cse541
LOGIC for COMPUTER SCIENCE

Professor Anita Wasilewska
LECTURE 5b
Chapter 5
Hilbert Proof Systems
Completeness of Classical Propositional Logic

Completeness Theorem
Proof Two: A Counter-Model Existence Method
Completeness Theorem Proof Two

Our goal is to prove the following

**Completeness Theorem** (Completeness Part)

For any formula $A \in \mathcal{F}$ of $H_2$

$$\text{if } \models A \text{ then } \vdash A$$

We do so by proving its logically equivalent opposite implication:

$$\text{If } \not\models A, \text{ then } \not\vdash A$$

Hence the **Proof Two** consists of using the information that a formula $A$ is not provable to show the existence of a counter-model for $A$
Completeness Theorem Proof Two

The Proof Two is more general and much more complicated than the Proof One.

The main point of the proof is a general, non-constructive method for proving existence of a counter-model for any non-provable formula $A$.

The generality of the method makes it possible to adopt it for other cases of predicate and some non-classical logics.

This is why we call the Proof Two a counter-model existence method.
Completeness Theorem Proof Two

The **Proof Two** construction of a **counter-model** for any non-provable formula $A$ is an abstract method that is not constructive as was the method used in the **Proof One**.

The **Proof Two** used the method can be **generalized** to the case of **predicate logic**, and many of **non-classical logics**; propositional and predicate.

This is the reason we present it here.
Proof Two Steps

We remind that \( \not\models A \) means that there is a truth assignment \( v : \text{VAR} \rightarrow \{T, F\} \), such that (as we are in classical semantics) \( v^*(A) = F \)

We assume that \( A \) does not have a proof i.e. \( \not\vdash A \) we use this information in order to define a general method of constructing \( v \), such that \( v^*(A) = F \)

This is done in the following steps.
Proof Two Steps

Step 1
Definition of a special set of formulas $\Delta^*$
We use the information $\not\models A$ to define a set of formulas $\Delta^*$ such that $\neg A \in \Delta^*$

Step 2
Definition of the counter-model
We define the variable truth assignment $v : \text{VAR} \rightarrow \{T, F\}$ as follows:

$$v(a) = \begin{cases} T & \text{if } \Delta^* \vdash a \\ F & \text{if } \Delta^* \vdash \neg a \end{cases}$$
Proof 2 Steps

Step 3
We prove that \( v \) is a **counter-model** for \( A \)
We first prove a following more general property of \( v \)

**Property**
The set \( \Delta^* \) and \( v \) defined in the Steps 1 and 2 are such that for every formula \( B \in \mathcal{F} \)

\[
    v^*(B) = \begin{cases} 
        T & \text{if } \Delta^* \vdash B \\
        F & \text{if } \Delta^* \vdash \neg B 
    \end{cases}
\]

We then use the **Step 3** to prove that \( v^*(A) = F \)
Main Notions

The definition, construction and the properties of the set $\Delta^*$ and hence the Step 1, are the most essential for the Proof Two.

The other steps have mainly technical character.

The main notions involved in the proof are: consistent set, complete set and a consistent complete extension of a set of formulas.

We are going prove some essential facts about them.
Consistent and Inconsistent Sets

There exist two definitions of consistency; semantical and syntactical

Semantical definition uses the notion of a model and says:

A set is consistent if it has a model

Syntactical definition uses the notion of provability and says:

A set is consistent if one can’t prove a contradiction from it
Consistent and Inconsistent Sets

In our proof of the **Completeness Theorem** we use the following formal **syntactical definition** of consistency of a set of formulas

**Definition** of a **consistent set**

We say that a set $\Delta \subseteq \mathcal{F}$ of formulas is **consistent** if and only if **there is no** a formula $A \in \mathcal{F}$ such that

$$\Delta \vdash A \quad \text{and} \quad \Delta \vdash \neg A$$
Consistent and Inconsistent Sets

