

cse541
LOGIC for Computer Science

Professor Anita Wasilewska

LECTURE 4a

Chapter 4 Review

PART 1: DEFINITIONS

PART 2: Problems

PART 1: Some Definitions from Chapter 4

Proof System

Definition 1

By a **proof system** we understand a quadruple

$$S = (\mathcal{L}, \mathcal{E}, LA, \mathcal{R})$$

where

$\mathcal{L} = \{\mathcal{A}, \mathcal{F}\}$ is a **language** of S with a set \mathcal{F} of formulas

\mathcal{E} is a set of **expressions** of S

In particular case $\mathcal{E} = \mathcal{F}$

$LA \subseteq \mathcal{E}$ is a **non-empty, finite** set of **logical axioms** of S

\mathcal{R} is a **non-empty, finite set** of **rules of inference** of S

Sound Rule of Inference

Definition 2

An inference rule

$$(r) \quad \frac{P_1 ; P_2 ; \dots ; P_m}{C}$$

is **sound** under a semantics **M** if and only if all **M - models** of the set $\{P_1, P_2, \dots, P_m\}$ of its **premisses** are also **M - models** of its **conclusion C**

In case of **propositional** semantics it means that for any truth assignment $v : VAR \rightarrow LV$, the following condition holds

If $v \models_M \{P_1, P_2, \dots, P_m\}$, **then** $v \models_M C$

STRONGLY Sound Rule of Inference

Definition 3

An inference rule

$$(r) \quad \frac{P_1 ; P_2 ; \dots ; P_m}{C}$$

is **STRONGLY sound** under a semantics **M** when

M is a model of the set $\{P_1, P_2, \dots, P_m\}$

if and only if **M** is a model of **C**

In case of **propositional** semantics it means that for any truth assignment $v : VAR \rightarrow LV$,

$$v \models_{\mathbf{M}} \{P_1, P_2, \dots, P_m\}, \quad \text{if and only if} \quad v \models_{\mathbf{M}} C$$

Direct Consequence

Definition 4

For any rule of inference $r \in \mathcal{R}$ of the form

$$(r) \quad \frac{P_1 ; P_2 ; \dots ; P_m}{C}$$

C is called a **direct consequence** of P_1, \dots, P_m by virtue of the rule $r \in \mathcal{R}$

Formal Proof

Definition 5

A **formal proof** of an expression $E \in \mathcal{E}$ in a proof system $S = (\mathcal{L}, \mathcal{E}, LA, \mathcal{R})$ is a sequence

$$A_1, A_2, \dots, A_n \text{ for } n \geq 1$$

of expressions from \mathcal{E} , such that

$$A_1 \in LA, \quad A_n = E$$

and for each $1 < i \leq n$, either $A_i \in LA$ or A_i is a **direct consequence** of some of the **preceding expressions** by virtue of **one** of the **rules of inference**

$n \geq 1$ is the **length** of the proof A_1, A_2, \dots, A_n

Provable Expressions

We write

$$\vdash_S E$$

to denote that the expression $E \in \mathcal{E}$ has a **formal proof** in the proof system S

A set

$$\mathbf{P}_S = \{E \in \mathcal{E} : \vdash_S E\}$$

is called the set of **all provable expressions** in S

Definition: Sound S

Definition 6

Given a proof system

$$S = (\mathcal{L}, \mathcal{E}, LA, \mathcal{R})$$

We say that the system S is **sound** under a semantics \mathbf{M} if and only if the following conditions hold

1. Logical axioms LA are **tautologies** of under the semantics \mathbf{M} , i.e.

$$LA \subseteq T_{\mathbf{M}}$$

2. Each **rule of inference** $r \in \mathcal{R}$ is **sound** under the semantics \mathbf{M}

Definition: STRONGLY Sound S

Definition 7

Given a proof system

$$S = (\mathcal{L}, \mathcal{E}, LA, \mathcal{R})$$

We say that the system **S** is **STRONGLY sound** under a semantics **M**

if and only if the following conditions hold

1. Logical axioms **LA** are **tautologies** of under the semantics **M**, i.e.

$$LA \subseteq T_M$$

2. Each **rule of inference** $r \in \mathcal{R}$ is **STRONGLY sound** under the semantics **M**

THEOREMS: Soundness Theorem

Let \mathbf{P}_S be the set of all provable expressions of S i.e.

$$\mathbf{P}_S = \{A \in \mathcal{E} : \vdash_S A\}$$

Let \mathbf{T}_M be a set of all expressions of S that are **tautologies** under a semantics \mathbf{M} , i.e.

$$\mathbf{T}_M = \{A \in \mathcal{E} : \models_M A\}$$

Our GOAL is to prove the following theorems:

