cse541
LOGIC for COMPUTER SCIENCE

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LECTURE 5a

Chapter 5
HILBERT PROOF SYSTEMS: Completeness of Classical Propositional Logic

Lecture 5a

PART 1: Introduction

PART 2: **Proof** of the **Main Lemma**

PART 3: **Proof 1**: Constructive Proof of **Completeness Theorem**

PART 1: Introduction

Two Proofs

There are many proof systems that describe classical propositional logic, i.e. that are **complete proof systems** with the respect to the classical semantics.

We present here a Hilbert proof system for the classical propositional logic and discuss two ways of proving the **Completeness Theorem** for it.

Any **proof** of the Completeness Theorem consists always of **two parts**.

Two Proofs

First we have show that all formulas that have a proof are tautologies.

This implication is also called a **Soundness Theorem**, or Soundness Part of the **Completeness Theorem**

The second implication says: if a formula is a tautology then it has a proof.

This alone is sometimes called a **Completeness Theorem** (on assumption that the system is sound)

Traditionally it is called a completeness part of the **Completeness Theorem**

Two Proofs

The **proof** of the soundness part is standard.

We concentrate here on the completeness part of the **Completeness Theorem** and present **two proofs** of it

The **first proof** is straightforward. It shows how one can use the assumption that a formula A is a tautology in order to **construct** its formal proof

It is hence called **a proof - construction method**.

Two Proofs

The **second proof** shows how one can **prove** that a formula A is not a tautology **from** the fact that it does not have a proof

It is hence called a **counter-model construction method**.

All these **proofs** and considerations are relative to proof systems and their semantics

At this moment the semantics is classical and the proof system is H_2

Reminder: we write $\models A$ to denote that A is a classical tautology

Proof System H_2

Reminder: H_2 is the following proof system:

$$H_2 = (\ \mathcal{L}_{\{\Rightarrow,\neg\}},\ \ \mathcal{F},\ \ \{A1, A2, A3\},\ \ MP\ )$$

The axioms $A1 - A3$ are defined as follows.

A1 $(A \Rightarrow (B \Rightarrow A))$,

A2 $((A \Rightarrow (B \Rightarrow C)) \Rightarrow ((A \Rightarrow B) \Rightarrow (A \Rightarrow C)))$,

A3 $((\neg B \Rightarrow \neg A) \Rightarrow ((\neg B \Rightarrow A) \Rightarrow B)))$

$$(MP)\ \frac{A\ ;\ (A \Rightarrow B)}{B}$$

Proof System H_2

Obviously, the selected axioms $A1, A2, A3$ are **tautologies**, and the MP rule leads from tautologies to tautologies.

Hence our proof system H_2 is **sound** and the following theorem holds.

Soundness Theorem

For every formula $A \in \mathcal{F}$,

If $\vdash_{H_2} A$, then $\models A$

System H_2 LEMMA

We have proved in Lecture 5 the following

Lemma

The following formulas a **are provable** in H_2

1. $(A \Rightarrow A)$

2. $(\neg\neg B \Rightarrow B)$

3. $(B \Rightarrow \neg\neg B)$

4. $(\neg A \Rightarrow (A \Rightarrow B))$

5. $((\neg B \Rightarrow \neg A) \Rightarrow (A \Rightarrow B))$

6. $((A \Rightarrow B) \Rightarrow (\neg B \Rightarrow \neg A))$

7. $(A \Rightarrow (\neg B \Rightarrow (\neg(A \Rightarrow B)))$

8. $((A \Rightarrow B) \Rightarrow ((\neg A \Rightarrow B) \Rightarrow B))$

9. $((\neg A \Rightarrow A) \Rightarrow A)$

First Proof

The **first proof** of **Completeness Theorem** presented here is very elegant and simple, but is applicable only to the **classical propositional logic**

This proof is, as was the proof of Deduction Theorem, a fully **constructive**

The technique it uses , because of its specifics can't be used even in a case of classical predicate logic, not to mention variaty of non-classical logics

Second Proof

The **second proof** is much more complicated.