Definition of an inconsistent set

A set $\Delta \subseteq F$ is inconsistent if and only if there is a formula $A \in F$ such that

$$\Delta \vdash A \quad \text{and} \quad \Delta \vdash \neg A$$

The notion of consistency, as defined above, is characterized by the following Consistency Lemma
Consistency Condition Lemma

Lemma  Consistency Condition

For every set  $\Delta \subseteq \mathcal{F}$  of formulas, the following conditions are equivalent

(i)  $\Delta$  is consistent

(ii) there is a formula  $A \in \mathcal{F}$  such that  $\Delta \nvDash A$
Proof of Consistency Lemma

Proof
To establish the equivalence of (i) and (ii) we prove the corresponding opposite implications.

We prove the following two cases:

Case 1: not (ii) implies not (i)

Case 2: not (i) implies not (ii)
Proof of Consistency Lemma

Case 1

Assume that not (ii)

It means that for all formulas $A \in \mathcal{F}$ we have that

$$\Delta \vdash A$$

In particular it is true for a certain $A = B$ and for a certain $A = \neg B$ i.e.

$$\Delta \vdash B \quad \text{and} \quad \Delta \vdash \neg B$$

and hence it proves that $\Delta$ is inconsistent

i.e. not (i) holds
Proof of Consistency Lemma

Case 2
Assume that not (i), i.e. that \( \Delta \) is inconsistent
Then there is a formula \( A \) such that \( \Delta \vdash A \) and \( \Delta \vdash \neg A \)
Let \( B \) be any formula
We proved (Lemma formula 6.) that \( \vdash (\neg A \Rightarrow (A \Rightarrow B)) \)
By monotonicity

\[
\Delta \vdash (\neg A \Rightarrow (A \Rightarrow B))
\]

Applying Modus Ponens twice to \( \neg A \) first, and to \( A \) next we get that \( \Delta \vdash B \) for any formula \( B \)
Thus not (ii) and it ends the proof of the Consistency Condition Lemma
Inconsistency Condition Lemma

Inconsistent sets are hence characterized by the following fact

Lemma Inconsistency Condition

For every set $\Delta \subseteq \mathcal{F}$ of formulas, the following conditions are equivalent:

(i) $\Delta$ is inconsistent,

(ii) for any formula $A \in \mathcal{F}$ $\Delta \vdash A$
Finite Consequence Lemma

We remind here property of the finiteness of the consequence operation.

Lemma  Finite Consequence

For every set $\Delta$ of formulas and for every formula $A \in \mathcal{F}$

$\Delta \vdash A$ if and only if there is a finite set $\Delta_0 \subseteq \Delta$ such that $\Delta_0 \vdash A$

Proof

If $\Delta_0 \vdash A$ for a certain $\Delta_0 \subseteq \Delta$,
hence by the monotonicity of the consequence, also $\Delta \vdash A$
Finite Consequence Lemma

Assume now that $\Delta \vdash A$ and let $A_1, A_2, ..., A_n$ be a formal proof of $A$ from $\Delta$

Let

$$\Delta_0 = \{A_1, A_2, ..., A_n\} \cap \Delta$$

Obviously, $\Delta_0$ is finite and $A_1, A_2, ..., A_n$ is a formal proof of $A$ from $\Delta_0$
Finite Inconsistency Theorem

The following theorem is a simple corollary of just proved Finite Consequence Lemma

**Theorem**  Finite Inconsistency

(1.) If a set $\Delta$ is **inconsistent**, then it has a finite inconsistent subset $\Delta_0$

(2.) If every finite subset of a set $\Delta$ is **consistent** then the set $\Delta$ is also **consistent**
Finite Inconsistency Theorem

Proof
If $\Delta$ is inconsistent, then for some formula $A$,

$$\Delta \vdash A \quad \text{and} \quad \Delta \vdash \neg A$$

By the Finite Consequence Lemma, there are finite subsets $\Delta_1$ and $\Delta_2$ of $\Delta$ such that

$$\Delta_1 \vdash A \quad \text{and} \quad \Delta_2 \vdash \neg A$$

The union $\Delta_1 \cup \Delta_2$ is a finite subset of $\Delta$ and by monotonicity

$$\Delta_1 \cup \Delta_2 \vdash A \quad \text{and} \quad \Delta_1 \cup \Delta_2 \vdash \neg A$$

Hence we proved that $\Delta_1 \cup \Delta_2$ is a finite inconsistent subset of $\Delta$.

