\to B
\end{array}$$

This proves the following theorem
Hilbert System R

**Provability Theorem**
For any formula $A \in \mathcal{F}$

$$\text{if } \vdash_R A, \text{ then } \vdash_{LK} A$$

Directly from the above provability theorem, the soundness of LK and the completeness of $R$ we get the following

**LK Completeness Theorem**
For any formula $A \in \mathcal{F}$

$$\vdash_{LK} A \text{ if and only if } \models A$$
Hauptzatz
Hauptzatz

Here is Gentzen original formulation of the Hauptzatz Theorems for classical LK and intuitionistic LI proof systems. They are also routinely called the Cut Elimination Theorems.

**LK Hauptzatz**

Every derivation in LK can be transformed into another LK derivation of the same sequent, in which no cuts occur.

**LI Hauptzatz**

Every derivation in LI can be transformed into another LI derivation of the same sequent, in which no cuts occur.
**Mix Rule**

**Hauptzatz** proof is quite long and very involved. We present its main and most important steps.

To facilitate the proof we introduce as Gentzen did, a general form of the cut rule, called a mix rule.

It is defined as follows:

\[
\text{(mix)} \quad \frac{\Gamma \rightarrow \Delta ; \Sigma \rightarrow \Theta}{\Gamma, \Sigma^* \rightarrow \Delta^*, \Theta}
\]

where \( \Sigma^*, \Delta^* \) are obtained from \( \Sigma, \Delta \) by removing all occurrences of a common formula \( A \).

The formula \( A \) is now called a mix formula.
Here are some examples of applications of the mix rule.

Observe that the mix rule applies, as the cut does, to only one mix formula at the time.

**b** is the mix formula in

\[
\begin{align*}
\text{(mix)} & \quad a \rightarrow b, \neg a ; (b \cup c), b, b, D, b \rightarrow \\
& \quad a, (b \cup c), D \rightarrow \neg a
\end{align*}
\]

**B** is the mix formula in

\[
\begin{align*}
\text{(mix)} & \quad A \rightarrow B, B, \neg A ; (b \cup c), B, B, D, B \rightarrow \neg B \\
& \quad A, (b \cup c), D \rightarrow \neg A, \neg B
\end{align*}
\]

**\neg A** is the mix formula in

\[
\begin{align*}
\text{(mix)} & \quad A \rightarrow B, \neg A, \neg A ; \neg A, B, B, \neg A, B \rightarrow \neg B \\
& \quad A, B, B \rightarrow B, \neg B
\end{align*}
\]

Mix Example
Mix and Cut

**Notice**, that every derivation with cut may be transformed into a derivation with mix.

We do so by means of a number of weakenings and interchanges, i.e. multiple application of the weakening rules exchange rules.

Conversely, every mix may be transformed into a cut derivation by means of a certain number of preceding exchanges and contractions, though we do not use this fact in the Hauptzatz proof.

Observe that cut is a particular case of mix.
Two Hauptzatz Theorems

There are two Hauptzatz theorems: classical **LK Hauptzatz** and **LI Hauptzatz**

The **proof** of intuitionistic **LI Hauptzatz** is basically **the same** as for **LK**

We must just be **careful** and **add**, at each step, the **restriction** made to the **ISQ sequents** and the form of the **LI** rules of inference. These **restrictions** do not alter the flow and **validity** of the **LK** proof

We discuss and present now the **proof** of **LK Hauptzatz** We leave it as a **homework exercise** to **re-write** this proof the case of for **LI**
Proof of LK Hauptzatz

Proof of LK Hauptzatz

We conduct the proof in three main steps

Step 1: we consider only derivations in which only mix rule is used

Step 2: we consider first derivation with a certain Property H (to be defined) and prove an H Lemma for them

The H Lemma is the most crucial for the proof of the Hauptzatz
We say that a derivation $D_{\Gamma \rightarrow \Delta}$ of a sequent $\Gamma \rightarrow \Delta$ has a Property H if it satisfies the following conditions:

1. The root $\Gamma \rightarrow \Delta$ of the derivation $D_{\Gamma \rightarrow \Delta}$ is obtained by direct use of the mix rule. It means that the mix rule is the last rule used in the derivation of $\Gamma \rightarrow \Delta$.

