

cse541  
LOGIC for COMPUTER SCIENCE

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## LECTURE 5

## Chapter 5

# HILBERT PROOF SYSTEMS: Completeness of Classical Propositional Logic

**PART 1:** Hilbert Proof System  $H_1$  and examples of applications of **Deduction Theorem**

**PART 2:** Proof of **Deduction Theorem** for System  $H_1$

**PART 3:** System  $H_2$  and examples of **formal proofs** in  $H_2$

## Hilbert Proof Systems

**Hilbert** proof systems are based on a **language** with **i and mplication** and **contain Modus Ponens** as a rule of inference

**Modus Ponens** is probably the **oldest** of all known rules of inference as it was already known to the **Stoics** in 3rd century B.C. and is also considered as the **most natural** to our **intuitive thinking**

The proof systems containing **Modus Ponens** as the inference rule play a **special role** in logic.

## Hilbert Proof Systems

**Hilbert systems** put major emphasis on **logical axioms** and keep the number of **rules** of inference at the **minimum**  
**Hilbert systems** often **admit** the **Modus Ponens** as the **sole rule** of inference

There are many proof systems that describe **classical propositional logic**, i.e. that are **complete** with respect to the **classical** semantics

We present a **Hilbert** proof system for the **classical propositional logic** and discuss **two ways** of proving the **Completeness Theorem** for it

## Hilbert Proof Systems

The **first proof** is based on the one included in Elliott Mendelson's book *Introduction to Mathematical Logic*

It is a **constructive** proof that shows how one can use the assumption that a formula  $A$  is a tautology in order to **construct** its formal **proof**

The **second proof** is **non-constructive**

Its importance lies in a fact that the **methods** it uses can be applied to the proof of **completeness** for classical **predicate** logic (chapter 9)

It also **generalizes** to some **non-classical** logics

## Hilbert Proof Systems

We prove **completeness part** of the **Completeness Theorem** by proving the **converse** implication to it

We show how one can **deduce** that a formula **A is not** a **tautology from** the fact that it **does not** have a **proof**

It is hence called a **counter-model** construction proof

**Both proofs** rely on the **Deduction Theorem** and so this is the **theorem** we are now going to **prove**

## Hilbert Proof System $H_1$

We consider now a **Hilbert** proof system  $H_1$  **based** on a language with **implication** as the **only** connective, with **two** logical **axioms**, and with **Modus Ponens** as a **sole rule** of inference



## Hilbert Proof System $H_1$

**We define** Hilbert system  $H_1$  as follows

$$H_1 = ( \mathcal{L}_{\{\Rightarrow\}}, \mathcal{F}, \{A1, A2\}, MP )$$

**A1** (Law of simplification)

$$(A \Rightarrow (B \Rightarrow A))$$

**A2** (Frege's Law)

$$((A \Rightarrow (B \Rightarrow C)) \Rightarrow ((A \Rightarrow B) \Rightarrow (A \Rightarrow C)))$$

**MP** is the **Modus Ponens** rule

$$(MP) \frac{A ; (A \Rightarrow B)}{B}$$

where  $A, B, C$  are any formulas from  $\mathcal{F}$

## Formal Proofs in $H_1$

Finding **formal proofs** in this system requires some ingenuity.  
**The formal proof** of  $(A \Rightarrow A)$  in  $H_1$  is a sequence

$$B_1, B_2, B_3, B_4, B_5$$

as defined below.

$B_1 : ((A \Rightarrow ((A \Rightarrow A) \Rightarrow A)) \Rightarrow ((A \Rightarrow (A \Rightarrow A)) \Rightarrow (A \Rightarrow A))),$   
axiom A2 for  $A = A$ ,  $B = (A \Rightarrow A)$ , and  $C = A$

$B_2 : (A \Rightarrow ((A \Rightarrow A) \Rightarrow A)),$   
axiom A1 for  $A = A$ ,  $B = (A \Rightarrow A)$

$B_3 : ((A \Rightarrow (A \Rightarrow A)) \Rightarrow (A \Rightarrow A)),$   
MP application to  $B_1$  and  $B_2$

$B_4 : (A \Rightarrow (A \Rightarrow A)),$   
axiom A1 for  $A = A$ ,  $B = A$

$B_5 : (A \Rightarrow A)$   
MP application to  $B_3$  and  $B_4$

## Searching for Proofs in a Proof System

A **general procedure** for **automated search** for proofs in a proof system **S** can be stated as follows

Let **B** be an expression of the system **S** that is not an axiom

If **B** has a **proof** in **S**, **B** must be the **conclusion** of one of the inference rules

Let's say it is a rule **r**

We **find** all its premisses, i.e. we evaluate  $r^{-1}(B)$

If **all premisses** are **axioms**, the proof is **found**

Otherwise we **repeat** the procedure for any **premiss** that **is not** an **axiom**

## Search for Proof by the Means of MP

The **MP** rule says:

given two formulas  $A$  and  $(A \Rightarrow B)$  we conclude a formula  $B$

**Assume** now that and want to find a **proof** of a formula  $B$

If  $B$  is an **axiom**, we have the **proof**; the formula itself

If  $B$  is **not an axiom**, it had to be obtained by the application of the **Modus Ponens** rule to certain two formulas

$A$  and  $(A \Rightarrow B)$  and there is **infinitely many** of such formulas!

The proof system  $H_1$  is **not syntactically decidable**

## Semantic Links

### Semantic Link 1

System  $H_1$  is **sound** under classical semantics and  
 $H_1$  is **not sound** under **K** semantics

### Soundness Theorem for $H_1$

For any  $A \in \mathcal{F}$ , if  $\vdash_{H_1} A$ , then  $\models A$

## Semantic Links

### Semantic Link 2

The system  $H_1$  **is not complete** under classical semantics

Not all classical **tautologies** have a proof in  $H_1$

For example we can't express negation in term of implication and a **tautology**  $(\neg\neg A \Rightarrow A)$  is not provable in  $H_1$  , i.e.

$$\not\vdash_{H_1} (\neg\neg A \Rightarrow A)$$

## Proof from Hypothesis

Given a proof system  $S = (\mathcal{L}, \mathcal{E}, LA, \mathcal{R})$

While proving expressions we often use **some extra information** available, besides the axioms of the proof system