The second implication (2.) is the opposite to the one just proved and hence also holds.
Consistency Lemma

The following Lemma links the notion of non-provability and consistency.

It will be used as an important step in our Proof Two of the Completeness Theorem.

Lemma

For any formula $A \in \mathcal{F}$,

if $\not\vdash A$ then the set $\{\neg A\}$ is consistent.
Consistency Lemma

Proof: We prove the opposite implication
If \( \{\neg A\} \) is inconsistent, then \( \vdash A \)
Assume that \( \{\neg A\} \) is inconsistent
By the Inconsistency Condition Lemma we have that \( \{\neg A\} \vdash B \) for any formula \( B \), and hence in particular
\[ \{\neg A\} \vdash A \]

By Deduction Theorem we get
\[ \vdash (\neg A \Rightarrow A) \]
We proved (Lemma formula 9.) that
\[ \vdash ((\neg A \Rightarrow A) \Rightarrow A) \]
By Modus Ponens we get
\[ \vdash A \]

This ends the proof
Another important notion, is that of a **complete set** of formulas.

**Complete sets**, as defined here are sometimes called **maximal**, but we use the first name for them.

They are defined as follows.

**Definition**  **Complete set**

A set $\Delta$ of formulas is called **complete** if for every formula $A \in \mathcal{F}$

$$\Delta \vdash A \quad \text{or} \quad \Delta \vdash \neg A$$

**Gödel** used this notion of complete sets in his **Incompleteness of Arithmetic Theorem**

The **complete sets** are characterized by the following fact.
Complete and Incomplete Sets

Complete Set Condition Lemma

For every set $\Delta \subseteq \mathcal{F}$ of formulas, the following conditions are equivalent

(i) The set $\Delta$ is complete

(ii) For every formula $A \in \mathcal{F}$,

if $\Delta \not\models A$ then the set $\Delta \cup \{A\}$ is inconsistent

Proof

We consider two cases

Case 1  We show that (i) implies (ii) and

Case 2  we show that (ii) implies (i)
Complete Set Condition Lemma

Proof of Case 1
Assume (i) and not(ii) i.e.
assume that $\Delta$ is complete and there is a formula $A \in \mathcal{F}$
such that $\Delta \not\vdash A$ and the set $\Delta \cup \{A\}$ is consistent
We have to show that we get a contradiction
But if $\Delta \not\vdash A$, then from the assumption that $\Delta$ is complete
we get that
\[ \Delta \vdash \neg A \]
By the monotonicity of the consequence we have that
\[ \Delta \cup \{A\} \vdash \neg A \]
Complete Set Condition Lemma

We proved (Lemma formula 4.) \( \vdash (A \Rightarrow A) \)

By monotonicity \( \Delta \vdash (A \Rightarrow A) \) and by Deduction Theorem

\[ \Delta \cup \{A\} \vdash A \]

We hence proved that that there is a formula \( A \in \mathcal{F} \) such that

\[ \Delta \cup \{A\} \quad \text{and} \quad \Delta \cup \{A\} \vdash \lnot A \]

i.e. that the set \( \Delta \cup \{A\} \) is inconsistent

Contradiction
Complete Set Condition Lemma

Proof of Case 2
Assume (ii), i.e. that for every formula $A \in \mathcal{F}$ if $\Delta \not\models A$ then the set $\Delta \cup \{A\}$ is inconsistent.
Let $A$ be any formula.
We want to show (i), i.e. to show that the following condition

$$C: \Delta \vdash A \text{ or } \Delta \vdash \neg A$$

is satisfied.