Soundness Theorem (for S and semantics \mathbf{M})

$$\mathbf{P}_S \subseteq \mathbf{T}_M$$

i.e. for any $A \in \mathcal{E}$, the following implication holds

$$\text{If } \vdash_S A \text{ then } \models_M A$$

THEOREMS: Completeness Theorem

Completeness Theorem (for \mathcal{S} and semantics \mathcal{M})

$$\mathbf{P_S = T_M}$$

i.e. for any $A \in \mathcal{E}$, the following holds

$$\vdash_{\mathcal{S}} A \quad \text{if and only if} \quad \models_{\mathcal{M}} A$$

The **Completeness Theorem** consists of two parts:

Part 1: Soundness Theorem

$$\mathbf{P_S \subseteq T_M}$$

Part 2: Completeness Part of the Completeness Theorem

$$\mathbf{T_M \subseteq P_S}$$

PART 2: Simple Problems

Formal Proofs

Given a proof system:

$$S = (\mathcal{L}_{\{\neg, \Rightarrow\}}, \mathcal{F}, \{(A \Rightarrow A), (A \Rightarrow (\neg A \Rightarrow B))\}, \mathcal{R} = \{(r)\})$$

$$\text{where } (r) \frac{(A \Rightarrow B)}{(B \Rightarrow (A \Rightarrow B))}$$

Write a **formal proof** in S with 2 applications of the rule (r)

Solution: There are many solutions. Here is one of them.

Required formal proof is a sequence A_1, A_2, A_3 , where

$$A_1 = (A \Rightarrow A)$$

(Axiom)

$$A_2 = (A \Rightarrow (A \Rightarrow A))$$

Rule (r) application 1 for $A = A, B = A$

$$A_3 = ((A \Rightarrow A) \Rightarrow (A \Rightarrow (A \Rightarrow A)))$$

Rule (r) application 2 for $A = A, B = (A \Rightarrow A)$

Soundness

Given a proof system:

$$S = (\mathcal{L}_{\{\neg, \Rightarrow\}}, \mathcal{F}, \{(A \Rightarrow A), (A \Rightarrow (\neg A \Rightarrow B))\}, (r) \frac{(A \Rightarrow B)}{(B \Rightarrow (A \Rightarrow B))})$$

Prove that S is **sound** under classical semantics.

Solution

1. Both axioms of S are basic classical tautologies
2. Consider the rule of inference of S

$$(r) \frac{(A \Rightarrow B)}{(B \Rightarrow (A \Rightarrow B))}$$

Assume that its premise (the only premise) is true, i.e. let v be any truth assignment, such that $v^*(A \Rightarrow B) = T$

We evaluate logical value of the conclusion under the truth assignment v as follows

$$v^*(B \Rightarrow (A \Rightarrow B)) = v^*(B) \Rightarrow T = T$$

for any B and any value of $v^*(B)$

Formal Proof

Given a proof system:

$$S = (\mathcal{L}_{\{\neg, \Rightarrow\}}, \mathcal{F}, \{(A \Rightarrow A), (A \Rightarrow (\neg A \Rightarrow B))\}, (r) \frac{(A \Rightarrow B)}{(B \Rightarrow (A \Rightarrow B))})$$

Write a **formal proof** of your choice in S with 2 applications of the rule (r)

Solution

There many of such proofs, of different length, with different choice if axioms - here is my choice: A_1, A_2, A_3 , where

$$A_1 = (A \Rightarrow A)$$

(Axiom)

$$A_2 = (A \Rightarrow (A \Rightarrow A))$$

Rule (r) application 1 for $A = A, B = A$

$$A_3 = ((A \Rightarrow A) \Rightarrow (A \Rightarrow (A \Rightarrow A)))$$

Rule (r) application 2 for $A = A, B = (A \Rightarrow A)$

Formal Proof

Given a proof system:

$$S = (\mathcal{L}_{\{\neg, \Rightarrow\}}, \mathcal{F}, \{(A \Rightarrow A), (A \Rightarrow (\neg A \Rightarrow B))\}, (r) \frac{(A \Rightarrow B)}{(B \Rightarrow (A \Rightarrow B))})$$

1. Prove, by constructing a **formal proof** that

$$\vdash_S ((\neg A \Rightarrow B) \Rightarrow (A \Rightarrow (\neg A \Rightarrow B)))$$

Solution Required formal proof is a sequence $A_1, A_2,$
where

$$A_1 = (A \Rightarrow (\neg A \Rightarrow B))$$

Axiom

$$A_2 = ((\neg A \Rightarrow B) \Rightarrow (A \Rightarrow (\neg A \Rightarrow B)))$$

Rule (r) application for $A = A, B = (\neg A \Rightarrow B)$

Soundness Theorem

2. Does above point 1. prove that

$$\models ((\neg A \Rightarrow B) \Rightarrow (A \Rightarrow (\neg A \Rightarrow B)))?$$

