cse541
LOGIC for Computer Science

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LECTURE 6b

Chapter 6
Automated Proof Systems
Completeness of Classical Propositional Logic

PART 5: Original Gentzen Systems **LK, LI**

Classical and Intiutionistic Completeness and Hauptzatz Theorem

Original Gentzen Systems **LK, LI**

The **original** systems **LK** and **LI** were created by **Gentzen** in 1935 for classical and intuitionistic **predicate** logics, respectively

We present now their **propositional** versions and use the same names **LK** and **LI**

The proof system **LI** for intuitionistic logic is a particular case of the proof system **LK**

Original Gentzen Systems **LK, LI**

Both systems **LK** and **LI** have **two groups** of the inference **rules**

They both have a special rule called a **cut rule**

First group consists of a set of rules similar to the rules of systems **GL** and **G** calllled **Logical Rules**

Second group contains a new type of rules
We call them **Structural Rules**

Original Gentzen Systems **LK, LI**

The **cut** rule in Gentzen sequent systems **corresponds** to the Modus Ponens rule in Hilbert proof systems

Modus Ponens is a particular case of the **cut** rule

The **cut** rule is needed to carry out the original Gentzen proof of the **completeness** of the system **LK** and for proving the **adequacy** of **LI** system for intituitionistic logic

Original Gentzen Systems **LK, LI**

Gentzen proof of **completeness** of **LK** was **not direct**

He used the **completeness** of already known Hilbert proof system H and **proved** that any formula that is provable in the systems H is also provable in **LK**

Hence the need of the **cut** rule

Original Gentzen Systems **LK, LI**

For the system **LI** he proved only the **adequacy** of **LI** system for intituitionistic logic since the **semantics** for the intuitionistic logic **didn't** yet **exist**

He used the **acceptance** of Heying intuitionistic axiom system as a **definition** of the intuitionistic logic and **proved** that any formula provable in the Heyting system is also provable in **LI**

Original Gentzen Systems **LK, LI**

Observe that by presence of the **cut** rule, Gentzen systems **LK** and **LI** **are also** Hilbert system

What **distinguishes** the Gentzen systems from all other known Hilbert proof systems is the **fact** that the **cut rule** could be eliminated from them, what is not the case of regular Hilbert proof systems

This is why Gentzen famous **Hauptzatz Theorem**, is also called **Cut Elimination Theorem**

Original Gentzen Systems **LK, LI**

The elimination of the **cut** rule and the structure of other **rules** makes it possible to define an effective automatic procedures for proof search, what is **impossible** in a case of the Hilbert style systems

Gentzen in his proof of **Hauptzatz Theorem** developed a powerful technique of proof **adaptable** to other logics

Original Gentzen Systems **LK, LI**

We present here the Gentzen **cut elimination** technique for the classical propositional case and show how to **adapt** it to the intuitionistic case

Gentzen proof is purely **syntactical**

The proof defines a **constructive** method of transformation of any formal **proof** (derivation) of a sequent $\Gamma \longrightarrow \Delta$ that uses the **cut** rule (and other rules) into its proof without use of the **cut** rule

Hence the English name **Cut Elimination Theorem**

Gentzen System **LK**

LK Components

Language

$$\mathcal{L} = \mathcal{L}_{\{\neg, \cap, \cup, \Rightarrow\}} \quad \text{and} \quad \mathcal{E} = SQ$$

for

$$SQ = \{\Gamma \longrightarrow \Delta : \quad \Gamma, \Delta \in \mathcal{F}^*\}$$

Logical Axioms

There is only one logical axiom, namely

$$A \longrightarrow A$$

where A is any formula of $\mathcal{L}$

LK Components

Rules of Inference

Group one: **Structural Rules**

Weakening

$$(weak \rightarrow) \quad \frac{\Gamma \longrightarrow \Delta}{A, \Gamma \longrightarrow \Delta}$$

$$(\rightarrow weak) \quad \frac{\Gamma \longrightarrow \Delta}{\Gamma \longrightarrow \Delta, A}$$

Contraction

$$(contr \rightarrow) \quad \frac{A, A, \Gamma \longrightarrow \Delta}{A, \Gamma \longrightarrow \Delta}$$

$$(\rightarrow contr) \quad \frac{\Gamma \longrightarrow \Delta, A, A}{\Gamma \longrightarrow \Delta, A}$$

LK Components

Exchange

$$(exch \rightarrow)\quad \frac{\Gamma_1,\ A, B,\ \Gamma_2\ \longrightarrow\ \Delta}{\Gamma_1,\ B, A,\ \Gamma_2\ \longrightarrow\ \Delta}$$

$$(\rightarrow exch)\quad \frac{\Delta \longrightarrow \Gamma_1,\ A, B,\ \Gamma_2}{\Delta \longrightarrow \Gamma_1,\ B, A,\ \Gamma_2}$$

Cut Rule

$$(cut)\quad \frac{\Gamma\ \longrightarrow\ \Delta,\ A\quad ;\quad A,\ \Sigma\ \longrightarrow\ \Theta}{\Gamma, \Sigma\ \longrightarrow\ \Delta, \Theta}$$

A is called a **cut formula**

LK Components

Group Two: Logical Rules

Conjunction rules

$$(\cap \rightarrow)_1 \quad \frac{A, \ \Gamma \ \longrightarrow \ \Delta}{(A \cap B), \ \Gamma \ \longrightarrow \ \Delta}$$

$$(\cap \rightarrow)_2 \quad \frac{B, \ \Gamma \ \longrightarrow \ \Delta}{(A \cap B), \ \Gamma \ \longrightarrow \ \Delta}$$

$$(\rightarrow \cap) \quad \frac{\Gamma \ \longrightarrow \Delta, \ A \quad ; \quad \Gamma \ \longrightarrow \Delta, \ B, \Delta}{\Gamma \ \longrightarrow \ \Delta, \ (A \cap B)}$$