This extra information is called **hypothesis** in the proof

Let  $\Gamma \subseteq \mathcal{E}$  be a set expressions called **hypothesis**

**Notation:**  $\Gamma \vdash_S E$

We read it :  $E$  has a proof in  $S$  from the set  $\Gamma$  and the logical axioms  $LA$

## Formal Definition

### Definition

We say that  $E \in \mathcal{E}$  has a **formal proof** in  $S$  from the set  $\Gamma$  and the logical axioms  $LA$  and denote it as  $\Gamma \vdash_S E$  if and only if there is a sequence

$$A_1, \dots, A_n$$

of expressions from  $\mathcal{E}$ , such that

$$A_1 \in LA \cup \Gamma, \quad A_n = E$$

and for each  $1 < i \leq n$ , either  $A_i \in LA \cup \Gamma$  or  $A_i$  is a **direct consequence** of some of the **preceding** expressions by virtue of **one of the rules** of inference of  $S$



## Special Cases

**Case 1:**  $\Gamma \subseteq \mathcal{E}$  is a **finite set** and  $\Gamma = \{B_1, B_2, \dots, B_n\}$

We write

$$B_1, B_2, \dots, B_n \vdash_S E$$

instead of  $\{B_1, B_2, \dots, B_n\} \vdash_S E$

**Case 2:**  $\Gamma = \emptyset$

By the **definition** of a proof of  $E$  from  $\Gamma$ ,  $\emptyset \vdash_S E$  means that in the proof of  $E$  we use **only** the logical axioms **LA** of  $S$

We hence write

$$\vdash_S E$$

to denote that  $E$  has a proof from  $\Gamma = \emptyset$

## Proof from Hypothesis in $H_1$

Show that

$$(A \Rightarrow B), (B \Rightarrow C) \vdash_{H_1} (A \Rightarrow C)$$

We construct a **formal proof**

$$B_1, B_2, \dots, B_7$$

$$B_1 : (B \Rightarrow C), \quad B_2 : (A \Rightarrow B),$$

hypothesis                  hypothesis

$$B_3 : ((A \Rightarrow (B \Rightarrow C)) \Rightarrow ((A \Rightarrow B) \Rightarrow (A \Rightarrow C))),$$

axiom A2

## Proof from Hypothesis in $H_1$

$B_4$  :  $((B \Rightarrow C) \Rightarrow (A \Rightarrow (B \Rightarrow C)))$ ,  
axiom A1 for  $A = (B \Rightarrow C)$ ,  $B = A$

$B_5$  :  $(A \Rightarrow (B \Rightarrow C))$ ,  
 $B_1$  and  $B_4$  and MP

$B_6$  :  $((A \Rightarrow B) \Rightarrow (A \Rightarrow C))$ ,       $B_7$  :  $(A \Rightarrow C)$   
MP

## Deduction Theorem

In mathematical arguments, one often **proves** a statement  $B$  on the **assumption** of some other statement  $A$  and then **concludes** that we have **proved** the implication "if  $A$ , then  $B$ "

This reasoning is justified a theorem, called a **Deduction Theorem**

### Reminder

We write  $\Gamma, A \vdash B$  for  $\Gamma \cup \{A\} \vdash B$

In general, we write  $\Gamma, A_1, A_2, \dots, A_n \vdash B$

for  $\Gamma \cup \{A_1, A_2, \dots, A_n\} \vdash B$

## Deduction Theorem for $H_1$

### Deduction Theorem for $H_1$

For any  $A, B \in \mathcal{F}$  and  $\Gamma \subseteq \mathcal{F}$

$\Gamma, A \vdash_{H_1} B$  if and only if  $\Gamma \vdash_{H_1} (A \Rightarrow B)$

In particular

$A \vdash_{H_1} B$  if and only if  $\vdash_{H_1} (A \Rightarrow B)$

## Formal Proofs

The proof of the following **Lemma** provides a good example of multiple **applications** of the **Deduction Theorem**

### Lemma

For any  $A, B, C \in \mathcal{F}$ ,

(a)  $(A \Rightarrow B), (B \Rightarrow C) \vdash_{H_1} (A \Rightarrow C)$ ,

(b)  $(A \Rightarrow (B \Rightarrow C)) \vdash_{H_1} (B \Rightarrow (A \Rightarrow C))$

**Observe** that by **Deduction Theorem** we can re-write (a) as

(a')  $(A \Rightarrow B), (B \Rightarrow C), A \vdash_{H_1} C$

## Formal Proofs

### Proof of (a')

We construct a formal proof

$B_1, B_2, B_3, B_4, B_5$

of  $(A \Rightarrow B), (B \Rightarrow C), A \vdash_{H_1} C$  as follows.

$B_1 : (A \Rightarrow B)$

hypothesis

$B_2 : (B \Rightarrow C)$

hypothesis

$B_3 : A$

hypothesis

$B_4 : B$

$B_1, B_3$  and MP

$B_5 : C$

$B_2, B_4$  and MP

## Formal Proofs

Thus we proved by **Deduction Theorem** that **(a)** holds, i.e.

$$(A \Rightarrow B), (B \Rightarrow C) \vdash_{H_1} (A \Rightarrow C)$$

**Proof** of **Lemma** part **(b)**

By **Deduction Theorem** we have that

$$(A \Rightarrow (B \Rightarrow C)) \vdash_{H_1} (B \Rightarrow (A \Rightarrow C))$$

if and only if

$$(A \Rightarrow (B \Rightarrow C)), B \vdash_{H_1} (A \Rightarrow C)$$



## Formal Proofs

We construct a formal proof

$$B_1, B_2, B_3, B_4, B_5, B_6, B_7$$

of  $(A \Rightarrow (B \Rightarrow C)), B \vdash_{H_1} (A \Rightarrow C)$  as follows.

$$B_1 : (A \Rightarrow (B \Rightarrow C))$$

hypothesis

$$B_2 : B$$

hypothesis

$$B_3 : ((B \Rightarrow (A \Rightarrow B)))$$

A1 for  $A = B, B = A$

$$B_4 : (A \Rightarrow B)$$

$B_2, B_3$  and MP

## Formal Proofs

$$B_5 : ((A \Rightarrow (B \Rightarrow C)) \Rightarrow ((A \Rightarrow B) \Rightarrow (A \Rightarrow C)))$$

axiom A2

$$B_6 : ((A \Rightarrow B) \Rightarrow (A \Rightarrow C))$$

$B_1, B_5$  and MP

$$B_7 : (A \Rightarrow C)$$

Thus we proved by **Deduction Theorem** that

$$(A \Rightarrow (B \Rightarrow C)) \vdash_{H_1} (B \Rightarrow (A \Rightarrow C))$$