Observe that if

$$\Delta \vdash \neg A$$

then the condition $C$ is obviously satisfied.
Complete Set Condition Lemma

If, on the other hand,
\[ \Delta \not\vdash \neg A \]

then we are going to show now that it must be, under the assumption of (ii), that \( \Delta \vdash A \) i.e. that (i) holds

Assume that
\[ \Delta \not\vdash \neg A \]

then by (ii) the set \( \Delta \cup \{\neg A\} \) is inconsistent
Complete Set Condition Lemma

The Inconsistency Condition Lemma says
For every set $\Delta \subseteq F$ of formulas, the following conditions are equivalent:

(i) $\Delta$ is inconsistent,
(ii) for any formula $A \in F$, $\Delta \vdash A$

We just proved that the set $\Delta \cup \{\neg A\}$ is inconsistent
So by the the above Lemma we get

$\Delta \cup \{\neg A\} \vdash A$
Complete Set Condition Lemma

By the **Deduction Theorem**  \( \Delta \cup \{\neg A\} \vdash A \)  implies that

\[ \Delta \vdash (\neg A \Rightarrow A) \]

**Observe** that by **Lemma** formula 4.

\[ \vdash ((\neg A \Rightarrow A) \Rightarrow A) \]

By monotonicity

\[ \Delta \vdash ((\neg A \Rightarrow A) \Rightarrow A) \]

Detaching, by **MP** the formula \((\neg A \Rightarrow A)\) we obtain that

\[ \Delta \vdash A \]

This **ends** the proof that (i) holds.
Incomplete Sets

Definition  Incomplete Set
A set \( \Delta \) of formulas is called incomplete if it is not complete i.e. when the following condition holds
There exists a formula \( A \in \mathcal{F} \) such that

\[
\Delta \not\vDash A \quad \text{and} \quad \Delta \not\vDash \neg A
\]
Incomplete Set Condition Lemma

We get as a direct consequence of the Complete Set Condition Lemma the following characterization of incomplete sets

**Lemma  Incomplete Set Condition**

For every set $\Delta \subseteq F$ of formulas, the following conditions are equivalent:

(i) $\Delta$ is incomplete,

(ii) there is formula $A \in F$ such that $\Delta \not\vdash A$ and the set $\Delta \cup \{A\}$ is consistent.
Main Lemma: Complete Consistent Extension

Now we are going to prove a **Main Lemma** that is **essential** to the construction of the special set $\Delta^*$ mentioned in the **Step 1** of the proof of the **Completeness Theorem** and hence to the **proof of the theorem** itself.

Let’s first introduce one more notion.
Complete Consistent Extension

Definition Extension $\Delta^*$ of the set $\Delta$

A set $\Delta^*$ of formulas is called an extension of a set $\Delta$ of formulas if the following condition holds

$$\{A \in \mathcal{F} : \Delta \vdash A\} \subseteq \{A \in \mathcal{F} : \Delta^* \vdash A\}$$

i.e.

$$\text{Cn}(\Delta) \subseteq \text{Cn}(\Delta^*)$$

In this case we say also that $\Delta$ extends to the set of formulas $\Delta^*$
Main Lemma
Main Lemma

Main Lemma  Complete Consistent Extension

Every consistent set $\Delta$ of formulas can be extended to a complete consistent set $\Delta^*$ of formulas i.e.

For every consistent set $\Delta$ there is a set $\Delta^*$ that is complete and consistent and is an extension of $\Delta$ i.e.

$$Cn(\Delta) \subseteq Cn(\Delta^*)$$
Proof of the Main Lemma

Proof
Assume that the lemma does not hold, i.e. that there is a consistent set $\Delta$, such that all its consistent extensions are not complete.

In particular, as $\Delta$ is an consistent extension of itself, we have that $\Delta$ is not complete.

The proof consists of a construction of a particular set $\Delta^*$ and proving that it forms a complete consistent extension of $\Delta$.