LK Components

Disjunction rules

$$(\rightarrow \cup)_1 \quad \frac{\Gamma \longrightarrow \Delta, A}{\Gamma \longrightarrow \Delta, (A \cup B)}$$

$$(\rightarrow \cup)_2 \quad \frac{\Gamma \longrightarrow \Delta, B}{\Gamma \longrightarrow \Delta, (A \cup B)}$$

$$(\cup \rightarrow) \quad \frac{A, \Gamma \longrightarrow \Delta \quad ; \quad B, \Gamma \longrightarrow \Delta}{(A \cup B), \Gamma \longrightarrow \Delta}$$

LK Components

Implication rules

$$(\longrightarrow\Rightarrow)\quad \frac{A,\ \Gamma\ \longrightarrow\ \Delta,\ B}{\Gamma\ \longrightarrow\ \Delta,\ (A \Rightarrow B)}$$

$$(\Rightarrow\longrightarrow)\quad \frac{\Gamma\ \longrightarrow\ \Delta,\ A \quad ; \quad B,\ \Gamma\ \longrightarrow\ \Delta}{(A \Rightarrow B),\ \Gamma\ \longrightarrow\ \Delta}$$

Negation rules

$$(\neg\longrightarrow)\quad \frac{\Gamma\ \longrightarrow\ \Delta,\ A}{\neg A,\ \Gamma\ \longrightarrow\ \Delta}$$

$$(\longrightarrow\neg)\quad \frac{A,\ \Gamma\ \longrightarrow\ \Delta}{\Gamma\ \longrightarrow\ \Delta,\ \neg A}$$

LK Definition

Classical System LK

We **define** the classical Gentzen system **LK** as

$$\mathbf{LK} = (\mathcal{L},\ SQ,\ LA, \mathcal{R}\ )$$

where

$$\mathcal{R} = \{\ \textbf{Structural}\ \text{ Rules},\ \ \textbf{Cut}\ \text{Rule},\ \ \textbf{Logical}\ \text{ Rules})$$

as defined by the **components** definitions

LI Definition

Intuitionistic System LI

We **define** the **intuitionistic** Gentzen system **LI** as

$$\mathbf{LI} = (\mathcal{L},\ ISQ,\ AL, \mathcal{R}\ )$$

$\mathcal{R} = \{$ **I - Structural** Rules, **I - Cut** Rule, **I -Logical** Rules)

where $\mathcal{R}$ are the **LK** rules restricted to the set ISQ of the **intuitionistic sequents** defined as follows

$$ISQ = \{\Gamma \longrightarrow \Delta : \quad \Delta \quad \text{consists of at most one formula}\ \}$$

We will study the intuitionistic system **LI** in Chapter 7

Classical System **LK**

We say that a formula $A \in \mathcal{F}$ has a **proof** in **LK** and **denote** it by

$$\vdash_{\mathbf{LK}} A$$

if the sequent $\longrightarrow A$ has a proof in **LK**, i.e. we write

$$\vdash_{\mathbf{LK}} A \quad \text{if and only if} \quad \vdash_{\mathbf{LK}} \longrightarrow A$$

LK Proof Trees

We write **formal proofs** in **LK**, as we did for other Gentzen style proof systems in a form of the **proof trees** defined as follows

Definition

By a **proof tree** of a sequent $\Gamma \longrightarrow \Delta$ in **LK** we understand a **tree**

$$\mathbf{D}_{\Gamma \longrightarrow \Delta}$$

satisfying the following conditions:

1. The topmost sequent, i.e the **root** of $\mathbf{D}_{\Gamma \longrightarrow \Delta}$ is $\Gamma \longrightarrow \Delta$

2. All **leaves** are axioms

3. The **nodes** are sequents such that each sequent on the tree **follows** from the ones immediately preceding it **by** one of the rules

Derivations in **LK**

Proofs are often called **derivations**

In particular, Gentzen, in his work used the term **derivation** for the proof and we will use this notion as well

This is why we **denote** the proof trees by **D**, for the **derivation**

Finding derivations **D** in **LK** is a complex **process**

LK **logical rules** are different, then in **GL** and **G**

Consequently, proofs rely strongly on use of the **structural rules**

Derivations in **LK**

For **example**, a **derivation** of Excluded Middle $(A \cup \neg A)$ formula in **LK** is as follows

D

$$\longrightarrow (A \cup \neg A)$$

$$\mid (\rightarrow contr)$$

$$\longrightarrow (A \cup \neg A), (A \cup \neg A)$$

$$\mid (\rightarrow \cup)_1$$

$$\longrightarrow (A \cup \neg A), A$$

$$\mid (\rightarrow exch)$$

$$\longrightarrow A, (A \cup \neg A)$$

$$\mid (\rightarrow \cup)_1$$

$$\longrightarrow A, \neg A$$

$$\mid (\rightarrow \neg)$$

$$A \longrightarrow A$$

axiom

Derivations in **LK**

Here is as yet another example a cut free derivation in **LK**

D

$$\longrightarrow (\neg(A \cap B) \Rightarrow (\neg A \cup \neg B))$$

$$\mid (\rightarrow\Rightarrow)$$

$$(\neg(A \cap B) \longrightarrow (\neg A \cup \neg B))$$

$$\mid (\rightarrow \neg)$$

$$\longrightarrow (\neg A \cup \neg B),\ (A \cap B)$$

$$\bigwedge(\Rightarrow\longrightarrow)$$

$$\longrightarrow (\neg A \cup \neg B),\ A \qquad\qquad \longrightarrow (\neg A \cup \neg B),\ B$$