## Simpler Proof

Here is a simpler proof of **Lemma** part (b)

We apply the **Deduction Theorem** twice, i.e. we get

$$(A \Rightarrow (B \Rightarrow C)) \vdash_{H_1} (B \Rightarrow (A \Rightarrow C))$$

if and only if

$$(A \Rightarrow (B \Rightarrow C)), B \vdash_{H_1} (A \Rightarrow C)$$

if and only if

$$(A \Rightarrow (B \Rightarrow C)), B, A \vdash_{H_1} C$$

## Simpler Proof

We now construct a proof of  $(A \Rightarrow (B \Rightarrow C)), B, A \vdash_{H_1} C$  as follows

$B_1 : (A \Rightarrow (B \Rightarrow C))$

hypothesis

$B_2 : B$

hypothesis

$B_3 : A$

hypothesis

$B_4 : (B \Rightarrow C)$

$B_1, B_3$  and (MP)

$B_5 : C$

$B_2, B_4$  and (MP)

# CONSEQUENCE OPERATION

## Review

Definition: Consequences of  $\Gamma$

Given a proof system

$$S = (\mathcal{L}, \mathcal{E}, LA, \mathcal{R})$$

For any  $\Gamma \subseteq \mathcal{E}$ , and  $A \in \mathcal{E}$ ,

If  $\Gamma \vdash_S A$ , then  $A$  is called a **consequence** of  $\Gamma$  in  $S$

We denote by  $\mathbf{Cn}_S(\Gamma)$  the **set of all consequences** of  $\Gamma$  in  $S$ , i.e. we put

$$\mathbf{Cn}_S(\Gamma) = \{A \in \mathcal{E} : \Gamma \vdash_S A\}$$

## Definition: Consequence Operation

**Observe** that by defining a consequence of  $\Gamma$  in  $S$ , we define in fact a **function** which to every set  $\Gamma \subseteq \mathcal{E}$  assigns a set of **all its consequences**  $\mathbf{Cn}_S(\Gamma)$

We denote this function by  $\mathbf{Cn}_S$  and adopt the following

### Definition

Any function

$$\mathbf{Cn}_S : 2^{\mathcal{E}} \longrightarrow 2^{\mathcal{E}}$$

such that for every  $\Gamma \in 2^{\mathcal{E}}$

$$\mathbf{Cn}_S(\Gamma) = \{A \in \mathcal{E} : \Gamma \vdash_S A\}$$

is called the **consequence operation determined by  $S$**

## Consequence Operation: Monotonicity

Take any **consequence operation** determined by  $S$

$$\mathbf{Cn}_S : 2^{\mathcal{E}} \longrightarrow 2^{\mathcal{E}}$$

### Monotonicity Property

For any sets  $\Gamma, \Delta$  of expressions of  $S$ ,

**if**  $\Gamma \subseteq \Delta$  **then**  $\mathbf{Cn}_S(\Gamma) \subseteq \mathbf{Cn}_S(\Delta)$

**Exercise:** write the proof;

it follows directly from the definition of  $\mathbf{Cn}_S$  and definition of the formal proof



## Consequence Operation: Transitivity

Take any **consequence operation**

$$\mathbf{Cn}_S : 2^{\mathcal{E}} \longrightarrow 2^{\mathcal{E}}$$

### Transitivity Property

For any sets  $\Gamma_1, \Gamma_2, \Gamma_3$  of expressions of  $S$ ,

**if**  $\Gamma_1 \subseteq \mathbf{Cn}_S(\Gamma_2)$  and  $\Gamma_2 \subseteq \mathbf{Cn}_S(\Gamma_3)$ , **then**  $\Gamma_1 \subseteq \mathbf{Cn}_S(\Gamma_3)$

**Exercise:** write the proof;

it follows directly from the definition of  $\mathbf{Cn}_S$  and definition of the formal proof

## Consequence Operation: Finiteness

Take any **consequence operation** determined by

$$\mathbf{Cn}_S : 2^{\mathcal{E}} \rightarrow 2^{\mathcal{E}}$$

### Finiteness Property

For any expression  $A \in \mathcal{E}$  and any set  $\Gamma \subseteq \mathcal{E}$ ,

$A \in \mathbf{Cn}_S(\Gamma)$  if and only if there is a **finite subset**  $\Gamma_0$  of  $\Gamma$  such that  $A \in \mathbf{Cn}_S(\Gamma_0)$

**Exercise:** write the proof;

it follows directly from the definition of  $\mathbf{Cn}_S$  and definition of the formal proof

## Proof Deduction Theorem for $H_1$

## The Deduction Theorem

As we now fix the proof system to be  $H_1$ , we write  $A \vdash B$  instead of  $A \vdash_{H_1} B$

**Deduction Theorem** (Herbrand, 1930) for  $H_1$

For any formulas  $A, B \in \mathcal{F}$ ,

If  $A \vdash B$ , then  $\vdash (A \Rightarrow B)$

**Deduction Theorem** (General case) for  $H_1$

For any formulas  $A, B \in \mathcal{F}$ ,  $\Gamma \subseteq \mathcal{F}$

$\Gamma, A \vdash B$  if and only if  $\Gamma \vdash (A \Rightarrow B)$

**Proof:**

**Part 1** We first prove the "if" part:

If  $\Gamma, A \vdash B$  then  $\Gamma \vdash (A \Rightarrow B)$

## Proof of The Deduction Theorem

Assume that

$$\Gamma, A \vdash B$$

i.e. that we have a formal proof

$$B_1, B_2, \dots, B_n$$

of  $B$  from the set of formulas  $\Gamma \cup \{A\}$

We have to show that

$$\Gamma \vdash (A \Rightarrow B)$$

## Proof of The Deduction Theorem

In order to prove that

$\Gamma \vdash (A \Rightarrow B)$  follows from  $\Gamma, A \vdash B$

we prove a **stronger statement**, namely that

$$\Gamma \vdash (A \Rightarrow B_i)$$

**for any**  $B_i$ ,  $1 \leq i \leq n$  in the formal proof  $B_1, B_2, \dots, B_n$  of  $B$   
also follows from  $\Gamma, A \vdash B$

Hence in **particular case**, when  $i = n$  we will obtain that

$\Gamma \vdash (A \Rightarrow B)$  follows from  $\Gamma, A \vdash B$

and that will end the proof of **Part 1**

## Base Step

The proof of **Part 1** is conducted by **mathematical induction** on  $i$ , for  $1 \leq i \leq n$

**Step 1**  $i = 1$  (base step)

**Observe** that when  $i = 1$ , it means that the **formal proof**  $B_1, B_2, \dots, B_n$  contains only **one element**  $B_1$

By the **definition** of the formal proof from  $\Gamma \cup \{A\}$ , we have that

- (1)  $B_1$  is a logical axiom, or  $B_1 \in \Gamma$ , or
- (2)  $B_1 = A$

This means that  $B_1 \in \{A_1, A_2\} \cup \Gamma \cup \{A\}$

## Base Step

Now we have **two cases** to consider.