2. The derivation $D_{\Gamma \rightarrow \Delta}$ does not contain any other application of the mix rule.
H Lemma

Any derivation that fulfills the **Property H** may be transformed into a derivation of the same sequent in which **no mix** occurs.

**Step 3:** we use the **H Lemma** and to prove the **Hauptzatz**
Proof of Hauptzatz

Step 3: Hauptzatz proof from H Lemma

Let \( D \) be any derivation (tree proof)
Let \( \Gamma \overset{\rightarrow}{\rightarrow} \Delta \) be any node on \( D \) such that its sub-tree \( D_{\Gamma \overset{\rightarrow}{\rightarrow} \Delta} \) has the Property H

By H Lemma the sub-tree \( D_{\Gamma \overset{\rightarrow}{\rightarrow} \Delta} \) can be replaced by a tree \( D^*_{\Gamma \overset{\rightarrow}{\rightarrow} \Delta} \) in which no mix occurs
The rest of \( D \) remains unchanged

We repeat this procedure for each node \( N \), such that the sub-tree \( D_N \) has the Property H until every application of mix rule has systematically been eliminated

This ends the proof of Hauptzatz provided the H Lemma has already been proved
Proof of H Lemma

Step 2: proof of H lemma

We consider derivation tree $D$ with the Property H. It means that $D$ is such that the mix rule is the last rule of inference used and $D$ does not contain any other application of the mix rule.

Observe that $D$ contains only one application of mix rule, and the mix rule, contains only one mix formula $A$. We call $A$ the mix formula of $D$.

We define two important notions: degree $n$ and rank $r$ of the derivation $D$. 
Degree of $D$

Definition

Given a derivation tree $D$ with the Property $H$

Let $A \in F$ be the mix formula of $D$ The degree $n \geq 0$ of $A$ is called the degree of the derivation $D$

We write it as

$$\text{deg}D = \text{deg} A = n$$
Degree of **D**

**Definition**
Given a derivation tree **D** with the **Property H**
We define the **rank** \( r \) of **D** as a sum of its **left rank** \( L_r \) and **right rank** \( R_r \) of **D**, i.e.

\[
r = L_r + R_r
\]

where:

1. **left rank** \( L_r \) of **D** is the largest number of consecutive nodes on the branch of **D** staring with the node containing the **left** premiss of the **mix rule**, such that each sequent on these nodes contains the **mix formula** in the **succeedent**;
2. **right rank** \( R_r \) of **D** is the largest number of consecutive nodes on the branch of **D** staring with the node containing the **right** premiss of the **mix rule**, such that each sequent on these nodes contains the **mix formula** in the **antecedent**.
Proof of H Lemma

We prove the H Lemma by carrying out two inductions. One on the degree $n$, the other on the rank $r$, of the derivation $D$.

It means we prove the H Lemma for a derivation of the degree $n$, assuming it to hold for derivations of a lower degree as long as $n \neq 0$, i.e. we assume that derivations of lower degree can be already transformed into derivations without mix.
Proof of H Lemma

The lowest possible rank is evidently 2.

We begin by considering the case 1 when the rank is $r = 2$. We carry induction with respect to the degree $n$ of the derivation $D$.

After that we examine the case 2 when the rank is $r > 2$ and we assume that the H Lemma already holds for derivations of the same degree, but a lower rank.
Proof of H Lemma

Case 1. Rank of $r = 2$

We carry induction with respect to the degree $n$ of derivation $D$, i.e. with respect to degree $n \geq 0$ of the mix formula

We split the induction cases to consider in two groups
GROUP 1. Axioms and Structural Rules
GROUP 2. Logical Rules

We present now some cases of rules of inference as examples. There are some more cases presented in the chapter, and the rest are left as exercises
Proof of H Lemma

Observe that first group contains cases that are especially simple in that they allow the mix to be immediately eliminated.

The second group contains the most important cases since their consideration brings out the basic idea behind the whole proof.

