CSE541 EXAMPLE 1: MIDTERM SOLUTIONS
(75pts)

PART 1: DEFINITIONS TOTAL **10pts**

DEF 1 Given a propositional language $\mathcal{L}_{CON}$ for $CON = C_1 \cup C_2$, where C_1 is the set of all unary connectives, and C_2 is the set of all binary connectives

1. (1pts) Write the **definition** of the set $\mathcal{F}$ of **all formulas** of $\mathcal{L}_{CON}$ for $C_1 = \{\neg, K\}$ and $C_2 = \{\cup\}$

Solution

$\mathcal{F} \subseteq \mathcal{A}^*$ and $\mathcal{F}$ is the **smallest** set for which the following conditions are satisfied

(1) $VAR \subseteq \mathcal{F}$ - ATOMIC FORMULAS

(2) If $A \in \mathcal{F}$, then $\neg A \in \mathcal{F}$ and $KA \in \mathcal{F}$

(3) If $A, B \in \mathcal{F}$, then $(A \cup B) \in \mathcal{F}$

2. (1pts) Write an **example** of 4 formulas, each of a different degree, of the language $\mathcal{L}_{\{\neg, K, \cup\}}$

Solution

Here are, for example, formulas of the degree 0, 1, 2, 3, respectively

$a, \quad \neg a, \quad K\neg a, \quad K(a \cup \neg b)$

DEF 2 Given the language $\mathcal{L}_{\{\neg, K, \cup\}}$ and a **M** truth assignment $v : VAR \longrightarrow LV$, where $LV \neq \emptyset$ is the set of logical values on the **extensional** semantics **M**. Let $T \in LV$ be its **distinguished** logical value.

1. (1pts) We say that a function v^* is the **M extension** of v to the set $\mathcal{F}$ of the language $\mathcal{L}_{\{\neg, K, \cup\}}$ if and only if the following conditions hold.

Solution

(i) for any $a \in VAR$, $v^*(a) = v(a)$; and

(ii) for any formulas $A, B \in \mathcal{F}$,

$$v^*(\neg A) = \neg v^*(A), \quad v^*(KA) = Kv^*(A), \quad \text{and} \quad v^*((A \cup B)) = \cup(v^*(A), v^*(B)$$

We also use standard notation $v^*((A \cup B)) = v^*(A) \cup v^*(B)$

2. (1pts) We say that $\models_{\mathbf{M}} A$ if and only if

Solution

$v^*(A) = T$ for all truth assignments $v : VAR \longrightarrow LV$

DEF 3 Given a language $\mathcal{L}_{\{\neg, \Rightarrow, \cup, \cap\}}$ and its **extensional** semantics **M**

1. (2pts) A formula $A \in \mathcal{F}$ is called **M independent** from a set $\mathcal{G} \subseteq \mathcal{F}$ if and only if

Solution

the sets $\mathcal{G} \cup \{A\}$ and $\mathcal{G} \cup \{\neg A\}$ are both **M** consistent.

I.e. when there are truth assignments v_1, v_2 such that $v_1 \models_{\mathbf{M}} \mathcal{G} \cup \{A\}$ and $v_2 \models_{\mathbf{M}} \mathcal{G} \cup \{\neg A\}$.

2. (2pts) Give and example of a set $\mathcal{G} \subseteq \mathcal{F}$ and a formula $A \in \mathcal{F}$ that is **classically independent** from $\mathcal{G}$

Solution

Here is a very simple example: $\mathcal{G} = \{a\}$ and $A = b$

Let $v_1,\ v_2$ be any truth assignments such that $v_1(a) = T,\ v_1(b) = T$ and $v_2(a) = T,\ v_2(b) = F$

Obviously, $\mathcal{G} \cup \{A\} = \{a, b\}$ and $\mathcal{G} \cup \{\neg A\} = \{a, \neg b\}$

$$v_1 \models \{a,\ b\} \quad \text{and} \quad v_2 \models \{a,\ \neg b\}$$

DEF 4 (2pts) Given a proof system $S = (\mathcal{L}_{\{\neg, \cup\}},\ \mathcal{F},\ LA,\ \mathcal{R})$. We write $\mathbf{P}_S = \{A \in \mathcal{F} :\ \vdash_S A\}$

The proof system S is **complete** under a semantics $\mathbf{M}$ if and only if the following condition holds.