This is contrary to the assumption that all its consistent extensions are not complete.
Construction of $\Delta^*$

As we know, the set $\mathcal{F}$ of all formulas is enumerable; they can hence be put in an infinite sequence

$$\mathcal{F} = A_1, A_2, \ldots, A_n, \ldots$$

such that every formula of $\mathcal{F}$ occurs in that sequence exactly once.

We define, by mathematical induction, an infinite sequence

$$D = \{\Delta_n\}_{n \in \mathbb{N}}$$

of consistent subsets of formulas together with a sequence

$$B = \{B_n\}_{n \in \mathbb{N}}$$

of formulas as follows.
Construction of $\Delta^*$

**Initial Step**

In this step we define the sets $\Delta_1, \Delta_2$ and the formula $B_1$

and prove that $\Delta_1$ and $\Delta_2$ are consistent, incomplete extensions of $\Delta$

We take as the first set in $D$ the set $\Delta$, i.e. we define $\Delta_1 = \Delta$
Construction of $\Delta^*$

By assumption the set $\Delta$, and hence also $\Delta_1$ is not complete.

From the Incomplete Set Condition Lemma we get that there is a formula $B \in \mathcal{F}$ such that

$$\Delta_1 \not\vdash B \quad \text{and} \quad \Delta_1 \cup \{B\} \quad \text{is consistent}$$

Let $B_1$ be the first formula with this property in the sequence $\mathcal{F}$ of all formulas.

We define

$$\Delta_2 = \Delta_1 \cup \{B_1\}$$
Construction of $\Delta^*$

Observe that the set $\Delta_2$ is consistent and 

$$\Delta_1 = \Delta \subseteq \Delta_2$$

By monotonicity $\Delta_2$ is a consistent extension of $\Delta$. Hence, as we assumed that all consistent extensions of $\Delta$ are not complete, we get that $\Delta_2$ cannot be complete, i.e.

$\Delta_2$ is incomplete
Construction of $\Delta^*$

**Inductive Step**

**Suppose** that we have defined a sequence

$$\Delta_1, \Delta_2, \ldots, \Delta_n$$

of **incomplete, consistent extensions** of $\Delta$ and a sequence

$$B_1, B_2, \ldots, B_{n-1}$$

of formulas, for $n \geq 2$
Construction of $\Delta^*$

Since $\Delta_n$ is incomplete, it follows from the Incomplete Set Condition Lemma that there is a formula $B \in \mathcal{F}$ such that

$$\Delta_n \nvdash B \quad \text{and} \quad \Delta_n \cup \{B\} \quad \text{is consistent}$$
Construction of $\Delta^*$

Let $B_n$ be the first formula with this property in the sequence $F$ of all formulas.

We define

$$\Delta_{n+1} = \Delta_n \cup \{B_n\}$$

By the definition

$$\Delta \subseteq \Delta_n \subseteq \Delta_{n+1}$$

and the set $\Delta_{n+1}$ is a consistent extension of $\Delta$.

Hence by our assumption that all consistent extensions of $\Delta$ are incomplete, we get that

$$\Delta_{n+1}$$

is an incomplete consistent extension of $\Delta$. 
Construction of $\Delta^*$

By the principle of mathematical induction we have defined an infinite sequence

$$D \quad \Delta = \Delta_1 \subseteq \Delta_2 \subseteq \ldots, \subseteq \Delta_n \subseteq \Delta_{n+1} \subseteq \ldots$$

such that for all $n \in \mathbb{N}$, $\Delta_n$ is consistent, and each $\Delta_n$ an incomplete consistent extension of $\Delta$

Moreover, we have also defined a sequence

$$B \quad B_1, B_2, \ldots, B_n, \ldots$$

of formulas, such that for all $n \in \mathbb{N}$,

$$\Delta_n \not\models B_n \quad \text{and} \quad \Delta_n \cup \{B_n\} \quad \text{is consistent}$$