$$\mid (\rightarrow exch) \qquad\qquad \mid (\rightarrow exch)$$

$$\longrightarrow A, (\neg A \cup \neg B) \qquad\qquad \longrightarrow B, (\neg A \cup \neg B)$$

$$\mid (\rightarrow \cup)_1 \qquad\qquad \mid (\rightarrow \cup)_1$$

$$\longrightarrow A, \neg A \qquad\qquad \longrightarrow B, \neg B$$

$$\mid (\rightarrow \neg) \qquad\qquad B \longrightarrow B$$

$$A \longrightarrow A \qquad\qquad \textit{axiom}$$

axiom

LK Soundness

LK Soundness

Observe that the **Logical Rules** of **LK** are similar in their structure to the rules of the system **G**

Hence **LK** **Logical Rules** admit similar proof of their **soundness**

The sound rules

$$(\rightarrow \cap)_1, \quad (\rightarrow \cap)_2 \quad \text{and} \quad (\rightarrow \cup)_1, \quad (\rightarrow \cup)_2$$

are not strongly sound because

$$A \not\equiv (A \cap B), \quad B \not\equiv (A \cap B) \quad \text{and} \quad A \not\equiv (A \cup B), \quad B \not\equiv (A \cup B)$$

All other **Logical Rules** are strongly sound.

LK Soundness

The Contraction and Exchange structural rules are **strongly sound** as for any formulas $A, B \in \mathcal{F}$,

$$A \equiv (A \cap A), \quad A \equiv (A \cup A) \quad \text{and}$$

$$(A \cap B) \equiv (B \cap A), \quad (A \cap B) \equiv (B \cap A)$$

The Weakening rule is **sound** because (we use shorthand notation)

$$\text{if} \quad (\Gamma \Rightarrow \Delta) = T \quad \text{then} \quad ((A \cap \Gamma) \Rightarrow \Delta) = T$$

for any logical value of the formula A

Obviously

$$(\Gamma \Rightarrow \Delta) \not\equiv ((A \cap \Gamma) \Rightarrow \Delta))$$

i.e. the Weakening rule **is not** strongly sound

LK Soundness

The Cut rule is **sound** as the fact that

$$(\Gamma \Rightarrow (\Delta \cup A)) = T \quad \text{and} \quad ((A \cap \Sigma) \Rightarrow \Lambda) = T$$

implies that

$$((\Gamma \cap \Sigma) \;\Rightarrow\; (\Delta \cup \Lambda)) = T$$

Cut rule **is not** strongly sound

Any truth assignment such that

$$\Gamma = T \quad \text{and} \quad \Delta = \Sigma = \Lambda = A = F$$

proves that

$$(\Gamma \;\longrightarrow\; \Delta,\; A) \cap (A,\; \Sigma \;\longrightarrow\; \Lambda) \not\equiv (\Gamma, \Sigma \;\longrightarrow\; \Delta, \Lambda)$$

LK Soundness

Obviously, the **Logical Axiom** is a tautology, i.e.

$$\models A \longrightarrow A$$

We have proved that **LK** is **sound** and the following theorem holds

Soundness Theorem

For any sequent $\Gamma \longrightarrow \Delta$,

$$\text{if } \vdash_{\mathbf{LK}} \Gamma \longrightarrow \Delta, \text{ then } \models \Gamma \longrightarrow \Delta$$

In particular, for any $A \in \mathcal{F}$,

$$\text{if } \vdash_{\mathbf{LK}} A, \text{ then } \models A$$

LK Completeness

LK Completeness

We follow Gentzen original proof of completeness of **LK**

We choose any **complete** Hilbert proof system for the **LK** language

$$\mathcal{L} = \mathcal{L}_{\{\neg, \cap, \cup, \Rightarrow\}}$$

and **prove**, after Gentzen, its **equivalency** with **LK**

Gentzen **referred** to the Hilbert-Ackerman (1920) system (axiomatization) included in chapter 5

We **choose** the Rasiowa-Sikorski (1952) formalization *R* also included in Chapter 5

LK Completeness

We **choose** the formalization R for two reasons

First, it reflexes a **connection** between classical and intuitionistic logics very much in a spirit Gentzen **relationship** between **LK** and **LI**

We obtain a **complete** proof system I from R by just **removing** the last axiom A12

Second, both sets of axioms reflect the best what set of rovable formulas is needed to conduct algebraic proofs of **completeness** of R and I, as discussed in Chapter 7

Hilbert System R

The set of **logical axioms** of the poof system *R*

A1 $((A \Rightarrow B) \Rightarrow ((B \Rightarrow C) \Rightarrow (A \Rightarrow C)))$

A2 $(A \Rightarrow (A \cup B))$

A3 $(B \Rightarrow (A \cup B))$

A4 $((A \Rightarrow C) \Rightarrow ((B \Rightarrow C) \Rightarrow ((A \cup B) \Rightarrow C)))$

A5 $((A \cap B) \Rightarrow A)$

A6 $((A \cap B) \Rightarrow B)$

A7 $((C \Rightarrow A) \Rightarrow ((C \Rightarrow B) \Rightarrow (C \Rightarrow (A \cap B)))$