Solution

Yes, it does because the system **S** is **sound** and we proved by Mathematical Induction over the length of a proof that if **S** is **sound**, then the **Soundness Theorem** holds for **S**

Soundness

Given a proof system:

$$S = (\mathcal{L}_{\{\neg, \Rightarrow\}}, \mathcal{F}, \{(A \Rightarrow A), (A \Rightarrow (\neg A \Rightarrow B))\}, (r) \frac{(A \Rightarrow B)}{(B \Rightarrow (A \Rightarrow B))})$$

Prove that **S** is **not sound** under **K** semantics

Solution

Axiom $(A \Rightarrow A)$ is not a **K** semantics tautology

Any truth assignment v such that $v^*(A) = \perp$ is a **counter-model** for it

This proves that **S** is **not sound** under **K** semantics

Soundness

Given a proof system

$$S = (\mathcal{L}_{\{\Rightarrow, \cup, \neg\}}, \mathcal{F}, \mathbf{A1}, (r1), (r2))$$

A1 $(A \Rightarrow (A \cup B))$, for any $A, B \in \mathcal{F}$

Rules of inference

$$(r1) \frac{A ; B}{(A \cup \neg B)}, \quad (r2) \frac{A ; (A \cup B)}{B},$$

for any formulas $A, B \in \mathcal{F}$

1. Verify whether S is **sound/not sound** under **classical** semantics.

Soundness

Rules of inference

$$(r1) \frac{A ; B}{(A \cup \neg B)}, \quad (r2) \frac{A ; (A \cup B)}{B}$$

The Logical Axiom **A1** is a **basic tautology**

The rule (r1) is **sound** because for any v ,

if $v^*(A) = v^*(B) = T$, then $v^*((A \cup \neg B)) = T \cup F = T$

The rule (r2) is **not sound**

Take any v such that it evaluates $A = T$ and $B = F$

The premiss $(A \cup B)$ of the rule (r2) is $T \cup F = T$ and the conclusion B is F

This proves that the system **S** is **not sound**

Formal Proof

Rules of inference

$$(r1) \frac{A ; B}{(A \cup \neg B)}, \quad (r2) \frac{A ; (A \cup B)}{B}$$

Write down a **formal proof** of the formula

$$\neg(a \Rightarrow (a \cup b))$$

in the **proof system** S , i.e. write all components

$$B_1, B_2, \dots B_n$$

of the proof with **comments** how they were obtained

Formal Proof

Rules of inference

$$(r1) \frac{A ; B}{(A \cup B)}, \quad (r2) \frac{A ; (A \cup B)}{B}$$

Axiom $(A \Rightarrow (A \cup B))$

Formal Proof of $\neg(a \Rightarrow (a \cup b))$ is:

B_1 : $(a \Rightarrow (a \cup b))$ Axiom or $A = a, B = b$

B_2 : $(a \Rightarrow (a \cup b))$ Axiom for $A = a, B = b$

B_3 : $((a \Rightarrow (a \cup b)) \cup \neg(a \Rightarrow (a \cup b)))$

rule (r1) application to B_1 and B_2 for

$A = (a \Rightarrow (a \cup b)), B = (a \Rightarrow (a \cup b))$

B_4 : $\neg(a \Rightarrow (a \cup b))$ rule (r2) application to B_1 and B_3 for

$A = (a \Rightarrow (a \cup b)), B = \neg(a \Rightarrow (a \cup b))$

Soundness

We proved that

$$\vdash_S \neg(a \Rightarrow (a \cup b))$$

Does the above prove that

$$\models \neg(a \Rightarrow (a \cup b))?$$

No, it doesn't

We proved that the proof system **S** is **not sound**, so the existence of a proof does not guarantee that what we proved is a tautology.

Moreover, the proof of $\neg(a \Rightarrow (a \cup b))$ used rule (r2) that is **not sound**

STRONG Soundness

Given a proof system

$$S = (\mathcal{L}_{\{\Rightarrow, \cup, \neg\}}, \mathcal{F}, \mathbf{A1}, (r1), (r2))$$

A1 $(A \Rightarrow (A \cup B))$, for any $A, B \in \mathcal{F}$

Rules of inference

$$(r1) \frac{\neg A ; B}{(A \Rightarrow B)}, \quad (r2) \frac{A ; \neg B}{\neg(A \Rightarrow B)},$$

for any formulas $A, B \in \mathcal{F}$

1. Prove that **S** is **sound** but not **strongly sound** under classical semantics.

2. Prove that $\mathbf{P}_S = \{(A \Rightarrow (A \cup B)) : A, B \in \mathcal{F}\}$