Its strength and importance lies in a fact that the methods it uses can be applied in an extended version to the **proof of completeness** for classical predicate logic and some non-classical propositional and predicate logics

The way **we define** a counter-model for any non-provable A is general and non- constructive

We call it a a counter-model existence method

PART 2: Proof of the MAIN LEMMA

Completeness Theorem

The proof of the **Completeness Theorem** presented here is similar in its structure to the proof of the **Deduction Theorem** and is due to Kalmar, 1935

It is a **constructive proof**

It shows how one can use the assumption that a formula A is a tautology in order to **construct** its formal proof.

We hence call it a **proof construction method**. It relies heavily on the Deduction Theorem

It is possible to prove the **Completeness Theorem** independently from the Deduction Theorem and we will present two of such a proofs in later chapters.

Introduction

We first present **one definition** and prove **one lemma**

We write $\vdash A$ instead of $\vdash_S A$ as the system S is fixed.

Let A be a formula and $b_1, b_2, ..., b_n$ be all propositional variables that occur in A, i.e.

$$A = A(b_1, b_2, ..., b_n)$$

MAIN LEMMA: Definition 1

Definition 1

Let v be a truth assignment $v : VAR \longrightarrow \{T, F\}$

We define, for $A, b_1, b_2, ..., b_n$ and truth assignment v corresponding formulas A', $B_1, B_2, ..., B_n$ as follows:

$$A' = \begin{cases} A & \text{if } v^*(A) = T \\ \neg A & \text{if } v^*(A) = F \end{cases}$$

$$B_i = \begin{cases} b_i & \text{if } v(b_i) = T \\ \neg b_i & \text{if } v(b_i) = F \end{cases}$$

for $i = 1, 2, ..., n$

Example 1

Let A be a formula $(a \Rightarrow \neg b)$

Let v be such that $v(a) = T, \ v(b) = F$

In this case we have that $b_1 = a, \ b_2 = b$, and

$v^*(A) = v^*(a \Rightarrow \neg b) = v(a) \Rightarrow \neg v(b) = T \Rightarrow \neg F = T$

The corresponding A', B_1, B_2 are:

$A' = A$ as $v^*(A) = T$

$B_1 = a$ as $v(a) = T$

$B_2 = \neg b$ as $v(b) = F$

Example 2

Let A be a formula $((\neg a \Rightarrow \neg b) \Rightarrow c)$

and let v be such that v(a)= T, v(b) =F, v(c)=F

Evaluate $A', B_1, ...B_n$ as defined by the **definition 1**

In this case $n = 3$ and $b_1 = a,\ b_2 = b, b_3 = c$

and we evaluate

$v^*(A) = v^*((\neg a \Rightarrow \neg b) \Rightarrow c) = ((\neg v(a) \Rightarrow \neg v(b)) \Rightarrow v(c)) = ((\neg T \Rightarrow \neg F) \Rightarrow F) = (T \Rightarrow F) = F$

The corresponding A', B_1, B_2, B_2 are:

$A' = \neg A = \neg((\neg a \Rightarrow \neg b) \Rightarrow c)$ as $v^*(A) = F$

$B_1 = a$ as $v(a) = T$, $B_2 = \neg b$ as $v(b) = F$, and

$B_3 = \neg c$ as $v(c) = F$

MAIN LEMMA

The lemma stated below describes a method of transforming a **semantic notion** of a tautology into a **syntactic notion** of provability

It **defines**, for any formula A and a truth assignment v a corresponding **deducibility relation**

Main Lemma

For any formula $A = A(b_1, b_2, ..., b_n)$ and any truth assignment v

If A', B_1, B_2, ..., B_n are corresponding formulas defined by **definition 1**, then

$$B_1, B_2, ..., B_n \vdash A'$$

Examples

Example 3

Let A, v be as defined in the **Example 1**, i.e. $A' = A$, $B_1 = a$, $B_2 = \neg b$

Main Lemma asserts that

$$a, \neg b \vdash (a \Rightarrow \neg b)$$

Example 4

Let A, v be defined as in **Example 2**, then the **Lemma** asserts that

$$a, \neg b, \neg c \vdash \neg((\neg a \Rightarrow \neg b) \Rightarrow c)$$

Proof of the Main Lemma

The proof is by induction on the **degree of the formula** A

Base Case $n = 0$

In this case A is atomic and so consists of a single propositional variable, say a