A8 $((A \Rightarrow (B \Rightarrow C)) \Rightarrow ((A \cap B) \Rightarrow C))$

A9 $(((A \cap B) \Rightarrow C) \Rightarrow (A \Rightarrow (B \Rightarrow C))$

A10 $(A \cap \neg A) \Rightarrow B)$

A11 $((A \Rightarrow (A \cap \neg A)) \Rightarrow \neg A)$

Hilbert System R

A12 $\quad (A \cup \neg A)$

where $A, B, C \in \mathcal{F}$ are any formulas

We adopt a Modus Ponens

$$(MP)\ \frac{A\ ;\ (A \Rightarrow B)}{B}$$

as the **only** inference rule

We **define** the proof system R as

$$R = (\ \mathcal{L}_{\{\neg, \cap, \cup, \Rightarrow\}},\ \ \mathcal{F},\ \ \{A1 - A12\},\ \ (MP)\ )$$

where A1 - A12 are **logical axioms** defined above

Hilbert System R

The system R is **complete**, i.e. we have the following

R **Completeness Theorem**

For any formula $A \in \mathcal{F}$,

$$\vdash_R A \text{ if and only if } \models A$$

We leave it as an exercise to show that all axioms A1 - A12 of the system R are **provable** in **LK**

Moreover, the Modus Ponens rule of R is a **particular case** of the Cut rule, namely

$$(MP) \quad \frac{\longrightarrow A \; ; \; A \longrightarrow B}{\longrightarrow B}$$

This proves the following theorem

Hilbert System R

Provability Theorem

For any formula $A \in \mathcal{F}$

$$\text{if} \quad \vdash_R A, \quad \text{then} \quad \vdash_{\mathbf{LK}} A$$

Directly from the above provability theorem, the soundness of **LK** and the completeness of R we get the following

LK Completeness Theorem

For any formula $A \in \mathcal{F}$

$$\vdash_{\mathbf{LK}} A \quad \text{if and only if} \quad \models A$$

Hauptzatz

Hauptzatz

Here is Gentzen **original** formulation of the **Hauptzatz Theorems** for classical **LK** and intuitionistic **LI** proof systems

They are also routinely called the **Cut Elimination Theorems**

LK Hauptzatz

Every derivation in **LK** can be transformed into another **LK** derivation of the same sequent, in which no cuts occur

LI Hauptzatz

Every derivation in **LI** can be transformed into another **LI** derivation of the same sequent, in which no cuts occur

Mix Rule

Hauptzatz proof is quite long and very involved. We present its **main** and most important **steps**

To facilitate the **proof** we introduce as Gentzen did, a general form of the **cut rule**, called a **mix rule**

It is defined as follows

$$(mix) \quad \frac{\Gamma \longrightarrow \Delta \quad ; \quad \Sigma \longrightarrow \Theta}{\Gamma, \Sigma^* \longrightarrow \Delta^*, \Theta}$$

where Σ^*, Δ^* are obtained from Σ, Δ by **removing** all occurrences of a common formula A

The formula A is now called a mix formula

Mix Example

Here are some **examples** of an applications of the mix rule

Observe t hat the mix rule applies, as the cut does, to only **one** mix formula at the time

b is the mix formula in

$$(mix)\ \frac{a \longrightarrow b,\ \neg a \quad ; \quad (b \cup c),\ b,\ b, D, b \longrightarrow}{a,\ (b \cup c),\ D \longrightarrow \neg a}$$

B is the mix formula in

$$(mix)\ \frac{A \longrightarrow B,\ B,\ \neg A \quad ; \quad (b \cup c),\ B,\ B, D,\ B \longrightarrow \neg B}{A,\ (b \cup c),\ D \longrightarrow \neg A, \neg B}$$

$\neg A$ is the mix formula in

$$(mix)\ \frac{A \longrightarrow B, \neg A,\ \neg A \quad ; \quad \neg A,\ B,\ B, \neg A, B \longrightarrow \neg B}{A,\ B,\ B \longrightarrow B, \neg B}$$

Mix and Cut

Notice, that every derivation with **cut** may be transformed into a derivation with **mix**

We do so by means of a number of **weakenings** and **interchanges**, i.e. multiple application of the **weakening** rules **exchange** rules

Conversely, every **mix** may be transformed into a **cut** derivation by means of a certain number of preceding **exchanges** and **contractions**, though we do not use this fact in the Hauptzatz proof

Observe that **cut** is a particular case of **mix**

Two Hauptzatz Theorems

There are two Hauptzatz theorems: classical **LK Hauptzatz** and **LI Hauptzatz**

The **proof** of intuitionistic **LI Hauptzatz** is basically **the same** as for **LK**

We must just be careful and **add,** at each step, the **restriction** made to the ISQ sequents and the form of the **LI** rules of inference. These **restrictions** do not alter the flow and validity of the **LK** proof

We discuss and present now the **proof** of **LK Hauptzatz**

We leave it as a homework exercise to **re-write** this proof the case of for **LI**

Proof of LK Hauptzatz

We conduct the proof in three **main steps**

Step 1: we consider only derivations in which only **mix rule** is used

Step 2: we consider first derivation with a certain **Property H** (to be defined) and prove an **H Lemma** for them

The **H Lemma** is the most crucial for the proof of the **Hauptzatz**

Property H

Property H

We say that a derivation $\mathbf{D}_{\Gamma \longrightarrow \Delta}$ of a sequent $\Gamma \longrightarrow \Delta$ has a **Property H** if it satisfies the the following conditions

1. The **root** $\Gamma \longrightarrow \Delta$ of the derivation $\mathbf{D}_{\Gamma \longrightarrow \Delta}$ is obtained by direct use of the **mix rule**

It means that the **mix** rule is the last rule used in the derivation of $\Gamma \longrightarrow \Delta$