**Case1:**  $B_1 \in \{A1, A2\} \cup \Gamma$

**Observe** that  $(B_1 \Rightarrow (A \Rightarrow B_1))$  is the axiom **A1**

By assumption  $B_1 \in \{A1, A2\} \cup \Gamma$

We get the **required proof** of  $(A \Rightarrow B_1)$  from  $\Gamma$

by the following application of the **Modus Ponens** rule

$$(MP) \frac{B_1 ; (B_1 \Rightarrow (A \Rightarrow B_1))}{(A \Rightarrow B_1)}$$



## Base Step

**Case 2:**  $B_1 = A$

When  $B_1 = A$  then to prove  $\Gamma \vdash (A \Rightarrow B_1)$

This means we have to prove

$$\Gamma \vdash (A \Rightarrow A)$$

This holds by **monotonicity** of the consequence and the fact that we have shown that

$$\vdash (A \Rightarrow A)$$

The above cases **conclude the proof** for  $i = 1$  of

$$\Gamma \vdash (A \Rightarrow B_i)$$

## Inductive Step

### Inductive Step

**Assume** that

$$\Gamma \vdash (A \Rightarrow B_k)$$

for **all**  $k < i$  (strong induction)

We will **show** that using this fact we can conclude that also

$$\Gamma \vdash (A \Rightarrow B_i)$$

## Inductive Step

Consider a formula  $B_i$  in the formal proof

$$B_1, B_2, \dots, B_n$$

By **definition** of the formal proof we have to show the following two cases

**Case 1 :**  $B_i \in \{A_1, A_2\} \cup \Gamma \cup \{A\}$  and

**Case 2:**  $B_i$  follows by **MP** from certain  $B_j, B_m$  such that  $j < m < i$

Consider now the **Case 1:**  $B_i \in \{A_1, A_2\} \cup \Gamma \cup \{A\}$

**The proof** of  $(A \Rightarrow B_i)$  from  $\Gamma$  in this case is **obtained** from the proof of the **Step**  $i = 1$  by replacement  $B_1$  by  $B_i$  and is omitted here as a **straightforward repetition**

## Inductive Step

### Case 2:

$B_i$  is a **conclusion** of (MP)

If  $B_i$  is a conclusion of (MP), then we must have two formulas  $B_j, B_m$  in the formal proof

$$B_1, B_2, \dots, B_n$$

such that  $j < i$ ,  $m < i$ ,  $j \neq m$  and

$$(MP) \frac{B_j ; B_m}{B_i}$$

## Inductive Step

By the **inductive assumption** the formulas  $B_j, B_m$  are such that  $\Gamma \vdash (A \Rightarrow B_j)$  and  $\Gamma \vdash (A \Rightarrow B_m)$

Moreover, by the definition of (MP) rule, the formula  $B_m$  has to have a form  $(B_j \Rightarrow B_i)$

This means that

$$B_m = (B_j \Rightarrow B_i)$$

**The inductive assumption** can be re-written as follows

$$\Gamma \vdash (A \Rightarrow (B_j \Rightarrow B_i))$$

for  $j < i$

## Inductive Step

**Observe** now that the formula

$$((A \Rightarrow (B_j \Rightarrow B_i)) \Rightarrow ((A \Rightarrow B_j) \Rightarrow (A \Rightarrow B_i)))$$

is a **substitution of the axiom A2** and hence **has a proof** in our system

By the monotonicity of the consequence, it also has a proof from the set  $\Gamma$ , i.e.

$$\Gamma \vdash ((A \Rightarrow (B_j \Rightarrow B_i)) \Rightarrow ((A \Rightarrow B_j) \Rightarrow (A \Rightarrow B_i)))$$

## Inductive Step

We know that

$$\Gamma \vdash ((A \Rightarrow (B_j \Rightarrow B_i)) \Rightarrow ((A \Rightarrow B_j) \Rightarrow (A \Rightarrow B_i)))$$

Applying the rule MP i.e. performing the following

$$\frac{(A \Rightarrow (B_j \Rightarrow B_i)) ; ((A \Rightarrow (B_j \Rightarrow B_i)) \Rightarrow ((A \Rightarrow B_j) \Rightarrow (A \Rightarrow B_i)))}{((A \Rightarrow B_j) \Rightarrow (A \Rightarrow B_i))}$$

we get that also

$$\Gamma \vdash ((A \Rightarrow B_j) \Rightarrow (A \Rightarrow B_i))$$

## Inductive Step

Applying again the rule **MP** i.e. performing the following

$$\frac{(A \Rightarrow B_j) ; ((A \Rightarrow B_j) \Rightarrow (A \Rightarrow B_i))}{(A \Rightarrow B_i)}$$

we get that

$$\Gamma \vdash (A \Rightarrow B_i)$$

what **ends the proof** of the **inductive step**



## Proof of the Deduction Theorem

By the mathematical induction principle, we have **proved** that

$$\Gamma \vdash (A \Rightarrow B_i), \quad \text{for all } 1 \leq i \leq n$$

In particular it is **true** for  $i = n$ , i.e. for  $B_n = B$  and we proved that

$$\Gamma \vdash (A \Rightarrow B)$$

This ends the proof of the **first part** of the **Deduction Theorem**:

$$\text{If } \Gamma, A \vdash B, \quad \text{then } \Gamma \vdash (A \Rightarrow B)$$

## Proof of the Deduction Theorem

The **proof** of the second part, i.e. of the inverse implication:

If  $\Gamma \vdash (A \Rightarrow B)$ , then  $\Gamma, A \vdash B$

is **straightforward** and goes as follows.