Here we use the induction hypothesis with respect do the degree of the derivation. We reduce each one of the cases to transformed derivations of a lower degree.
Proof of H Lemma

GROUP 1. Axioms and Structural Rules

1. The left premiss of the mix rule is an axiom

   \[ A \rightarrow A \]

Then the sub-tree of D containing mix is as follows

\[ A, \Sigma^* \rightarrow \Delta \]

\[ \bigwedge (\text{mix}) \]

\[ A \rightarrow A \]
\[ \Sigma \rightarrow \Delta \]
Proof of H Lemma

We transform it, and replace it in the derivation tree $D$ by

$$A, \Sigma^* \rightarrow \Delta$$

(possibly several exchanges and contractions)

$$\Sigma \rightarrow \Delta$$

Such obtained tree $D^*$ proves the same sequent as $D$ and contains no mix
Proof of H Lemma

2. The right premiss of the **mix rule** is an axiom $A \rightarrow A$
Then the **sub-tree** of $D$ containing **mix** is as follows

$$\Sigma \rightarrow \Delta^*, A$$

$$\bigwedge (\text{mix})$$

$$\Sigma \rightarrow \Delta \quad A \rightarrow A$$

We **transform** it, and **replace** it in $D$ by

$$\Sigma \rightarrow \Delta^*, A$$

*(possibly several exchanges and contractions)*

$$\Sigma \rightarrow \Delta$$

Such obtained $D^*$ proves the same sequent and contains no mix
Proof of H Lemma

Suppose that neither of premisses of mix is an axiom
As the rank is \( r=2 \), the right and left ranks are equal 1

This means that in the sequents on the nodes directly below left premiss of the mix, the mix formula \( A \) does not occur in the succedent; in the sequents on the nodes directly below right premiss of the mix, the mix formula \( A \) does not occur in the antecedent

In general, if a formula occurs in the antecedent (succedent) of a conclusion of a rule of inference, it is either obtained by a logical rule or by a contraction rule
Proof of H Lemma

3. The left premiss of the mix rule is the conclusion of a contraction rule. The sub-tree of $D$ containing mix is:

$$
\Gamma, \Sigma^* \rightarrow \Delta, \Theta
$$

$$
\bigwedge (mix)
$$

$$
\Gamma \rightarrow \Delta, A \quad \Sigma \rightarrow \Theta
$$

$$
\mid (\rightarrow contr)
$$

$$
\Gamma \rightarrow \Delta
$$
Proof of H Lemma

We transform it, and replace it in $D$ by

$$\Gamma, \Sigma^* \rightarrow \Delta, \Theta$$

(possibly several weakenings and exchanges)

$$\Gamma \rightarrow \Delta$$

Such obtained $D^*$ contains no mix

Observe that the whole branch of $D$ that starts with the node $\Sigma \rightarrow \Theta$ disappears

4. The right premiss of the mix rule is the conclusion of a contraction rule ($\rightarrow$ contr). It is a dual case to 3. s left as an exercise
Proof of H Lemma

GROUP 2. Logical Rules

1. The mix formula is \((A \cap B)\) The left premiss of the mix rule is the conclusion of a rule \((\to \land)\). The right premiss of the mix rule is the conclusion of a rule \((\land \to)_1\). The sub-tree \(T\) of \(D\) containing mix is:

\[
\Gamma, \Sigma \rightarrow \Delta, \Theta \\
\land (mix)
\]

\[
\Gamma \rightarrow \Delta, (A \cap B) \\
\land (\to \land)
\]

\[
(A \cap B), \Sigma \rightarrow \Theta \\
| (\land \to)_1 \\
A, \Sigma \rightarrow \Theta
\]

\[
\Gamma \rightarrow \Delta, A \\
\Gamma \rightarrow \Delta, B
\]
Proof of H Lemma

We transform \( T \) into \( T^\ast \) as follows.

\[
\Gamma, \Sigma \rightarrow \Delta, \Theta \\
(\text{possibly several weakenings and exchanges})
\]

\[
\Gamma, \Sigma^\ast \rightarrow \Delta^\ast, \Theta \\
\wedge (mix)
\]

\[
\Gamma \rightarrow \Delta, A \quad A, \Sigma \rightarrow \Theta
\]

We replace \( T \) by \( T^\ast \) in \( D \) and obtain \( D^\ast \)
Proof of H Lemma

Now we can apply induction hypothesis with respect to the degree of the mix formula.

The mix formula $A$ in $D^*$ has a lower degree then the mix formula $(A \cap B)$.

By the inductive assumption the derivation $D^*$, and hence the derivation $D$ may be transformed into one without mix.