If $v^*(A) = T$ then we have by **definition 1**

$A' = A = a, \; B_1 = a$

We obtain, by **definition of provability** from a set Γ of hypothesis for $\Gamma = \{a\}$ that

$$a \vdash a$$

Proof of the Main Lemma

If $v^*(A) = F$ we have by **Definition 1** that

$$A' = \neg A = \neg a \quad \text{and} \quad B_1 = \neg a$$

We obtain, by **definition of provability** from a set Γ of hypothesis for $\Gamma = \{\neg a\}$ that

$$\neg a \vdash \neg a$$

This **proves** that **Lemma** holds for n=0

Proof of the Main Lemma

Inductive Step

Now **assume** that the **Main Lemma** holds for any formula with $j < n$ connectives

Need to prove: the **Main Lemma** holds for A with n connectives

There are several sub-cases to deal with

Proof of the Main Lemma

Case: A is $\neg A_1$

By the inductive assumption we have the formulas

$$A_1^{'}, \quad B_1, B_2, ..., B_n$$

corresponding to the A_1 and the propositional variables $b_1, b_2, ..., b_n$ in A_1, such that

$$B_1, B_2, ..., B_n \vdash A_1^{'}$$

Observe that the formulas A and $\neg A_1$ have the same propositional variables

So the corresponding formulas B_1 , B_2, ..., B_n are the same for both of them.

Proof of the Main Lemma

We are going to show that the inductive assumption allows us to prove that

$$B_1, B_2, ..., B_n \vdash A'$$

There are two cases to consider.

Case: $v^*(A_1) = T$

If $v^*(A_1) = T$ then by **definition 1** $A_1' = A_1$ and by the inductive assumption

$$B_1, B_2, ..., B_n \vdash A_1$$

In this case: $v^*(A) = v^*(\neg A_1) = \neg v^*(T) = F$

So we have that $A' = \neg A = \neg\neg A_1$

Proof of the Main Lemma

By Lemma **3.** we have that that $\vdash (A \Rightarrow \neg\neg A)$, so in particular

$$\vdash (A_1 \Rightarrow \neg\neg A_1)$$

we obtain by the monotonicity that also

$$B_1, B_2, ..., B_n \vdash (A_1 \Rightarrow \neg\neg A_1)$$

By **inductive assumption** $B_1, B_2, ..., B_n \vdash A_1$ and by MP we have

$$B_1, B_2, ..., B_n \vdash \neg\neg A_1$$

and as $A' = \neg A = \neg\neg A_1$ we get

$$B_1, B_2, ..., B_n \vdash \neg A \quad \text{and so} \quad B_1, B_2, ..., B_n \vdash A'$$

Proof of the Main Lemma

Case: $v^*(A_1) = F$

If $v^*(A_1) = F$ then $A_1' = \neg A_1$ and $v^*(A) = T$ so $A' = A$

Therefore by the **inductive assumption** we have that

$$B_1, B_2, ..., B_n \vdash \neg A_1$$

that is as $A = \neg A_1$

$$B_1, B_2, ..., B_n \vdash A'$$

Proof of the Main Lemma

Case: A is $(A_1 \Rightarrow A_2)$

If A is $(A_1 \Rightarrow A_2)$ then A_1 and A_2 have less than n connectives

$A = A(b_1, \dots b_n)$ so there are some **subsequences** $c_1, \dots, c_k$ and $d_1, \dots d_m$ for $k, m \leq n$ of the sequence $b_1, \dots, b_n$ such that

$$A_1 = A_1(c_1, \dots, c_k) \quad \text{and} \quad A_2 = A(d_1, \dots d_m)$$

Proof of the Main Lemma

A_1 and A_2 have less than n connectives and so by the **inductive assumption** we have appropriate formulas $C_1, ..., C_k$ and $D_1, ...D_m$ such that

$$C_1, C_2, \ldots, C_k \vdash A_1{}' \quad \text{and} \quad D_1, D_2, \ldots, D_m \vdash A_2{}'$$

and $C_1, C_2, ..., C_k$, $D_1, D_2, ..., D_m$ are **subsequences** of formulas $B_1, B_2, ..., B_n$ corresponding to the propositional variables in A