Observe that $B_n \in \Delta_{n+1}$ for all $n \geq 1$
Definition of $\Delta^*$

Now we are ready to define $\Delta^*$

**Definition of $\Delta^*$**

$$\Delta^* = \bigcup_{n \in N} \Delta_n$$

To complete the proof our theorem we have now to prove that $\Delta^*$ is a **complete consistent extension** of $\Delta$
Δ* Consistent

Obviously directly from the definition $\Delta \subseteq \Delta^*$ and hence we have the following

Fact 1 $\Delta^*$ is an extension of $\Delta$

By Monotonicity of Consequence $Cn(\Delta) \subseteq Cn(\Delta^*)$, hence extension

As the next step we prove

Fact 2 The set $\Delta^*$ is consistent
\[ \Delta^* \text{ Consistent} \]

**Proof** that \( \Delta^* \) is consistent

Assume that \( \Delta^* \) is inconsistent

By the Finite Inconsistency Theorem there is a finite subset \( \Delta_0 \) of \( \Delta^* \) that is inconsistent, i.e.

\[
\Delta_0 \subseteq \bigcup_{n \in \mathbb{N}} \Delta_n, \quad \Delta_0 = \{C_1, ..., C_n\}, \quad \Delta_0 \text{ is inconsistent}
\]
Proof of $\Delta^*$ Consistent

We have $\Delta_0 = \{C_1, \ldots, C_n\}$

By the definition of $\Delta^*$ for each formula $C_i \in \Delta_0$

$C_i \in \Delta_{k_i}$

for certain $\Delta_{k_i}$ in the sequence

$D \quad \Delta = \Delta_1 \subseteq \Delta_2 \subseteq \ldots, \subseteq \Delta_n \subseteq \Delta_{n+1} \subseteq \ldots$

Hence $\Delta_0 \subseteq \Delta_m$ for $m = \text{max}\{k_1, k_2, \ldots k_n\}$
Proof of $\Delta^*$ Consistent

But we proved that all sets of the sequence $D$ are consistent

This contradicts the fact that $\Delta_m$ is consistent as it contains an inconsistent subset $\Delta_0$

This contradiction ends the proof that $\Delta^*$ is consistent
Proof of $\Delta^*$ Complete

**Fact 3** The set $\Delta^*$ is **complete**

**Proof** Assume that $\Delta^*$ is **not complete**.

By the **Incomplete Set Condition**, there is a formula $B \in F$ such that $\Delta^* \not\models B$, and the set $\Delta^* \cup \{B\}$ is **consistent**.

By definition of the sequence $D$ and the sequence $B$ of formulas, we have that for every $n \in N$

$$\Delta_n \not\models B_n \quad \text{and the set } \Delta_n \cup \{B_n\} \text{ is consistent}$$

Moreover $B_n \in \Delta_{n+1}$ for all $n \geq 1$
Proof of $\Delta^*$ Complete

Since the formula $B$ is one of the formulas of the sequence $B$ so we get that $B = B_j$ for certain $j$
By definition, $B_j \in \Delta_{j+1}$ and it proves that

$$B \in \Delta^* = \bigcup_{n\in\mathbb{N}} \Delta_n$$

But this means that $\Delta^* \vdash B$
This is a contradiction with the assumption $\Delta^* \nvdash B$ and it ends the proof of the Fact 3
Main Lemma

Facts 1-3 prove that that $\Delta^*$ is a complete consistent extension of $\Delta$.

We hence completed the proof of the Main Lemma.