**Assume** that  $\Gamma \vdash (A \Rightarrow B)$

By the monotonicity of the consequence we have also that  
 $\Gamma, A \vdash (A \Rightarrow B)$

Obviously  $\Gamma, A \vdash A$

Applying **Modus Ponens** to the above, we get the proof of  
 $B$  from  $\{\Gamma, A\}$

We have hence proved that

$$\Gamma, A \vdash B$$

## Proof of the Deduction Theorem

This **ends** the proof of

**Deduction Theorem** (General case) for  $H_1$

For any formulas  $A, B \in \mathcal{F}$  and any  $\Gamma \subseteq \mathcal{F}$

$$\Gamma, A \vdash B \quad \text{if and only if} \quad \Gamma \vdash (A \Rightarrow B)$$

The particular case we get also the particular case

**Deduction Theorem** (Herbrand, 1930) for  $H_1$

For any formulas  $A, B \in \mathcal{F}$ ,

$$\text{If } A \vdash B, \text{ then } \vdash (A \Rightarrow B)$$

is obtained from the above by assuming that the set  $\Gamma$  is empty

## Classical Propositional Proof System $H_2$

## Hilbert System $H_2$

The proof system  $H_1$  is **sound** and strong enough to prove the **Deduction Theorem**, but it is **not complete**

We **extend** now its **language** and the set of **logical axioms** to a **complete set of axioms**

**We define** a system  $H_2$  that is **complete** with respect to the classical semantics

The **proof of completeness theorem** is be presented in the next chapter.

## Hilbert System $H_2$ Definition

### Definition

$$H_2 = ( \mathcal{L}_{\{\Rightarrow, \neg\}}, \mathcal{F}, \{A1, A2, A3\} (MP) )$$

**A1** (Law of simplification)

$$(A \Rightarrow (B \Rightarrow A))$$

**A2** (Frege's Law)

$$((A \Rightarrow (B \Rightarrow C)) \Rightarrow ((A \Rightarrow B) \Rightarrow (A \Rightarrow C)))$$

**A3**  $((\neg B \Rightarrow \neg A) \Rightarrow ((\neg B \Rightarrow A) \Rightarrow B))$

**MP** (Rule of inference)

$$(MP) \frac{A ; (A \Rightarrow B)}{B}$$

where  $A, B, C$  are any formulas of the propositional language  $\mathcal{L}_{\{\Rightarrow, \neg\}}$

## Deduction Theorem for System $H_2$

### Observation 1

The proof system  $H_2$  is obtained by adding axiom  $A_3$  to the system  $H_1$

### Observation 2

The language of  $H_2$  is obtained by adding the connective  $\neg$  to the language of  $H_1$

### Observation 3

The use of axioms  $A_1, A_2$  in the proof of **Deduction Theorem** for the system  $H_1$  is independent of the connective  $\neg$  added to the language of  $H_1$

### Observation 4

Hence the proof of the **Deduction Theorem** for the system  $H_1$  can be repeated **as it is** for the system  $H_2$

## Deduction Theorem for System $H_2$

**Observations 1-4** prove that the **Deduction Theorem** holds for system  $H_2$

**Deduction Theorem** for  $H_2$

For any  $\Gamma \subseteq \mathcal{F}$  and  $A, B \in \mathcal{F}$

$\Gamma, A \vdash_{H_2} B$  if and only if  $\Gamma \vdash_{H_2} (A \Rightarrow B)$

In particular

$A \vdash_{H_2} B$  if and only if  $\vdash_{H_2} (A \Rightarrow B)$



## Soundness and Completeness Theorems

We get by easy verification

**Soundness Theorem**  $H_2$

For every formula  $A \in \mathcal{F}$

if  $\vdash_{H_2} A$  then  $\models A$

We prove in the next chapter 10, that  $H_2$  is also complete, i.e. we prove

**Completeness Theorem** for  $H_2$

For every formula  $A \in \mathcal{F}$ ,

$\vdash_{H_2} A$  if and only if  $\models A$

## Completeness Theorems

**The proof** of completeness theorem (for a given semantics) is always a **main point** in **creation** of any new **logic**

There are **many techniques** to prove it, depending on the proof system, and on the **semantics** we define for it

We **present** in **Lecture 5a** and **Lecture 5b** two proofs of the **Completeness Theorem** for the system  $H_2$

These proofs use very different **techniques**, hence the **reason** of presenting **both** of them

## FORMAL PROOFS IN $H_2$

## Examples and Exercises

We present now some examples of **formal proofs** in  $H_2$

There are **two reasons** for presenting them.

**First reason** is that all formulas we prove here to be provable play a **crucial role** in the **proof** of **Completeness Theorem** for  $H_2$

**The second reason** is that they provide a "training ground" for a reader to **learn** how to develop formal proofs

For this reason we write some proofs in a **full detail** and we leave some for the reader to **complete** in a way explained in the following example.

## Important Lemma

We write  $\vdash$  instead of  $\vdash_{H_2}$  for the sake of simplicity

### Reminder

In the construction of the formal proofs we **often use** the **Deduction Theorem** and the following **Lemma 1** they was proved in previous section

### Lemma 1

$$(a) \quad (A \Rightarrow B), (B \Rightarrow C) \vdash_{H_2} (A \Rightarrow C)$$

$$(b) \quad (A \Rightarrow (B \Rightarrow C)) \vdash_{H_2} ((B \Rightarrow (A \Rightarrow C)))$$

## Example 1

### Example 1

Here are consecutive steps

$B_1, \dots, B_5, B_6$

of the proof in  $H_2$  of  $(\neg\neg B \Rightarrow B)$

$$B_1 : ((\neg B \Rightarrow \neg\neg B) \Rightarrow ((\neg B \Rightarrow \neg B) \Rightarrow B))$$

$$B_2 : ((\neg B \Rightarrow \neg B) \Rightarrow ((\neg B \Rightarrow \neg\neg B) \Rightarrow B))$$

$$B_3 : (\neg B \Rightarrow \neg B)$$

$$B_4 : ((\neg B \Rightarrow \neg\neg B) \Rightarrow B)$$

$$B_5 : (\neg\neg B \Rightarrow (\neg B \Rightarrow \neg\neg B))$$

$$B_6 : (\neg\neg B \Rightarrow B)$$

## Exercise 1

### Exercise 1

**Complete the proof** presented in **Example 1** by providing **comments** how each step of the proof was obtained.

### ATTENTION

**The solution** presented on the next slide **shows you** how you will have to write details of your solutions on the **TESTS**