By **monotonicity** we have the also

$$B_1, B_2, ..., B_n \vdash A_1{}' \quad \text{and} \quad B_1, B_2, ..., B_n \vdash A_2{}'$$

Now we have the following sub-cases to consider

Proof of the Main Lemma

Case: $v^*(A_1) = v^*(A_2) = T$

If $v^*(A_1) = T$ then $A_1{}' = A_1$ and

if $v^*(A_2) = T$ then $A_2{}' = A_2$

We also have $v^*(A_1 \Rightarrow A_2) = T$ and so $A' = (A_1 \Rightarrow A_2)$

By the above and the **inductive assumption**

$$B_1, B_2, ..., B_n \vdash A_2$$

and By Axiom 1 and by **monotonicity** we have

$$B_1, B_2, ..., B_n \vdash (A_2 \Rightarrow (A_1 \Rightarrow A_2))$$

By above and MP we have $B_1, B_2, ..., B_n \vdash (A_1 \Rightarrow A_2)$
that is

$$B_1, B_2, ..., B_n \vdash A'$$

Proof of the Main Lemma

Case: $v^*(A_1) = T, \; v^*(A_2) = F$

If $v^*(A_1) = T$ then $A_1{}' = A_1$ and

if $v^*(A_2) = F$ then $A_2{}' = \neg A_2$

Also we have in this case $v^*(A_1 \Rightarrow A_2) = F$ and so

$A' = \neg(A_1 \Rightarrow A_2)$

By the **above**, the **inductive assumption** and **monotonicity**

$B_1, B_2, ..., B_n \vdash \neg A_2$

By Lemma **7.** we have $\vdash (A \Rightarrow (\neg B \Rightarrow \neg(A \Rightarrow B)))$. By **monotonicity** we have in our particular case

$$B_1, B_2, ..., B_n \vdash (A_1 \Rightarrow (\neg A_2 \Rightarrow \neg(A_1 \Rightarrow A_2)))$$

By above and MP **twice** we have

$B_1, B_2, ..., B_n \vdash \neg(A_1 \Rightarrow A_2)$ that is

$$B_1, B_2, ..., B_n \vdash A'$$

Proof of the Main Lemma

Case: $v^*(A_1) = F$

Observe that if $v^*(A_1) = F$ then $A_1{}'$ is $\neg A_1$ and, whatever value v gives A_2, we have

$$v^*(A_1 \Rightarrow A_2) = T$$

So A' is $(A_1 \Rightarrow A_2)$

Therefore

$$B_1, B_2, \ldots, B_n \vdash \neg A_1$$

We have that $\vdash (\neg A \Rightarrow (A \Rightarrow B))$ by Lemma **4.** and so by monotonicity we have

$$B_1,\ B_2, \ldots,\ B_n \vdash (\neg A_1 \Rightarrow (A_1 \Rightarrow A_2))$$

Proof of the Main Lemma

By Modus Ponens we get that

$$B_1, B_2, ..., B_n \vdash (A_1 \Rightarrow A_2)$$

that is

$$B_1, B_2, ..., B_n \vdash A^{'}$$

We have covered **all cases** and, by **mathematical induction** on the degree of the formula A we got

$$B_1, B_2, ..., B_n \vdash A^{'}$$

The proof of the **Main Lemma** is complete

PART3

Proof 1: Constructive Proof of Completeness Theorem

Proof of Completeness Theorem

Now we use the **Main Lemma** to prove the **Completeness Theorem** i.e. to prove the following implication

For any formula $A \in \mathcal{F}$

$$if \models A \ then \vdash A$$

Proof

Assume that $\models A$

Let $b_1, b_2, ..., b_n$ be all propositional variables that occur in the formula A, i.e.