2. The derivation $\mathbf{D}_{\Gamma \longrightarrow \Delta}$ **does not** contain any other application of the **mix rule**

H Lemma

H Lemma

Any derivation that fulfills the **Property H** may be transformed into a derivation of the same sequent in which **no mix** occurs

Step 3: we use the **H Lemma** and to prove the **Hauptzatz**

Proof of Hauptzatz

Step 3: **Hauptzatz** proof from **H Lemma**

Let $\mathbf{D}$ be any derivation (tree proof)
Let $\Gamma \longrightarrow \Delta$ be any node on $\mathbf{D}$ such that its **sub-tree** $\mathbf{D}_{\Gamma \longrightarrow \Delta}$ has the **Property H**

By **H Lemma** the sub-tree $\mathbf{D}_{\Gamma \longrightarrow \Delta}$ can be **replaced** by a tree $\mathbf{D}^*_{\Gamma \longrightarrow \Delta}$ in which no mix occurs
The rest of $\mathbf{D}$ remains unchanged

We **repeat** this procedure for each node N, such that the **sub-tree** $\mathbf{D}_N$ has the **Property H** until every application of **mix** rule has systematically been eliminated

This **ends** the proof of **Hauptzatz** provided the **H Lemma** has already been proved

Proof of H Lemma

Step 2: proof of **H lemma**

We consider derivation tree **D** with the **Property H**

It means that **D** is such that the **mix rule** is the last rule of inference **used** and **D** **does not** contain any other application of the **mix** rule

Observe that **D** contains only one application of **mix** rule, and the **mix** rule, contains only one mix formula A

Mix rule used may contain many copies of A, but there always is only one **mix** formula A. We call A the **mix formula** of **D**

We **define** two important notions: degree n and rank r of the derivation **D**

Degree of **D**

Definition

Given a derivation tree **D** with the **Property H**

Let $A \in \mathcal{F}$ be the mix formula of **D** The degree $n \geq 0$ of A is called the **degree** of the **derivation** **D**

We write it as

$$\deg \mathbf{D} = \deg A = n$$

Degree of D

Definition

Given a derivation tree **D** with the **Property H**

We define the **rank** r of **D** as a sum of its **left rank** Lr and **right rank** Rr of **D**, i.e.

$$r = Lr + Rr$$

where:

1. **left rank** Lr of **D** is the largest number of consecutive nodes on the branch of **D** staring with the node containing the **left** premiss of the **mix rule**, such that each sequent on these nodes contains the mix formula in the **succedent**;

2. **right rank** Rr of **D** is the largest number of consecutive nodes on the branch of **D** staring with the node containing the **right** premiss of the **mix rule**, such that each sequent on these nodes contains the mix formula in the **antecedent**.

Proof of H Lemma

We prove the **H Lemma** by carrying out two inductions

One on the **degree** n, the other on the **rank** r, of the derivation **D**

It means we prove the **H Lemma** for a derivation of the degree n, assuming it **to hold** for derivations of a lower degree as long as $n \neq 0$, i.e. we assume that derivations of lower degree **can** be already **transformed** into derivations without mix

Proof of H Lemma

The lowest possible **rank** is evidently 2

We begin by considering the **case 1** when the rank is $r = 2$
We carry induction with respect to the degree n of the derivation **D**

After that we examine the **case 2** when the rank is $r > 2$
and we assume that the **H Lemma** already **holds** for derivations of the same degree, but a lower rank

Proof of H Lemma

Case 1. Rank of r =2

We carry induction with respect to the degree n of derivation **D**, i.e. with respect to degree $n \geq 0$ of the **mix formula**

We split the induction cases to consider in two groups

GROUP 1. Axioms and Structural Rules

GROUP 2. Logical Rules

We present now some cases of rules of inference as examples. There are some more cases presented in the chapter, and the rest are left as exercises

Proof of H Lemma

Observe that first group contains cases that are especially simple in that they allow the **mix** to be immediately eliminated

The **second group** contains the most important cases since their consideration brings out the **basic idea** behind the whole proof

Here we use the induction hypothesis with respect do the **degree** of the derivation. We reduce each one of the cases to **transformed** derivations of a lower degree

Proof of H Lemma

GROUP 1. Axioms and Structural Rules

1. The left premiss of the **mix rule** is an axiom

$$A \longrightarrow A$$

Then the **sub-tree** of **D** containing **mix** is as follows

$$A,\ \Sigma^* \longrightarrow \Delta$$

$$\bigwedge(mix)$$

$$A \longrightarrow A \qquad\qquad \Sigma \longrightarrow \Delta$$

Proof of H Lemma

We **transform** it, and **replace** it in the derivation tree **D** by

$$A, \Sigma^* \longrightarrow \Delta$$

(possibly several exchanges and contractions)

$$\Sigma \longrightarrow \Delta$$

Such obtained tree **D*** proves the same sequent as **D** and contains **no mix**

Proof of H Lemma

2 . The right premiss of the **mix rule** is an axiom $A \longrightarrow A$

Then the **sub-tree** of **D** containing **mix** is as follows

$$\Sigma \longrightarrow \Delta^*, A$$

$$\bigwedge (mix)$$

$$\Sigma \longrightarrow \Delta \qquad\qquad A \longrightarrow A$$

We **transform** it, and **replace** it in **D** by

$$\Sigma \longrightarrow \Delta^*, A$$

(possibly several exchanges and contractions)

$$\Sigma \longrightarrow \Delta$$

Such obtained $\mathbf{D}^*$ proves the same sequent and contains no mix

Proof of H Lemma

Suppose that **neither** of premisses of **mix** is an axiom

As the **rank** is r=2 , the right and left **ranks** are requal 1

This means that in the sequents on the nodes directly below **left** premiss of the **mix**, the mix formula A **does not** occur in the succedent; in the sequents on the nodes directly below **right** premiss of the **mix**, the mix formula A **does not** occur in the antecedent