Main Lemma:
Every consistent set $\Delta$ of formulas can be extended to a complete consistent set $\Delta^*$ of formulas.
Proof Two of Completeness Theorem
Proof Two of Completeness Theorem

We proved already that $H_2$ is **sound**, so we have to prove only the **Completeness part** of the **Completeness Theorem**:

For any formula $A \in \mathcal{F}$,

If $\models A$, then $\vdash A$

We prove it by **proving** its logically equivalent **opposite implication form**, i.e we prove now the following **Completeness Theorem**

For any formula $A \in \mathcal{F}$,

If $\not\models A$, then $\not\vdash A$
Proof Two of Completeness Theorem

Proof
Assume that $A$ does not have a proof, we want to define a counter-model for $A$.
But if $\not\models A$, then by the Inconsistency Lemma the set $\{\neg A\}$ is consistent.
By the Main Lemma there is a complete, consistent extension of the set $\{\neg A\}$.
This means that there is a set $\Delta^*$ such that $\{\neg A\} \subseteq \Delta^*$, i.e.

$$E \quad \neg A \in \Delta^* \text{ and } \Delta^* \text{ is complete and consistent}$$
Proof Two of Completeness Theorem

Since $\Delta^*$ is a consistent, complete set, it satisfies the following form of Consistency Condition

For any $A \in \mathcal{F}$,

$$\Delta^* \not\vdash A \quad \text{or} \quad \Delta^* \not\vdash \neg A$$

$\Delta^*$ is also complete i.e. satisfies Completeness Condition

For any $A \in \mathcal{F}$,

$$\Delta^* \vdash A \quad \text{or} \quad \Delta^* \vdash \neg A$$
Proof Two of Completeness Theorem

Directly from the **Completeness** and **Consistency** Conditions we get the following

**Separation Condition**

For any $A \in \mathcal{F}$, exactly one of the following conditions is satisfied:

(1) $\Delta^* \vdash A$, or (2) $\Delta^* \vdash \neg A$

In particular case we have that for every propositional variable $a \in \text{VAR}$ exactly one of the following conditions is satisfied:

(1) $\Delta^* \vdash a$, or (2) $\Delta^* \vdash \neg a$

This justifies the **correctness** of the following definition
Definition

We define the variable truth assignment

\[ v : \text{VAR} \rightarrow \{ T, F \} \]

as follows:

\[ v(a) = \begin{cases} 
T & \text{if } \Delta^* \vdash a \\
F & \text{if } \Delta^* \vdash \neg a. 
\end{cases} \]

We show, as a separate Lemma below, that such defined variable assignment \( v \) has the following property
Property of $v$ Lemma

Lemma Property of $v$

Let $v$ be the variable assignment defined above and $v^*$ its extension to the set $\mathcal{F}$ of all formulas $B \in \mathcal{F}$, the following is true

$$v^*(B) = \begin{cases} T & \text{if } \Delta^* \vdash B \\ F & \text{if } \Delta^* \vdash \neg B \end{cases}$$
Proof 2 of Completeness Theorem

Given the **Property of \( \nu \) Lemma** (still to be proved)
we now **prove** that the \( \nu \) is in fact, a **counter model** for any formula \( A \), such that \( \not \models A \)
Let \( A \) be such that \( \not \models A \)
By the Property \( \mathbf{E} \) we have that \( \neg A \in \Delta^* \)
So obviously
\[
\Delta^* \vdash \neg A
\]
Hence by the **Property of \( \nu \) Lemma**
\[
\nu^*(A) = F
\]
what **proves** that \( \nu \) is a **counter-model** for \( A \) and it **ends the proof** of the **Completeness Theorem**
Proof of Property of \( v \) Lemma

Proof of the Property of \( v \) Lemma

The proof is conducted by the induction on the degree of the formula \( A \)

Initial step \( A \) is a propositional variable so the Lemma holds by definition of \( v \)

Inductive Step

If \( A \) is not a propositional variable, then \( A \) is of the form \( \neg C \) or \( (C \Rightarrow D) \), for certain formulas \( C, D \)

By the inductive assumption the Lemma holds for the formulas \( C \) and \( D \)
Proof of Property of \( v \) Lemma

**Case** \( A = \neg C \)

By the **Separation Condition** for \( \Delta^* \) we consider two possibilities

1. \( \Delta^* \vdash A \)
2. \( \Delta^* \vdash \neg A \)