2. The case when the left premiss of the mix rule is the conclusion of a rule $(\rightarrow \cap)$ and right premiss of the mix rule is the conclusion of a rule $(\cap \rightarrow)_2$ is dual to 1. and is left as exercise.
Proof of H Lemma

3. The main connective of the mix formula is $\cup$, i.e. the mix formula is $(A \cup B)$
This case is to be dealt with symmetrically to the $\cap$ cases and is presented in the book chapter 6

4. The main connective of the mix formula is $\neg$, i.e. the mix formula is $\neg A$
This case is also presented in the book chapter 6

We consider now a slightly more complicated case of the implication, i.e. the case of the mix formula $(A \Rightarrow B)$
5. The main connective of the **mix** formula is $\implies$, i.e. the **mix** formula is $(A \implies B)$

Here is the **sub-tree** $T$ of $D$ containing the application of the **mix** rule

$$\Gamma, \Sigma \rightarrow \Delta, \Theta$$

$$\bigwedge (\text{mix})$$

$$\Gamma \rightarrow \Delta, (A \implies B)$$

$$| (\rightarrow \rightarrow)$$

$$A, \Gamma \rightarrow \Delta, B$$

$$(A \implies B), \Sigma \rightarrow \Theta$$

$$\bigwedge ((\implies \rightarrow))$$

$$\Sigma \rightarrow \Theta, A \quad B, \Sigma \rightarrow \Theta,$$
Proof of H Lemma

We transform $T$ into $T^*$ as follows.

$$\Gamma, \Sigma \longrightarrow \Delta, \Theta$$

*(possibly several weakenings and exchanges)*

$$\Sigma, \Gamma^*, \Sigma^{**} \longrightarrow \Theta^*, \Delta^*, \Theta$$

$$\bigwedge (mix)$$

$$\Sigma \longrightarrow \Theta, A$$

$$A, \Gamma, \Sigma^* \longrightarrow \Delta^*, \Theta$$

$$\bigwedge (mix)$$

$$A, \Gamma \longrightarrow \Delta, B$$

$$B, \Sigma \longrightarrow \Theta,$$
Proof of H Lemma

The asteriks are, of course, intended as follows

$\Sigma^*, \Delta^*$ results from $\Sigma, \Delta$ by the omission of all formulas $B$

$\Gamma^*, \Sigma^{**}, \Theta^*$ results from $\Gamma, \Sigma^*, \Theta$ by the omission of all formulas $A$
Proof of H Lemma

We replace the sub-tree $T$ by $T^*$ in $D$ and obtain $D^*$

Now we have **two mixes**, but both **mix** formulas $A$ and $B$ are of a **lower** degree then $n$

We first apply the **inductive assumption** to the lower mix (formula $B$) and the lower mix is **eliminated**
We then apply by the **inductive assumption** and **eliminate** the upper mix (formula $A$)

This **ends** the proof of the **case** of the rank $r=2$
Proof of H Lemma

Case \( r > 2 \)

In the case \( r = 2 \), we reduced the derivation to one of lower degree. Now we proceed to reduce the derivation to one of the same degree, but of a lower rank.

This allows us to be able to carry the induction with respect to the rank \( r \) of the derivation.

We use the inductive assumption in all cases except, as before, a case of an axiom or structural rules.

In these cases the mix can be eliminated immediately, as it was eliminated in the previous case of rank \( r = 2 \).
Proof of H Lemma

In a case of logical rules we obtain the reduction of the mix to derivations with mix of a lower ranks which consequently can be eliminated by the inductive assumption.

We carry proofs for two logical rules $(\to \cap)$ and $(\cup \to)$. The proof for all other rules is similar and is left as exercise.

We consider only the case of left rank $L_r = 1$ and right rank $R_r > 1$.

The symmetrical case of left rank $L_r > 1$ and right rank $R_r = 1$ is left as an exercise.
Proof of H Lemma

Case: \( Lr = 1 \) and \( Rr = r > 1 \)

The right premiss of the mix is a conclusion of the inference rule \((\to \cap)\), i.e. it is of a form

\[ \Gamma \to \Delta, (A \cap B) \]

where \( \Gamma \) contains a mix formula \( M \)