In general, if a formula occurs in the antecedent (succedent) of a conclusion of a rule of inference, it is either obtained by a **logical** rule or by a **contraction** rule

Proof of H Lemma

3. The left premiss of the **mix rule** is the conclusion of a contraction rule. The sub-tree of **D** containing **mix** is:

$$\Gamma,\ \Sigma^* \ \longrightarrow\ \Delta,\ \Theta$$

$$\bigwedge(mix)$$

$$\Gamma \longrightarrow\ \Delta,\ A \qquad\qquad \Sigma \longrightarrow \Theta$$

$$|\,(\rightarrow contr)$$

$$\Gamma \longrightarrow\ \Delta$$

Proof of H Lemma

We **transform** it, and **replace** it in **D** by

$$\Gamma,\ \Sigma^{*}\ \longrightarrow\ \Delta,\ \Theta$$

(possibly several weakenings and exchanges)

$$\Gamma \longrightarrow \Delta$$

Such obtained **D*** contains **no mix**

Observe that the whole **branch** of **D** that starts with the node $\Sigma \longrightarrow \Theta$ **disappears**

4. The right premiss of the **mix rule** is the conclusion of a contraction rule $(\rightarrow \mathit{contr})$. It is a dual case to **3.** s left as an exercise

Proof of H Lemma

GROUP 2. Logical Rules

1. The mix formula is $(A \cap B)$ The **left** premiss of the **mix** rule is the conclusion of a rule $(\rightarrow \cap)$. The **right** premiss of the **mix** rule is the conclusion of a rule $(\cap \rightarrow)_1$

The **sub-tree T** of **D** containing **mix** is:

$$\Gamma, \Sigma \longrightarrow \Delta, \Theta$$

$$\bigwedge (mix)$$

$$\Gamma \longrightarrow \Delta, (A \cap B) \qquad\qquad (A \cap B), \Sigma \longrightarrow \Theta$$

$$\bigwedge (\rightarrow \cap) \qquad\qquad \mid (\cap \rightarrow)_1$$

$$\Gamma \longrightarrow \Delta, A \quad \Gamma \longrightarrow \Delta, B \qquad\qquad A, \Sigma \longrightarrow \Theta$$

Proof of H Lemma

We transform **T** into **T*** as follows.

$$\Gamma,\ \Sigma \ \longrightarrow\ \Delta,\ \Theta$$

(possibly several weakenings and exchanges)

$$\Gamma,\ \Sigma^* \ \longrightarrow\ \Delta^*,\ \Theta$$

$$\bigwedge(mix)$$

$$\Gamma \longrightarrow\ \Delta, A \qquad\qquad A, \Sigma \longrightarrow \Theta$$

We replace **T** by **T*** in **D** and obtain **D***

Proof of H Lemma

Now we can apply induction hypothesis with respect to the **degree** of the **mix** formula

The **mix** formula A in $\mathbf{D}^*$ has a lower degree then the **mix** formula $(A \cap B)$

By the inductive assumption the derivation $\mathbf{D}^*$, and hence the derivation $\mathbf{D}$ may be **transformed** into one without mix

2. The case when the left premiss of the**mix** rule is the conclusion of a rule $(\rightarrow \cap)$ and right premiss of the **mix** rule is the conclusion of a rule $(\cap \rightarrow)_2$ is dual to **1.** and is left as exercise

Proof of H Lemma

3. The main connective of the mix formula is $\cup$, i.e. the mix formula is $(A \cup B)$

This case is to be dealt with symmetrically to the $\cap$ cases and is presented in the book chapter 6

4. The main connective of the mix formula is $\neg$, i.e. the **mix** formula is $\neg A$

This case is also presented in the book chapter 6

We consider now a slightly more complicated case of the implication, i.e. the case of the **mix** formula $(A \Rightarrow B)$

Proof of H Lemma

5. The main connective of the **mix** formula is $\Rightarrow$, i.e. the **mix** formula is $(A \Rightarrow B)$

Here is the **sub-tree T** of **D** containing the application of the **mix** rule

$$\Gamma, \Sigma \longrightarrow \Delta, \Theta$$

$$\bigwedge(mix)$$

$$\Gamma \longrightarrow \Delta, (A \Rightarrow B) \qquad\qquad (A \Rightarrow B), \Sigma \longrightarrow \Theta$$

$$\mid (\rightarrow\Rightarrow) \qquad\qquad \bigwedge((\Rightarrow\rightarrow)$$

$$A, \Gamma \longrightarrow \Delta, B$$

$$\Sigma \longrightarrow \Theta, A \qquad B, \Sigma \longrightarrow \Theta,$$

Proof of H Lemma

We transform **T**into $\mathbf{T}^*$ as follows.

$$\Gamma,\ \Sigma\ \longrightarrow\ \Delta,\ \Theta$$

(possibly several weakenings and exchanges)

$$\Sigma,\ \Gamma^*, \Sigma^{**}\ \longrightarrow\ \Theta^*, \Delta^*, \Theta$$

$$\bigwedge(mix)$$

$$\Sigma \longrightarrow \Theta,\ A \qquad\qquad A,\ \Gamma,\ \Sigma^*, \longrightarrow \Delta^*,\ \Theta$$

$$\bigwedge(mix)$$

$$A,\ \Gamma\ \longrightarrow \Delta,\ B \quad B,\ \Sigma \longrightarrow \Theta,$$