Solution

$$\mathbf{P}_S = \mathbf{T}_M \quad \text{for} \quad \mathbf{T}_M = \{A \in \mathcal{F} :\ \models_{\mathbf{M}} A\}$$

PART 2: PROBLEMS (65 pts)

QUESTION 1 (25pts) By a m-valued semantics S_m, for a propositional language $\mathcal{L} = \mathcal{L}_{\{\neg, \cap, \cup, \Rightarrow\}}$ we understand any definition of connectives $\neg, \cap, \cup, \Rightarrow$ as operations on a set $L_m = \{l_1, l_2, ...l_m\}$ of logical values (for $m \geq 2$).

We assume that $l_1 \leq l_2 \leq ... \leq l_m$, i.e. the set $L_m = \{l_1 \leq l_2 \leq ... \leq l_m\}$ is totally ordered by a certain relation $\leq$ with l_1, l_m being smallest and greatest elements, respectively. We denote $l_1 = F,\ l_m = T$ and call them (total) False and Truth, respectively. For example, when $m = 2$, $L_2 = \{F, T\}$, $F \leq T$. Semantics S_2 is called a classical semantics if the connectives are defined as $x \cup y = max\{x, y\}$, $x \cap y = min\{x, y\}$, $\neg T = F, \neg F = T$, and $x \Rightarrow y = \neg x \cup y$, for any $x, y \in L_2$.

Let VAR be a set of propositional variables of $\mathcal{L}$ and let S_m be any m-valued semantics for $\mathcal{L}$. A truth assignment $v : VAR \longrightarrow L_m$ is called a S_m model for a formula A of $\mathcal{L}$ if and only if $v^*(A) = T$ and logical value $v^*(A)$ is evaluated accordingly to the semantics S_m. We denote is symbolically as $v \models_{S_m} A$.

1. Let S_3 be a 3-valued semantics for $\mathcal{L}_{\{\neg, \cap, \cup, \Rightarrow\}}$ defined as follows.

$L_3 = \{F, U, T\}$, for $F \leq U \leq T$, and for any $x,\ y \in L_3$ we put $x \cap y = min\{x, y\}$, $x \Rightarrow y = \neg x \cup y$, where

∪	F	U	T
F	F	U	T
U	U	U	U
T	T	U	T

¬	F	U	T
	T	*F*	U

Consider the following classical tautologies: $A_1 = (A \cup \neg A)$, $A_2 = (A \Rightarrow (B \Rightarrow A))$

(a) (5pts) Find S_3 counter-models for A_1, A_2, if exist. Use shorthand notation.

Solution

Any v such that $v^*(A) = v^*(B) = U$ is a counter-model for both, A_1 and A_2.

(b) (5pts) Define a 2-valued semantics C_2 for $\mathcal{L}$, such that **none of** A_1, A_2 is a S_2 tautology. Verify your results. Use shorthand notation.

Solution

This is not the only solution, but it is the simplest and most obvious. Here it is.

We define $\neg x = F$, $x \Rightarrow y = F$ for all $x, y \in \{F, T\}$, and $x \cap y$, $x \cup y$ can be defined in the same, or another way, as these connectives do not appear in our formulas.

2. Let $S = (\mathcal{L}_{\{\neg,\Rightarrow\}}, \mathcal{F}, \mathbf{A1}, \mathbf{A2}, \mathbf{A3}, MP)$ be a proof system with axioms:

A1 $(A \Rightarrow (B \Rightarrow A))$,

A2 $((A \Rightarrow (B \Rightarrow C)) \Rightarrow ((A \Rightarrow B) \Rightarrow (A \Rightarrow C)))$,

A3 $((\neg B \Rightarrow \neg A) \Rightarrow ((\neg B \Rightarrow A) \Rightarrow B))$,

(a) (5pts) The system S is **complete** with respect to classical semantics.