Consider case 1. i.e. we assume that \( \Delta^* \vdash A \)

It means that

\[ \Delta^* \vdash \neg C \]

Then from the fact that \( \Delta^* \) is **consistent** it must be that

\[ \Delta^* \not\vdash C \]
Proof of Property of \( \nu \) Lemma

By the inductive assumption we have that \( \nu^*(C) = F \) and accordingly \( \nu^*(A) = \nu^*(\neg C) = \neg \nu^*(C) = \neg F = T \)

**Consider** case 2. i.e. we assume that \( \Delta^* \vdash \neg A \)

Then from the fact that \( \Delta^* \) is **consistent** it must be that \( \Delta^* \not\vdash A \) and

\[
\Delta^* \not\vdash \neg C
\]

If so, then \( \Delta^* \vdash C \), as the set \( \Delta^* \) is **complete**

By the inductive assumption, \( \nu^*(C) = T \), and accordingly

\[
\nu^*(A) = \nu^*(\neg C) = \neg \nu^*(C) = \neg T = F
\]

Thus \( A \) **satisfies** the Property of \( \nu \) Lemma
Proof of Property of \( \nu \) Lemma

**Case** \( A = (C \Rightarrow D) \)

As in the previous case, we assume that the Lemma holds for the formulas \( C, D \) and we consider by the **Separation Condition** for \( \Delta^* \) two possibilities:

1. \( \Delta^* \vdash A \) and 2. \( \Delta^* \vdash \neg A \)

**Case 1.** Assume \( \Delta^* \vdash A \)

It means that \( \Delta^* \vdash (C \Rightarrow D) \)

If at the same time \( \Delta^* \nvDash C \), then \( \nu^*(C) = F \), and accordingly

\[
\nu^*(A) = \nu^*(C \Rightarrow D) = \\
\nu^*(C) \Rightarrow \nu^*(D) = F \Rightarrow \nu^*(D) = T
\]
Proof of Property of $v$ Lemma

If at the same time $\Delta^* \vdash C$, then since $\Delta^* \vdash (C \Rightarrow D)$, we infer, by Modus Ponens, that

$$\Delta^* \vdash D$$

If so, then $v^*(C) = v^*(D) = T$ and accordingly

$$v^*(A) = v^*(C \Rightarrow D) =$$

$$v^*(C) \Rightarrow v^*(D) = T \Rightarrow T = T$$

Thus if $\Delta^* \vdash A$, then $v^*(A) = T$
Case 2. Assume now, as before, that $\Delta^* \vdash \neg A$,

Then from the fact that $\Delta^*$ is consistent it must be that $\Delta^* \not\vdash A$, i.e.,

$$\Delta^* \not\vdash (C \Rightarrow D)$$

It follows from this that $\Delta^* \not\vdash D$

For if $\Delta^* \vdash D$, then, as $(D \Rightarrow (C \Rightarrow D))$ is provable formula 1. in $S$, by monotonicity also

$$\Delta^* \vdash (D \Rightarrow (C \Rightarrow D))$$

Applying Modus Ponens we obtain

$$\Delta^* \vdash (C \Rightarrow D)$$

which is contrary to the assumption, so it must be $\Delta^* \not\vdash D$
Proof of Property of $\nu$ Lemma

Also we must have

$$\Delta^* \vdash C$$

for otherwise, as $\Delta^*$ is complete we would have $\Delta^* \vdash \neg C$

This is impossible since by Lemma formula 9.

$$\vdash (\neg C \Rightarrow (C \Rightarrow D))$$

By monotonicity

$$\Delta^* \vdash (\neg C \Rightarrow (C \Rightarrow D))$$

Applying Modus Ponens we would get

$$\Delta^* \vdash (C \Rightarrow D)$$

which is contrary to the assumption $\Delta^* \not\vdash (C \Rightarrow D)$
Proof Two of Completeness Theorem

This ends the proof of the Property of v Lemma and the Proof Two of the Completeness Theorem is also completed.