QUESTION 1  (5pts)

Write the following natural language statement:

_It is possible that one likes when it rains and from the fact that it is necessary to buy a raincoat, we conclude the following: one does not like when it rains or one likes when it is not necessary to buy a raincoat_ in the following two ways.

1. (3pts) Formula \( A_1 \in F_1 \) of a language \( L_{\neg \cup \cap \Rightarrow} \) where \( L_A \) represents statement "one likes A", and \( \Box, \Diamond \) are modal connectives of necessity, possibility, respectively.

Solution  We translate our statement into a formula \( A_1 \in F_1 \) of a language \( L_{\neg \cup \cap \Rightarrow} \) as follows.

**Propositional Variables:** \( a, b \)

\( a \) denotes statement: _when it rains_,

\( b \) denotes a statement: _buy a raincoat_

**Translation 1**

\[
A_1 = (\Diamond L_a \cap (\Box b \Rightarrow (\neg L_a \cup L_{\neg \Box b})))
\]

2. Formula \( A_2 \in F_2 \) of a language \( L_{\neg \cup \Rightarrow} \).

Solution  We translate our statement into a formula \( A_2 \in F_2 \) of a language \( L_{\neg \cup \Rightarrow} \) as follows.

**Propositional Variables:** \( a, b, c, d \)

\( a \) denotes statement: _It is possible that one likes when it rains_,

\( b \) denotes a statement: _it is necessary to buy a raincoat_,

\( c \) denotes a statement: _one likes when it rains_

\( d \) denotes a statement: _one likes when it is not necessary to buy a raincoat_

**Translation 2:**

\[
A_2 = (a \cup (b \Rightarrow (c \cup d)))
\]

QUESTION 2  (5pts)

Here is a mathematical statement \( S \):

_For all real numbers \( x \in R \) the following holds: If \( x < 0 \), then there is a rational number \( q \in Q \), such that \( x + q < 0 \)_.

1. (2pts) Re-write \( S \) as a symbolic mathematical statement \( SM \) that only uses mathematical and logical symbols.

Solution  \( S \) becomes a symbolic mathematical statement

\[
SM : \quad \forall x \in \mathbb{R} (x < 0 \Rightarrow \exists q \in \mathbb{Q} \quad x + q < 0)
\]

2. (2pts) Translate the symbolic statement \( SM \) into a corresponding formula with restricted quantifiers. Explain your choice of symbols.
Solution  We write R(x) for \( x \in R \), Q(y) for \( y \in Q \), a constant c for the number 0. We use \( L \in P \) to denote the relation <, we use f \( \in F \) to denote the function +.

The statement \( x < 0 \) becomes an atomic formula \( L(x, c) \). The statement \( x + q < 0 \) becomes an atomic formula \( L(f(x,y), c) \).

The symbolic mathematical statement \( SM \) becomes a restricted quantifiers formula

\[ \forall_{R(x)}(L(x, c) \Rightarrow \exists_{Q(y)}L(f(x,y), c)) \]

3. (1pts) Translate your restricted domain quantifiers formula into a correct formula \( A \) of the predicate language \( L \)

**Solution**  We apply now the transformation rules and get a corresponding formula \( A \in F \):

\[ \forall x(R(x) \Rightarrow (L(x, c) \Rightarrow \exists y(Q(y) \cap L(f(x,y), c))) \]

QUESTION 3 (5pts)

Given a predicate language \( L(P, F, C) \) and a structure \( M = [U, I] \) such that

\( U = Z \) and \( P_I: =, g_I: +, a_I: 0 \), where \( Z \) is the set of integers.

For the following formula A

\[ \forall x \exists y(P(g(x, y), a) \Rightarrow (P(x, a) \cap P(y, a))) \]

decide whether \( M \models A \) or not.

Do so by examining the corresponding mathematical statement defined by \( M \).