Verify whether S is complete with respect to 3-valued semantics S_3, defined in **1.**

Solution

(This is a kind of a "free points" problem.

System S is **not complete** because it is **not sound**, as we have shown in **1.**(a)

(b) (5pts) Define your own S_2 semantics under which S is not sound.

Solution

This is not the only solution, but it is the simplest and most obvious. Here it is.

We define $x \Rightarrow y = F$ for all $x, y \in \{F, T\}$, and $\neg$ in anyway, as for such defined $\Rightarrow$ anv v is obviously a counter-model for **A1** and in fact, for all axioms.

(c) (5pts) Define your own S_n semantics such that S is sound for all for $2 \leq n \leq m$

Solution

This is not, again the only solution, but it is the simplest and most obvious. Here it is.

Consider $n \in N$, such that $2 \leq n \leq m$. We define a semantics S_n as follows.

Let $L_n = \{l_1, l_2, ... l_n\}$ be the set of logical values of S_n. We put

$$\neg x = T \quad \text{and} \quad x \Rightarrow y = T \quad \text{for all } x \in L_n$$

QUESTION 2 (15pts)

S is the following proof system:

$$S = (\mathcal{L}_{\{\Rightarrow,\cup,\neg\}}, \ \mathcal{F}, \ LA, \ \{(r1), (r2)\})$$

Logical Axioms

$$LA; \quad (a \Rightarrow (a \cup b)), \quad \text{where } a, b \in VAR$$

Rules of inference:

$$(r1)\ \frac{A \,;\, B}{(A \cup \neg B)}, \qquad (r2)\ \frac{A \,;\, (A \cup B)}{B} \qquad \text{where} \quad A, B \in \mathcal{F}$$

1. (10pts) Find a formal proof of $\neg(c \Rightarrow (c \cup a))$ in S, i. e. show that

$$\vdash_S \neg(c \Rightarrow (c \cup a))$$

Solution

The formal proof B_1, B_2, B_3, B_4 of $\neg(c \Rightarrow (c \cup a))$ in S is as follows

B_1**:** $(c \Rightarrow (c \cup a))$

Axiom LA for $a = c, b = a$

B_2**:** $(c \Rightarrow (c \cup a))$

Axiom LA for $a = c$, $b = a$

B_3**:** $((c \Rightarrow (c \cup a)) \cup \neg(c \Rightarrow (c \cup a)))$

Rule $(r1)$ application to B_1 and B_2

B_4**:** $\neg(c \Rightarrow (c \cup a))$

Rule $(r2)$ application to B_1 and B_3

2. (5pts) Does above point **1.** prove that $\models \neg(c \Rightarrow (c \cup a))$? Justify your answer

Solution

The system S is not sound. Consider rule $(r2)$.

Take any v, such that it evaluates $A = T$ and $B = F$.

The premiss $(A \cup B)$ of $(r2)$ is T and the conclusion B is F.

Moreover, the proof B_1, B_2, B_3, B_4 of $((c \Rightarrow (c \cup a)) \cup \neg(c \Rightarrow (c \cup a)))$ used the rule $(r2)$ that is not sound.

QUESTION 3 (10pts)

Consider any Hilbert proof system $S = (\mathcal{L}_{\{\cap,\cup,\Rightarrow,\neg\}}, \mathcal{F}, LA, \{MP\})$ that is **complete** under classical classical semantics.