Proof of H Lemma

The asteriks are, of course, intended as follows

$\Sigma^*, \ \Delta^*$ **results** from Σ, Δ by the **omission** of all formulas B

$\Gamma^*, \ \Sigma^{**}, \ \Theta^*$ **results** from $\Gamma, \ \Sigma^*, \ \Theta$ by the **omission** of all formulas A

Proof of H Lemma

We replace the sub-tree **T** by **T*** in **D** and obtain **D***

Now we have **two mixes**, but both **mix** formulas A and B are of a lower degree then n

We first apply the inductive assumption to the lower **mix** (formula B) and the lower **mix** is **eliminated**

We then apply by the inductive assumption and **eliminate** the upper **mix** (formula A)

This **ends** the proof of the **case** of the rank r=2

Proof of H Lemma

Case $r > 2$

In the case $r = 2$, we **reduced** the derivation to one of lower degree. Now we proceed to **reduce** the derivation to one of the same degree, but of a **lower rank**

This allows us to to be able to carry the **induction** with respect to the rank r of the **derivation**

We use the inductive assuption in all cases **except**, as before, a **case** of an axiom or **structural rules**

In these cases the **mix** can be eliminated immediately, as it was **eliminated** in the previous case of rank $r = 2$

Proof of H Lemma

In a case of **logical rules** we obtain the reduction of the **mix** to derivations with **mix** of a lower ranks which consequently can be **eliminated** by the inductive assumption

We carry proofs for two **logical rules** $(\rightarrow \cap)$ and $(\cup \rightarrow)$

The proof for all other rules is similar and is left as exercise

We consider only the **case** of left rank Lr= 1 and right rank $Rr > 1$

The symmetrical **case** of left rank $Lr > 1$ and right rank Rr = 1 is left as an exercise

Proof of H Lemma

Case: $Lr = 1$ and $Rr = r > 1$

The right premiss of the **mix** is a conclusion of the inference rule $(\rightarrow \cap)$, i.e. it is of a form

$$\Gamma \longrightarrow \Delta, (A \cap B)$$

where Γ contains a **mix** formula M

The left premiss of the **mix** is a sequent

$$\Theta \longrightarrow \Sigma$$

and Σ contains the **mix** formula M

Proof of H Lemma

The **sub-tree T** of **D** containing the application of the **mix** rule is

$$\Theta, \Gamma^* \longrightarrow \Sigma^*, \Delta, (A \cap B)$$

$$\bigwedge (mix)$$

$$\Theta \longrightarrow \Sigma \qquad\qquad \Gamma \longrightarrow \Delta, (A \cap B)$$

$$\bigwedge (\rightarrow \cap)$$

$$\Gamma \longrightarrow \Delta, A \qquad \Gamma \longrightarrow \Delta, B$$

Proof of H Lemma

We **transform** **T** into **T*** as follows

$$\Theta,\ \Gamma^* \longrightarrow \Sigma^*, \Delta, (A \cap B)$$

$$\bigwedge(\to \cap)$$

$$\Theta,\ \Gamma^* \longrightarrow \Sigma^*, \Delta, A \qquad\qquad \Theta,\ \Gamma^* \longrightarrow \Sigma^*, \Delta, B$$

We perform **mix** on the left branch

$$\Theta,\ \Gamma^* \longrightarrow \Sigma^*, \Delta, A$$

$$\bigwedge(mix)$$

$$\Theta \longrightarrow \Sigma \qquad\qquad \Gamma \longrightarrow \Delta, A$$

Proof of H Lemma

We perform **mix** on the right branch

$$\Theta,\ \Gamma^* \longrightarrow \Sigma^*, \Delta, B$$

$$\bigwedge (mix)$$

$$\Theta \longrightarrow \Sigma \qquad\qquad \Gamma \longrightarrow \Delta, B$$

We replace **T** by **T*** in **D** and obtain **D***

Now we have **two mixes**, but both have the right rank Rr = r-1 and both of them can be **eliminated** by the inductive assumption

Proof of H Lemma

Case: $Lr = 1$ and $Rr = r > 1$

The right premiss of the **mix** is a conclusion of the rule $(\cup \rightarrow)$, i.e. it is of a form

$$(A \cup B), \Gamma \longrightarrow \Delta$$

and Γ contains a **mix formula** M

The left premiss of the **mix** is a sequent

$$\Theta \longrightarrow \Sigma$$

and Σ contains the **mix formula** M

Proof of H Lemma

The **sub-tree T** of **D** containing the application of the **mix** rule is

$$\Theta,\ (A \cup B)^*, \Gamma^* \longrightarrow \Sigma^*, \Delta$$

$$\bigwedge (mix)$$

$$\Theta \longrightarrow \Sigma \qquad\qquad (A \cup B), \Gamma \longrightarrow \Delta$$

$$\bigwedge (\cup \rightarrow)$$

$$A, \Gamma \longrightarrow \Delta \qquad B, \Gamma \longrightarrow \Delta$$

Proof of H Lemma

$(A \cup B)^*$ stands **either** for $(A \cup B)$ **or** for nothing according as $(A \cup B)$ is unequal or equal to the **mix formula** M

The **mix formula** M certainly occurs in Γ

For otherwise M would been equal to $(A \cup B)$ and the right rank Rr would be equal to 1 **contrary** to the assumption that $Rr > 1$