The left premiss of the mix is a sequent

\[ \Theta \to \Sigma \]

and \( \Sigma \) contains the mix formula \( M \)
Proof of H Lemma

The **sub-tree** $T$ of $D$ containing the application of the \texttt{mix} rule is

\[
\Theta, \Gamma^* \rightarrow \Sigma^*, \Delta, (A \cap B)
\]

\[\bigwedge (\text{mix})\]

\[
\Theta \rightarrow \Sigma \quad \Gamma \rightarrow \Delta, (A \cap B)
\]

\[\bigwedge (\rightarrow \cap)\]

\[
\Gamma \rightarrow \Delta, A \quad \Gamma \rightarrow \Delta, B
\]
Proof of H Lemma

We transform $T$ into $T^*$ as follows

$$
\Theta, \Gamma^* \rightarrow \Sigma^*, \Delta, (A \cap B)
$$

$$\wedge (\rightarrow \land)$$

$$
\Theta, \Gamma^* \rightarrow \Sigma^*, \Delta, A
\quad \Theta, \Gamma^* \rightarrow \Sigma^*, \Delta, B
$$

We perform mix on the left branch

$$
\Theta, \Gamma^* \rightarrow \Sigma^*, \Delta, A
$$

$$\wedge (\text{mix})$$

$$
\Theta \rightarrow \Sigma
\quad \Gamma \rightarrow \Delta, A
$$
Proof of H Lemma

We perform \textbf{mix} on the right branch

\[ \Theta, \Gamma^* \rightarrow \Sigma^*, \Delta, B \]

\[ \bigwedge (\text{mix}) \]

\[ \Theta \rightarrow \Sigma \quad \Gamma \rightarrow \Delta, B \]

We replace \( T \) by \( T^* \) in \( D \) and obtain \( D^* \)

Now we have \textbf{two mixes}, but both have the right rank \( Rr = r-1 \) and both of them can be \textbf{eliminated} by the \textbf{inductive assumption}
Proof of H Lemma

Case: \( Lr = 1 \) and \( Rr = r > 1 \)

The right premiss of the \textbf{mix} is a conclusion of the rule \((\cup \rightarrow)\), i.e. it is of a form

\[
(A \cup B), \Gamma \rightarrow \Delta
\]

and \( \Gamma \) contains a \textbf{mix formula} \( M \)

The left premiss of the \textbf{mix} is a sequent

\[
\Theta \rightarrow \Sigma
\]

and \( \Sigma \) contains the \textbf{mix formula} \( M \)
Proof of H Lemma

The **sub-tree** $T$ of $D$ containing the application of the **mix** rule is

\[
\Theta, \ (A \cup B)^*, \Gamma^* \rightarrow \Sigma^*, \Delta
\]

\[\wedge (\text{mix})\]

\[
\Theta \rightarrow \Sigma
\]

\[(A \cup B), \Gamma \rightarrow \Delta
\]

\[\wedge (\cup \rightarrow)\]

\[
A, \Gamma \rightarrow \Delta \quad B, \Gamma \rightarrow \Delta
\]
Proof of H Lemma

\((A \cup B)^*\) stands either for \((A \cup B)\) or for nothing according as \((A \cup B)\) is unequal or equal to the mix formula \(M\)

The mix formula \(M\) certainly occurs in \(\Gamma\)

For otherwise \(M\) would been equal to \((A \cup B)\) and the right rank \(R_r\) would be equal to 1 contrary to the assumption that \(R_r > 1\)
Proof of H Lemma

We transform $T$ into $T^*$ as follows

$$\Theta, (A \cup B), \Gamma^* \rightarrow \Sigma^*, \Delta$$

$$\bigwedge (\cup \rightarrow)$$

$$A, \Theta, \Gamma^* \rightarrow \Sigma^*, \Delta \quad B, \Theta, \Gamma^* \rightarrow \Sigma^*, \Delta$$

We perform **mix** on the left branch

$$A, \Theta, \Gamma^* \rightarrow \Sigma^*, \Delta$$

(some weakenings, exchanges)

$$\Theta, A^*, \Gamma^* \rightarrow \Sigma^*, \Delta$$

$$\bigwedge (mix)$$

$$\Theta \rightarrow \Sigma \quad A, \Gamma \rightarrow \Delta$$
Proof of H Lemma

We perform \textbf{mix} on the right branch

\[
B, \Theta, \Gamma^* \rightarrow \Sigma^*, \Delta
\]