$$A = A(b_1, b_2, ..., b_n)$$

By the **Main Lemma** we know that, for any truth assignment v, the corresponding formulas $A^{'}$, B_1, B_2, ..., B_n can be found such that

$$B_1, B_2, ..., B_n \vdash A^{'}$$

Proof

Note that in this case $A' = A$ for any v since $\models A$
We have two cases.
1. If v is such that $v(b_n) = T$, then $B_n = b_n$ and

$$B_1, B_2, ..., b_n \vdash A$$

2. If v is such that $v(b_n) = F$, then $B_n = \neg b_n$ and by the **Main Lemma**

$$B_1, B_2, ..., \neg b_n \vdash A$$

So, by the **Deduction Theorem** we have

$$B_1, B_2, ..., B_{n-1} \vdash (b_n \Rightarrow A)$$

and

$$B_1, B_2, ..., B_{n-1} \vdash (\neg b_n \Rightarrow A)$$

Proof of Completeness Theorem

By Lemma **8.**

$$\vdash((A \Rightarrow B) \Rightarrow ((\neg A \Rightarrow B) \Rightarrow B))$$

for $A = b_n,\ B = A$
and by monotonicity we have that

$$B_1, B_2, ..., B_{n-1} \vdash ((b_n \Rightarrow A) \Rightarrow ((\neg b_n \Rightarrow A) \Rightarrow A))$$

Applying Modus Ponens twice we get that

$$B_1, B_2, ..., B_{n-1} \vdash A$$

Similarly, $v^*(B_{n-1})$ may be T or F
Applying the **Main Lemma** , the **Deduction Theorem**, monotonicity, formula **8.** and Modus Ponens twice we can eliminate B_{n-1} just as we have eliminated B_n
After n steps, we finally obtain proof of A in S, i.e. we have that

$$\vdash A$$

Constructiveness of the Proof

Observe that our proof of the Completeness Theorem is a **constructive** one.

Moreover, we have used in it only Main Lemma and Deduction Theorem which both have a **constructive proofs**

We **can** hence reconstruct proofs in each case when we apply these theorems back to the original axioms of H_2

The same applies to the **proofs** in H_2 of all formulas **1. - 9.**

It means that for any A, such that $\models A$, the set V_A of all v restricted to A **provides** us a method of a **construction** of the **formal proof** of A in H_2.

Example

Example

The proof of **Completeness Theorem** defines a **method** of efficiently combining $v \in V_A$ while **constructing** the proof of A

Let's consider the following **tautology** $A = A(a, b, c)$

$$((\neg a \Rightarrow b) \Rightarrow (\neg(\neg a \Rightarrow b) \Rightarrow c))$$

We present on the next slides all steps of the **Proof 1** as applied to A

Example

Given

$$A(a,b,c) = ((\neg a \Rightarrow b) \Rightarrow (\neg(\neg a \Rightarrow b) \Rightarrow c)$$

By the **Main Lemma** and the assumption that

$$\models A(a,b,c)$$

any $v \in V_A$ **defines** formulas B_a , B_b, B_c such that

$$B_a, B_b, B_c \vdash A$$

The proof is based on a method of using all $v \in V_A$ (there is 8 of them) to **define** a process of elimination of all hypothesis B_a, B_b, B_c to **construct** the proof of A , ı.e. to prove that

$$\vdash A$$

Example

Step 1: elimination of B_c

Observe that by definition, B_c is c or $\neg c$ depending on the **choice** of $v \in V_A$

We **choose** two truth assignments $v_1 \neq v_2 \in V_A$ such that

$$v_1 \mid \{a, b\} = v_2 \mid \{a, b\} \quad \text{and} \quad v_1(c) = T, \;\; v_2(c) = F$$

Case 1: $v_1(c) = T$

By by definition $B_c = c$

By our choice, the assumption that $\models A$ and the **Main Lemma** applied to v_1

$$B_a, B_b, c \;\vdash\; A$$

By **Deduction Theorem** we have that

$$B_a, B_b \;\vdash\; (c \Rightarrow A)$$

Example

Case 2: $v_2(c) = F$

By definition $B_c = \neg c$

By our **choice**, assumption that $\models A$, and the **Main Lemma** applied to v_2

$$B_a, B_b, \neg c \vdash A$$

By the **Deduction Theorem** we have that

$$B_a, B_b \vdash (\neg c \Rightarrow A)$$

Example

By Lemma **8.** for $A = c$, $B = A$ we have that

$$\vdash ((c \Rightarrow A) \Rightarrow ((\neg c \Rightarrow A) \Rightarrow A))$$