**Solutions** of other problems presented later are **less detailed**  
Use them as **exercises** to write a detailed, **complete solutions**

## Exercise 1 Solution

### Solution

The comments that complete the proof are as follows.

$$B_1 : ((\neg B \Rightarrow \neg\neg B) \Rightarrow ((\neg B \Rightarrow \neg B) \Rightarrow B))$$

Axiom A3 for  $A = \neg B, B = B$

$$B_2 : ((\neg B \Rightarrow \neg B) \Rightarrow ((\neg B \Rightarrow \neg\neg B) \Rightarrow B))$$

$B_1$  and **Lemma 1 (b)** for

$A = (\neg B \Rightarrow \neg\neg B), B = (\neg B \Rightarrow \neg B), C = B$ , i.e. we have

$$((\neg B \Rightarrow \neg\neg B) \Rightarrow ((\neg B \Rightarrow \neg B) \Rightarrow B)) \vdash ((\neg B \Rightarrow \neg B) \Rightarrow ((\neg B \Rightarrow \neg\neg B) \Rightarrow B))$$



## Exercise 1 Solution

$$B_3 : (\neg B \Rightarrow \neg B)$$

We proved for  $H_1$  and hence for  $H_2$  that  $\vdash (A \Rightarrow A)$  and we substitute  $A = \neg B$

$$B_4 : ((\neg B \Rightarrow \neg\neg B) \Rightarrow B)$$

$B_2, B_3$  and MP

$$B_5 : (\neg\neg B \Rightarrow (\neg B \Rightarrow \neg\neg B))$$

Axiom A1 for  $A = \neg\neg B, B = \neg B$

$$B_6 : (\neg\neg B \Rightarrow B)$$

$B_4, B_5$  and **Lemma 1 (a)** for

$A = \neg\neg B, B = (\neg B \Rightarrow \neg\neg B), C = B$ ; i.e.

$$(\neg\neg B \Rightarrow (\neg B \Rightarrow \neg\neg B)), ((\neg B \Rightarrow \neg\neg B) \Rightarrow B) \vdash (\neg\neg B \Rightarrow B)$$

## Proofs from Axioms Only

### General remark

**Observe** that in steps  $B_2, B_3, B_5, B_6$  we **call on previously proved facts** and use them as a part of our proof.

We can **obtain** a proof that uses **only axioms** by **inserting** previously constructed formal proofs of these facts into the places occupying by the steps  $B_2, B_3, B_5, B_6$

**For example** in **previously constructed** proof of  $(A \Rightarrow A)$  we **replace**  $A$  by  $\neg B$  and **insert** such constructed proof of  $(\neg B \Rightarrow \neg B)$  after step  $B_2$

The **last step** of the inserted proof becomes now "old" step  $B_3$  and we **re-numerate** all other steps accordingly

## Proofs from Axioms Only

Here are consecutive first THREE steps of the proof of  $(\neg\neg B \Rightarrow B)$

$$B_1 : ((\neg B \Rightarrow \neg\neg B) \Rightarrow ((\neg B \Rightarrow \neg B) \Rightarrow B))$$

$$B_2 : ((\neg B \Rightarrow \neg B) \Rightarrow ((\neg B \Rightarrow \neg\neg B) \Rightarrow B))$$

$$B_3 : (\neg B \Rightarrow \neg B)$$

We **insert** now the proof of  $(\neg B \Rightarrow \neg B)$  after step  $B_2$  and **erase** the  $B_3$

The **last step** of the **inserted proof** becomes the **erased**  $B_3$

## Proofs from Axioms Only

A part of new **transformed** proof is

$$B_1 : ((\neg B \Rightarrow \neg\neg B) \Rightarrow ((\neg B \Rightarrow \neg B) \Rightarrow B)) \quad (\text{Old } B_1)$$

$$B_2 : ((\neg B \Rightarrow \neg B) \Rightarrow ((\neg B \Rightarrow \neg\neg B) \Rightarrow B)) \quad (\text{Old } B_2)$$

We insert here the proof from axioms only of **Old  $B_3$**

$$B_3 : ((\neg B \Rightarrow ((\neg B \Rightarrow \neg B) \Rightarrow \neg B)) \Rightarrow ((\neg B \Rightarrow (\neg B \Rightarrow \neg B)) \Rightarrow (\neg B \Rightarrow \neg B))), \quad (\text{New } B_3)$$

$$B_4 : (\neg B \Rightarrow ((\neg B \Rightarrow \neg B) \Rightarrow \neg B))$$

$$B_5 : ((\neg B \Rightarrow (\neg B \Rightarrow \neg B)) \Rightarrow (\neg B \Rightarrow \neg B))$$

$$B_6 : (\neg B \Rightarrow (\neg B \Rightarrow \neg B))$$

$$B_7 : (\neg B \Rightarrow \neg B) \quad (\text{Old } B_3)$$

## Proofs from Axioms Only

**We repeat** our procedure by **replacing** the step  $B_2$  by its formal proof as defined in **the proof** of the **Lemma 1 (b)**

We **continue the process** for all other steps which involved application of the **Lemma 1** until we get a full **formal proof** from the **axioms** of  $H_2$  only

Usually we **don't do** it and we **don't need** to do it, but it is important to remember that **it always can be done**

## Example 2

### Example 2

Here are consecutive steps

$B_1, B_2, \dots, B_5$

in a proof of  $(B \Rightarrow \neg\neg B)$

$B_1$   $((\neg\neg\neg B \Rightarrow \neg B) \Rightarrow ((\neg\neg\neg B \Rightarrow B) \Rightarrow \neg\neg B))$

$B_2$   $(\neg\neg\neg B \Rightarrow \neg B)$

$B_3$   $((\neg\neg\neg B \Rightarrow B) \Rightarrow \neg\neg B)$

$B_4$   $(B \Rightarrow (\neg\neg\neg B \Rightarrow B))$

$B_5$   $(B \Rightarrow \neg\neg B)$

## Exercise 2

### Exercise 2

**Complete** the proof presented in **Example 2** by providing **detailed comments** how each step of the proof was obtained.