**Solution**  \( M \not\models A \) because the corresponding mathematical statement defined by \( M \) (written with logical symbols) is

\[ \forall_{n \in Z} \exists_{m \in Z} (n + m = 0 \Rightarrow (n = 0 \cap m = 0)) \]

Observe that

\[ \forall_{n \in Z} \exists_{m \in Z} (n + m = 0 \Rightarrow (n = 0 \cap m = 0)) \equiv \forall_{n \in Z} \exists_{m \in Z} (n + m \neq 0 \cup (n = 0 \cap m = 0)) \]

Consider

\[ \forall_{n \in Z} \exists_{m \in Z} (n + m \neq 0 \cup (n = 0 \cap m = 0)) \]

For a any \( n \in Z \) there is integer \( m = n + 1 \), such that \( (n + m = n + n + 1 = 2n + 1 \neq 0) \)

We rewrite our statement as

\[ \forall_{n \in Z} \exists_{m}(m = n + 1) \cap (n + m \neq 0) \cup (n = 0 \cap m = 0)) \]

By distributivity of conjunction over disjunction we get

\[ ((m = n + 1) \cap (n + m \neq 0) \cup (n = 0 \cap m = 0)) \equiv ((m = n + 1) \cap (n + m \neq 0)) \cup (m = n + 1) \cap (n = 0 \cap m = 0)) \]

Substituting \( n + 1 \) for \( m \) we get equivalent statement

\[ (2n + 1 \neq 0) \cup (n = 0 \cap n + 1 = 0) \]

that is TRUE for ALL \( n \in Z \), as \( 2n + 1 \neq 0 \) is TRUE for ALL \( n \in Z \)
Hence we proved that \( M \models A \)

**QUESTION 4 (10pts)**

We define a 3 valued extensional semantics \( M \) for the language \( \mathcal{L}_{\{\neg, \cup, \Rightarrow\}} \) by **defining the connectives** \( \neg, \cup, \Rightarrow \) on a set \( \{F, \bot, T\} \) of logical values by the following truth tables.

**I. Connective**

**Negation** :

<table>
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<th>( \mathcal{L} )</th>
<th>F</th>
<th>( \bot )</th>
<th>T</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \neg )</td>
<td>F</td>
<td>( \bot )</td>
<td>T</td>
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</tbody>
</table>

**Implication**

<table>
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<th>( \Rightarrow )</th>
<th>F</th>
<th>( \bot )</th>
<th>T</th>
</tr>
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<tbody>
<tr>
<td>F</td>
<td>T</td>
<td>T</td>
<td>T</td>
</tr>
<tr>
<td>( \bot )</td>
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<td>( \bot )</td>
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<tr>
<td>T</td>
<td>F</td>
<td>F</td>
<td>T</td>
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</tbody>
</table>

**Disjunction**

<table>
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<th>F</th>
<th>( \bot )</th>
<th>T</th>
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<tr>
<td>F</td>
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<td>( \bot )</td>
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1. (5pts) Verify whether formulas

\[ A_1 = (La \cup (b \Rightarrow (\neg La \cup L\neg a))) \quad \text{and} \quad A_2 = (a \cup (b \Rightarrow (\neg a \cup c))) \]

have a model/ counter model under the semantics \( M \). You can use **shorthand notation**. You must show your evaluation.

**Solution**

Any \( \nu \), such that \( \nu(a) = T \) is a \( M \) model for \( A_1 \) and for \( A_2 \). Students must show evaluation.

2. (5pts) Verify whether the following set \( G \) is \( M \)-consistent. You can use **shorthand notation**

\[ G = \{ La, (a \cup \neg Lb), (a \Rightarrow b), b \} \]

**Solution**

Any \( \nu \), such that \( \nu(a) = T, \nu(b) = T \) is a \( M \) model for \( G \) as

\[ LT = T, \quad (T \cup \neg LT) = T, \quad (T \Rightarrow T) = T, \quad b = T \]