By monotonicity we have that

$$B_a, B_b \vdash ((c \Rightarrow A) \Rightarrow ((\neg c \Rightarrow A) \Rightarrow A))$$

Applying Modus Ponens twice to the above property and properties on the previous slide we get that

$$B_a, B_b \vdash A$$

We have **eliminated** B_c

Example

Step 2: elimination of B_b from $B_a, B_b \vdash A$

We repeat the **Step 1**

As before we have 2 cases to consider: $B_b = b$ or $B_b = \neg b$

We **choose** two truth assignments $w_1 \neq w_2 \in V_A$ such that

$$w_1 | \{a\} = w_2 | \{a\} = v_1 | \{a\} = v_2 | \{a\} \text{ and } w_1(b) = T, \; w_2(b) = F$$

Case 1: $w_1(b) = T$ and by definition $B_b = b$

By our choice, assumption that $\models A$ and the **Main Lemma** applied to w_1

$$B_a, b \vdash A$$

By **Deduction Theorem** we have that

$$B_a \vdash (b \Rightarrow A)$$

Example

Case 2: $w_2(b) = F$ and by definition $B_b = \neg b$

By choice, assumption that $\models A$ and the **Main Lemma** applied to
w_2

$$B_a, \neg b \vdash A$$

By the **Deduction Theorem** we have that

$$B_a \vdash (\neg b \Rightarrow A)$$

Example

By Lemma **8.** for $A = b,\ B = A$ we have that

$$\vdash ((b \Rightarrow A) \Rightarrow ((\neg b \Rightarrow A) \Rightarrow A))$$

By monotonicity

$$B_a \vdash ((b \Rightarrow A) \Rightarrow ((\neg b \Rightarrow A) \Rightarrow A))$$

Applying Modus Ponens twice to the above property and properties from the previous slide we get that

$$B_a \vdash A$$

We have **eliminated** B_b

Example

Step 3: elimination] of B_a from $B_a \vdash A$

We repeat the **Step 2**

As before we have 2 cases to consider: $B_a = a$ or $B_a = \neg a$

We choose two truth assignments $g_1 \neq g_2 \in V_A$ such that

$$g_1(a) = T \quad \text{and} \quad g_2(a) = F$$

Case 1: $g_1(a) = T$, and by definition $B_a = a$

By the choice, assumption that $\models A$, and the **Main Lemma** applied to g_1

$$a \vdash A$$

By **Deduction Theorem** we have that

$$\vdash (a \Rightarrow A)$$

Example

Case 2: $g_2(a) = F$ and by definition $B_a = \neg a$

By the choice, assumption that $\models A$, and the **Main Lemma** applied to g_2

$$\neg a \vdash A$$

By the **Deduction Theorem** we have that

$$\vdash (\neg a \Rightarrow A)$$

Example

By Lemma **8.** for $A = a, \; B = A$ we have that

$$\vdash ((a \Rightarrow A) \Rightarrow ((\neg a \Rightarrow A) \Rightarrow A))$$

Applying Modus Ponens twice to the above property and properties from previous slides we get that

$$\vdash \; A$$

We have **eliminated** $B_a, \; B_b, \; B_c$ and constructed the **proof** of A in S

Exercises

Exercise 1

The **Lemma** listed formulas **1.** - **9.** that we said they were needed for **both** proofs of the **Completeness Theorem**.

List all the formulas from t**Lemma** that are are **needed** for the **Proof One** alone

Exercises

Exercise 2

The system H_2 was defined and the **Proof One** was carried out for the language $\mathcal{L}_{\{\Rightarrow,\neg\}}$

Extend the system H_2 and the **Proof One** to the language $\mathcal{L}_{\{\Rightarrow,\cup,\neg\}}$ by **adding** all new cases concerning the new connective $\cup$

List all new formulas needed to be **added** as new Axioms to H_2 to be able to follow the methods of the original **Proof One**

Exercise 3

Repeat the **Exercise 2** for he language

$$\mathcal{L}_{\{\Rightarrow,\ \cup,\ \cap\ \ \neg\}}$$