### Solution

**The comments** that complete the proof are as follows.

$$B_1 \quad ((\neg\neg\neg B \Rightarrow \neg B) \Rightarrow ((\neg\neg\neg B \Rightarrow B) \Rightarrow \neg\neg B))$$

Axiom A3 for  $A = B, B = \neg\neg B$

$$B_2 \quad (\neg\neg\neg B \Rightarrow \neg B)$$

Example 1 for  $B = \neg B$

## Exercise 2

$B_3$   $((\neg\neg\neg B \Rightarrow B) \Rightarrow \neg\neg B)$

$B_1, B_2$  and **MP**, i.e.

$$\frac{(\neg\neg B \Rightarrow \neg B); ((\neg\neg B \Rightarrow \neg B) \Rightarrow ((\neg\neg B \Rightarrow B) \Rightarrow \neg\neg B))}{((\neg\neg B \Rightarrow B) \Rightarrow \neg\neg B)}$$

$B_4$   $(B \Rightarrow (\neg\neg\neg B \Rightarrow B))$

Axiom A1 for  $A = B$ ,  $B = \neg\neg\neg B$

$B_5$   $(B \Rightarrow \neg\neg B)$

$B_3, B_4$  and lemma 1a for  $A = B, B = (\neg\neg\neg B \Rightarrow B), C = \neg\neg B$ ,  
i.e.

$$(B \Rightarrow (\neg\neg\neg B \Rightarrow B)), ((\neg\neg\neg B \Rightarrow B) \Rightarrow \neg\neg B) \vdash (B \Rightarrow \neg\neg B)$$



## Example 3

### Example 3

Here are consecutive steps

$B_1, B_2, \dots, B_{12}$  in a proof of  $(\neg A \Rightarrow (A \Rightarrow B))$

$$B_1 \quad \neg A$$

$$B_2 \quad A$$

$$B_3 \quad (A \Rightarrow (\neg B \Rightarrow A))$$

$$B_4 \quad (\neg A \Rightarrow (\neg B \Rightarrow \neg A))$$

$$B_5 \quad (\neg B \Rightarrow A)$$

$$B_6 \quad (\neg B \Rightarrow \neg A)$$

$$B_7 \quad ((\neg B \Rightarrow \neg A) \Rightarrow ((\neg B \Rightarrow A) \Rightarrow B))$$

### Example 3

$$B_8 \quad ((\neg B \Rightarrow A) \Rightarrow B)$$

$$B_9 \quad B$$

$$B_{10} \quad \neg A, A \vdash B$$

$$B_{11} \quad \neg A \vdash (A \Rightarrow B)$$

$$B_{12} \quad (\neg A \Rightarrow (A \Rightarrow B))$$

### Exercise 3

1. **Complete** the proof from the **Example 3** by providing comments how each step of the proof was obtained.
2. **Prove** that

$$\neg A, A \vdash B$$

## Exercise 4

### Example 4

Here are consecutive steps  $B_1, \dots, B_7$   
in a proof of  $((\neg B \Rightarrow \neg A) \Rightarrow (A \Rightarrow B))$

$$B_1 \quad (\neg B \Rightarrow \neg A)$$

$$B_2 \quad ((\neg B \Rightarrow \neg A) \Rightarrow ((\neg B \Rightarrow A) \Rightarrow B))$$

$$B_3 \quad (A \Rightarrow (\neg B \Rightarrow A))$$

$$B_4 \quad ((\neg B \Rightarrow A) \Rightarrow B)$$

$$B_5 \quad (A \Rightarrow B)$$

$$B_6 \quad (\neg B \Rightarrow \neg A) \vdash (A \Rightarrow B)$$

$$B_7 \quad ((\neg B \Rightarrow \neg A) \Rightarrow (A \Rightarrow B))$$

### Exercise 4

**Complete** the proof from **Example 4** by providing comments  
how each step of the proof was obtained

## Example 5

### Example 5

Here are consecutive steps  $B_1, \dots, B_9$   
in a proof of  $((A \Rightarrow B) \Rightarrow (\neg B \Rightarrow \neg A))$

$$B_1 \quad (A \Rightarrow B)$$

$$B_2 \quad (\neg\neg A \Rightarrow A)$$

$$B_3 \quad (\neg\neg A \Rightarrow B)$$

$$B_4 \quad (B \Rightarrow \neg\neg B)$$

$$B_5 \quad (\neg\neg A \Rightarrow \neg\neg B)$$

$$B_6 \quad ((\neg\neg A \Rightarrow \neg\neg B) \Rightarrow (\neg B \Rightarrow \neg A))$$

$$B_7 \quad (\neg B \Rightarrow \neg A)$$

$$B_8 \quad (A \Rightarrow B) \vdash (\neg B \Rightarrow \neg A)$$

$$B_9 \quad ((A \Rightarrow B) \Rightarrow (\neg B \Rightarrow \neg A))$$

## Exercise 5

### Exercise 5

**Complete** the proof of **Example 5** by providing comments how each step of the proof was obtained.

### Solution

$$B_1 \quad (A \Rightarrow B)$$

Hypothesis

$$B_2 \quad (\neg\neg A \Rightarrow A)$$

Example 1 for  $B = A$

$$B_3 \quad (\neg\neg A \Rightarrow B)$$

Lemma 1 a for  $A = \neg\neg A, B = A, C = B$

$$B_4 \quad (B \Rightarrow \neg\neg B)$$

Example 2

## Exercise 5

$B_5$  ( $\neg\neg A \Rightarrow \neg\neg B$ )

Lemma 1 a for  $A = \neg\neg A, B = B, C = \neg\neg B$

$B_6$  ( $((\neg\neg A \Rightarrow \neg\neg B) \Rightarrow (\neg B \Rightarrow \neg A))$ )

Example 4 for  $B = \neg A, A = \neg B$

$B_7$  ( $\neg B \Rightarrow \neg A$ )

$B_5, B_6$  and **MP**

$B_8$  ( $(A \Rightarrow B) \vdash (\neg B \Rightarrow \neg A)$ )

$B_1 - B_7$

$B_9$  ( $((A \Rightarrow B) \Rightarrow (\neg B \Rightarrow \neg A))$ )