(some weakenings, exchanges)

\[
\Theta, B^*, \Gamma^* \rightarrow \Sigma^*, \Delta
\]

\[
\wedge (\text{mix})
\]

\[
\Theta \rightarrow \Sigma
\]

\[
B, \Gamma \rightarrow \Delta
\]
Proof of H Lemma

Now we have two mixes
But both have the right rank $R_r = r-1$ and hence both of them can be eliminated by the inductive assumption

We replace $T$ by $T^*$ in $D$ and obtain $D^*$

This ends the proof of the Hauptzatz Lemma
We have hence completed the proof of the Hauptzatz Theorem
**LK and LI Hauptzatz Theorems**
**LK and LI Hauptzatz Theorems**

Let’s denote by $\text{LK} - \text{c}$ and $\text{LI} - \text{c}$ the systems $\text{LK}$, $\text{LI}$ without the cut rule, i.e. we put

$$\text{LK} - \text{c} = \text{LK} - \{(\text{cut})\}$$

$$\text{LI} - \text{c} = \text{LI} - \{(\text{cut})\}$$

We re-write the **Hauptzatz Theorems** as follows.
**LK and LI Hauptzatz Theorem**

**LK Hauptzatz**
For every **LK** sequent $\Gamma \rightarrow \Delta$,

$$\vdash_{LK} \Gamma \rightarrow \Delta \text{ if and only if } \vdash_{LK-c} \Gamma \rightarrow \Delta$$

**LI Hauptzatz**
For every **LI** sequent $\Gamma \rightarrow \Delta$,

$$\vdash_{LI} \Gamma \rightarrow \Delta \text{ if and only if } \vdash_{LI-c} \Gamma \rightarrow \Delta$$

This is why the **cut-free** Gentzen systems **LK-c** and **LI-c** are just called **LK**, **LI**, respectively.
LK-c Completeness

Directly from the **LK Completeness Theorem** and the **LK Hauptzatz Theorem** we get that the following.

**LK-c Completeness Theorem**

For any sequent $\Gamma \rightarrow \Delta$,

$$\vdash_{LK-c} \Gamma \rightarrow \Delta \text{ if and only if } \models \Gamma \rightarrow \Delta$$
LK and GK Systems Equivalency
Let $G$ be the Gentzen sequents proof system defined previously. We replace the logical axiom of $G$

$$\Gamma', a, \Gamma' \rightarrow \Delta', a, \Delta'$$

where $a \in \text{VAR}$ is any propositional variable and

$$\Gamma', \Gamma', \Delta', \Delta' \in \text{VAR}^*$$

are any indecomposable sequences, by a new logical axiom

$$\Gamma, A, \Gamma \rightarrow \Delta, A, \Delta$$

for any $A \in \mathcal{F}$ and any sequences

$$\Gamma, \Gamma, \Delta, \Delta \in \text{SQ}$$
We call a resulting proof system \textbf{GK}, i.e. we defined it as follows

\[
\textbf{GK} = ( \mathcal{L}_{\cup, \cap, \Rightarrow, \neg}, \text{SQ}, \text{LA}, \mathcal{R} )
\]

where \text{LA} is the \textit{new axiom} defined above and \text{R} is the set of rules of the system \textbf{G}.

\textbf{Observe} that the \textit{only difference} between the systems \textbf{GK} and \textbf{G} is the form of their logical \textit{axioms}, both being \textit{tautologies}.

We get the proof of \textbf{completeness} of \textbf{GK} in the same way as we proved it for \textbf{G}, i.e. we have the following
GK Completeness

**GK Completeness Theorem**

For any formula $A \in \mathcal{F}$,

$\vdash_{\text{GK}} A$ if and only if $\models A$

For any sequent $\Gamma \rightarrow \Delta \in \mathcal{SQ}$

$\vdash_{\text{GK}} \Gamma \rightarrow \Delta$ if and only if $\models \Gamma \rightarrow \Delta$
LK and GK Systems Equivalency

By the GK, LK-c Completeness Theorems we get the equivalency of GK and the cut free LK-c proof systems.

LK, GK Equivalency Theorem
The proof systems GK and the cut free LK are equivalent, i.e for any sequent \( \Gamma \rightarrow \Delta \),

\[ \vdash_{LK} \Gamma \rightarrow \Delta \text{ if and only if } \vdash_{GK} \Gamma \rightarrow \Delta \]