Proof of H Lemma

We transform **T** into **T*** as follows

$$\Theta, \ (A \cup B), \Gamma^* \ \longrightarrow \ \Sigma^*, \Delta$$

$$\bigwedge (\cup \rightarrow)$$

$$A, \Theta, \ \Gamma^* \ \longrightarrow \ \Sigma^*, \Delta \qquad\qquad B, \Theta, \ \Gamma^* \ \longrightarrow \ \Sigma^*, \Delta$$

We perform **mix** on the left branch

$$A, \Theta, \ \Gamma^* \ \longrightarrow \ \Sigma^*, \Delta$$

(some weakenings, exchanges)

$$\Theta, A^*, \Gamma^* \ \longrightarrow \ \Sigma^*, \Delta$$

$$\bigwedge (mix)$$

$$\Theta \longrightarrow \Sigma \qquad\qquad A, \Gamma \longrightarrow \ \Delta$$

Proof of H Lemma

We perform **mix** on the right branch

$$B, \Theta, \Gamma^* \longrightarrow \Sigma^*, \Delta$$

(some weakenings, exchanges)

$$\Theta, B^*, \Gamma^* \longrightarrow \Sigma^*, \Delta$$

$$\bigwedge (mix)$$

$$\Theta \longrightarrow \Sigma \qquad\qquad B, \Gamma \longrightarrow \Delta$$

Proof of H Lemma

Now we have **two mixes**

But both have the right rank $Rr = r\text{-}1$ and hence both of them can be **eliminated** by the inductive assumption

We **replace** **T** by $\mathbf{T}^*$ in **D** and **obtain** $\mathbf{D}^*$

This **ends** the proof of the **Hauptzatz Lemma**

We have hence **completed** the proof of the **Hauptzatz Theorem**

LK and **LI** Hauptzatz Theorems

LK and **LI** Hauptzatz Theorems

Let's denote by **LK - c** and **LI - c** the systems **LK**, **LI** without the cut rule, i.e. we put

$$\mathbf{LK} - \mathbf{c} = \mathbf{LK} - \{(cut)\}$$

$$\mathbf{LI} - \mathbf{c} = \mathbf{LI} - \{(cut)\}$$

We re-write the **Hauptzatz Theorems** as follows.

LK and **LI** Hauptzatz Theorem

LK Hauptzatz

For every **LK** sequent $\Gamma \longrightarrow \Delta$,

$$\vdash_{LK} \Gamma \longrightarrow \Delta \quad \textit{if and only if} \quad \vdash_{LK-c} \Gamma \longrightarrow \Delta$$

LI Hauptzatz

For every **LI** sequent $\Gamma \longrightarrow \Delta$,

$$\vdash_{LI} \Gamma \longrightarrow \Delta \quad \textit{if and only if} \quad \vdash_{LI-c} \Gamma \longrightarrow \Delta$$

This is why the **cut-free** Gentzen systems **LK-c** and **LI -c** are just **called** **LK, LI**, respectively

LK-c Completeness

Directly from the **LK Completeness Theorem** and the **LK Hauptzatz Theorem** we get that the following.

LK-c Completeness Theorem

For any sequent $\Gamma \longrightarrow \Delta$,

$$\vdash_{\mathbf{LK-c}} \Gamma \longrightarrow \Delta \quad \text{if and only if} \quad \models \Gamma \longrightarrow \Delta$$

LK and **GK** Systems Equivalency

GK System

Let **G** be the Gentzen sequents proof system defined previously

We **replace** the logical axiom of **G**

$$\Gamma'_1,\ a,\ \Gamma'_2 \longrightarrow \Delta'_1,\ a,\ \Delta'_2$$

where $a \in VAR$ is any propositional variable and

$$\Gamma'_1, \Gamma'_2,\ \Delta'_1,\ \Delta'_2 \in VAR^*$$

are any **indecomposable sequences**, by a new logical axiom

$$\Gamma_1,\ A,\ \Gamma_2 \longrightarrow \Delta_1,\ A,\ \Delta_2$$

for any $A \in \mathcal{F}$ and any sequences

$$\Gamma_1, \Gamma_2, \Delta_1, \Delta_2 \in SQ$$

GK System

We call a resulting proof system **GK**, i.e. we defined it as follows

$$\mathbf{GK} = (\ \mathcal{L}_{\{\cup,\cap,\Rightarrow,\neg\}},\ SQ,\ LA,\ \mathcal{R}\ )$$

where LA is the new axiom defined above and $\mathcal{R}$ is the set of rules of the system **G**

Observe that the only difference between the systems**GK** and **G** is the form of their logical axioms, both being **tautologies**

We get the proof of **completeness** of **GK** in the same way as we proved it for **G**, i.e. we have the following

GK Completeness

GK Completeness Theorem

For any formula $A \in \mathcal{F}$,

$$\vdash_{\mathbf{GK}} A \quad \text{if and only if} \quad \models A$$

For any sequent $\Gamma \longrightarrow \Delta \in SQ$

$$\vdash_{\mathbf{GK}} \Gamma \longrightarrow \Delta \quad \text{if and only if} \quad \models \Gamma \longrightarrow \Delta$$

LK and **GK** Systems Equivalency

By the **GK**, **LK-c** **Completeness Theorems** we get the **equivalency** of **GK** and the **cut free** **LK-c** proof systems

LK, GK Equivalency Theorem

The proof systems **GK** and the **cut free** **LK** are **equivalent**, i.e for any sequent $\Gamma \longrightarrow \Delta$,

$$\vdash_{\mathbf{LK}} \Gamma \longrightarrow \Delta \quad \text{if and only if} \quad \vdash_{\mathbf{GK}} \Gamma \longrightarrow \Delta$$