Deduction Theorem

## Example 6

### Example 6

**Prove** that  $\vdash (A \Rightarrow (\neg B \Rightarrow (\neg(A \Rightarrow B))))$

**Solution** Here are consecutive steps of building the formal proof.

$B_1$   $A, (A \Rightarrow B) \vdash B$

by MP

$B_2$   $A \vdash ((A \Rightarrow B) \Rightarrow B)$

Deduction Theorem

$B_3$   $\vdash (A \Rightarrow ((A \Rightarrow B) \Rightarrow B))$

Deduction Theorem

$B_4$   $\vdash (((A \Rightarrow B) \Rightarrow B) \Rightarrow (\neg B \Rightarrow \neg(A \Rightarrow B)))$

Example 5 for  $A = (A \Rightarrow B), B = B$

$B_5$   $\vdash (A \Rightarrow (\neg B \Rightarrow (\neg(A \Rightarrow B))))$

$B_3$  and  $B_4$  and lemma 2a for

$A = A, B = ((A \Rightarrow B) \Rightarrow B), C = (\neg B \Rightarrow (\neg(A \Rightarrow B)))$

## Example 7

### Example 7

Here are consecutive steps  $B_1, \dots, B_{12}$

in a proof of  $((A \Rightarrow B) \Rightarrow ((\neg A \Rightarrow B) \Rightarrow B))$

$$B_1 \quad (A \Rightarrow B)$$

$$B_2 \quad (\neg A \Rightarrow B)$$

$$B_3 \quad ((A \Rightarrow B) \Rightarrow (\neg B \Rightarrow \neg A))$$

$$B_4 \quad (\neg B \Rightarrow \neg A)$$

$$B_5 \quad ((\neg A \Rightarrow B) \Rightarrow (\neg B \Rightarrow \neg\neg A))$$

$$B_6 \quad (\neg B \Rightarrow \neg\neg A)$$

$$B_7 \quad ((\neg B \Rightarrow \neg\neg A) \Rightarrow ((\neg B \Rightarrow \neg A) \Rightarrow B)))$$



## Example 7

$$B_8 \quad ((\neg B \Rightarrow \neg A) \Rightarrow B)$$

$$B_9 \quad B$$

$$B_{10} \quad (A \Rightarrow B), (\neg A \Rightarrow B) \vdash B$$

$$B_{11} \quad (A \Rightarrow B) \vdash ((\neg A \Rightarrow B) \Rightarrow B)$$

$$B_{12} \quad ((A \Rightarrow B) \Rightarrow ((\neg A \Rightarrow B) \Rightarrow B))$$

### Exercise 7

**Complete** the proof in **Example 7** by providing comments how each step of the proof was obtained.

## Exercise 7

### Exercise 7

#### Solution

$$B_1 \quad (A \Rightarrow B)$$

Hypothesis

$$B_2 \quad (\neg A \Rightarrow B)$$

Hypothesis

$$B_3 \quad ((A \Rightarrow B) \Rightarrow (\neg B \Rightarrow \neg A))$$

Example 5

$$B_4 \quad (\neg B \Rightarrow \neg A)$$

$B_1, B_3$  and MP

$$B_5 \quad ((\neg A \Rightarrow B) \Rightarrow (\neg B \Rightarrow \neg\neg A))$$

Example 5 for  $A = \neg A, B = B$

$$B_6 \quad (\neg B \Rightarrow \neg\neg A)$$

$B_2, B_5$  and MP

## Exercise 7

$$B_7 \quad ((\neg B \Rightarrow \neg\neg A) \Rightarrow ((\neg B \Rightarrow \neg A) \Rightarrow B)))$$

Axiom A3 for  $B = B, A = \neg A$

$$B_8 \quad ((\neg B \Rightarrow \neg A) \Rightarrow B)$$

$B_6, B_7$  and MP

$$B_9 \quad B$$

$B_4, B_8$  and MP

$$B_{10} \quad (A \Rightarrow B), (\neg A \Rightarrow B) \vdash B$$

$B_1 - B_9$

$$B_{11} \quad (A \Rightarrow B) \vdash ((\neg A \Rightarrow B) \Rightarrow B)$$

Deduction Theorem

$$B_{12} \quad ((A \Rightarrow B) \Rightarrow ((\neg A \Rightarrow B) \Rightarrow B))$$

Deduction Theorem

## Exercise 8

### Example 8

Here are consecutive steps  $B_1, \dots, B_3$   
in a proof of  $((\neg A \Rightarrow A) \Rightarrow A)$

$$B_1 \quad ((\neg A \Rightarrow \neg A) \Rightarrow ((\neg A \Rightarrow A) \Rightarrow A)))$$

$$B_1 \quad (\neg A \Rightarrow \neg A)$$

$$B_1 \quad ((\neg A \Rightarrow A) \Rightarrow A))$$

### Exercise 8

Complete the proof of example 8 by providing comments how each step of the proof was obtained.

### Solution

$$B_1 \quad ((\neg A \Rightarrow \neg A) \Rightarrow ((\neg A \Rightarrow A) \Rightarrow A)))$$

Axiom A3 for  $B = A$

$$B_1 \quad (\neg A \Rightarrow \neg A)$$

Already proved  $(A \Rightarrow A)$  for  $A = \neg A$

$$B_1 \quad ((\neg A \Rightarrow A) \Rightarrow A))$$

$B_1, B_2$  and MP

## LEMMA

We summarize all the formal proofs in  $H_2$  provided in our Examples and Exercises in a form of a following Lemma

### Lemma

The following formulas are **provable** in  $H_2$

1.  $(A \Rightarrow A)$
2.  $(\neg\neg B \Rightarrow B)$
3.  $(B \Rightarrow \neg\neg B)$
4.  $(\neg A \Rightarrow (A \Rightarrow B))$
5.  $((\neg B \Rightarrow \neg A) \Rightarrow (A \Rightarrow B))$
6.  $((A \Rightarrow B) \Rightarrow (\neg B \Rightarrow \neg A))$
7.  $(A \Rightarrow (\neg B \Rightarrow (\neg(A \Rightarrow B))))$
8.  $((A \Rightarrow B) \Rightarrow ((\neg A \Rightarrow B) \Rightarrow B))$
9.  $((\neg A \Rightarrow A) \Rightarrow A)$

## Proof of Completeness Theorem

**Formulas 1, 3, 4, and 7-9** from the set of **provable formulas** from the **Lemma** are all formulas **we need** together with  $H_2$  axioms to **execute two proofs** of the **Completeness Theorem** for  $H_2$

We present these proofs in **Lecture 5a** and **Lecture 5b**  
The two proofs represent two different **methods of proving** the **Completeness Theorem**