We define a set $Cn(X)$ of all **consequences** of the set $X \subseteq \mathcal{F}$ as follows

$$Cn(X) = \{A \in F : X \vdash_S A\}$$

i.e. $Cn(X)$ is the set of all formulas that can be proved in S from the set $(LA \cup X)$ using the Modus Ponens rule MP as the only rule of inference

1. (5pts) Prove that for any $A, B \in F$ and any $X \subseteq F$,

$$\text{if } A \in Cn(X) \text{ or } B \in Cn(X), \quad \text{then} \quad (A \cup B) \in Cn(X)$$

Solution

Assume that $A \in Cn(X)$ or $B \in Cn(X)$.

We have, by the $Cn(X)$ definition, that

$$(*) \quad X \vdash_S A \quad \text{or} \quad X \vdash_S B$$

From Completeness of S, the fact that

$$\models (A \Rightarrow (A \cup B)), \quad \models (B \Rightarrow (A \cup B))$$

and from monotonicity of the Cn operation we get that

$$(**) \quad X \vdash_S (A \Rightarrow (A \cup B)) \quad \text{and} \quad X \vdash_S (B \Rightarrow (A \cup B))$$

Applying the Modus Ponens rule MP to $(*)$ and $(**)$ we get $X \vdash_S (A \cup B)$.

This proves

$$X \vdash_S (A \Rightarrow (A \cup B))$$

2. (5pts) Prove that the **inverse implication** to **2.** does not hold, i.e. prove that it is **not true** that for any $A,\ B \in F$ and any $X \subseteq F$,

$$\text{if} \quad (A \cup B) \in Cn(X), \quad \text{then} \quad A \in Cn(X) \ \text{ or } \ B \in Cn(X)$$

Solution

We have to show that **there are** $A,\ B \in F$ and $X \subseteq F$, such that

$$(A \cup B) \in Cn(X), \quad \text{and} \quad A \notin Cn(X) \ \text{ and } \ B \notin Cn(X)$$

This is not the only solution, but it is the simplest and most obvious **counter - example** Here it is.

Take $A = a,\ \ B = \neg a$ and $X = \emptyset$

Obviously, in classical semantics $\models (a \cup \neg a)$ and $\not\models a$ and $\not\models \neg a$

By the **completeness** of S we get

$$\vdash_S (a \cup \neg a) \quad \text{and} \quad \nvdash_S a \ \text{ and } \quad \nvdash_S \neg a$$

This proves that

$$(a \cup \neg a) \in Cn(\emptyset), \quad \text{and} \quad a \notin Cn(\emptyset) \quad \text{and} \quad \notin Cn(\emptyset)$$

Remark

By the monotonicity argument we get that $A = a,\ \ B = \neg a$ and X any subset of $\mathcal{F}$ is also a **counter - example**

QUESTION 4 (15pts)

Consider the Hilbert system $H1 = (\ \mathcal{L}_{\{\Rightarrow\}},\ \mathcal{F},\ \{A1, A2\},\ (MP)\ \frac{A\ ;\ (A \Rightarrow B)}{B}\ )$ where for any $A, B \in \mathcal{F}$

$A1;\ \ (A \Rightarrow (B \Rightarrow A)),\quad A2:\ \ ((A \Rightarrow (B \Rightarrow C)) \Rightarrow ((A \Rightarrow B) \Rightarrow (A \Rightarrow C)))$.

1. (10pts) The **Deduction Theorem** holds for $H1$.

Use the **Deduction Theorem** to show that $\quad (A \Rightarrow (C \Rightarrow B)) \vdash_{H1} (C \Rightarrow (A \Rightarrow B))$

Solution

We apply the **Deduction Theorem** twice, i.e. we get

$(A \Rightarrow (C \Rightarrow B)) \vdash_H (C \Rightarrow (A \Rightarrow B))$ if and only if

$(A \Rightarrow (C \Rightarrow B)),\ C \vdash_H (A \Rightarrow B)$ if and only if

$(A \Rightarrow (C \Rightarrow B)),\ C,\ A \vdash_H B$

We now construct a proof of $(A \Rightarrow (C \Rightarrow B)), C, A \vdash_H B$ as follows

$B_1:\quad (A \Rightarrow (C \Rightarrow B))$ hypothesis

$B_2:\quad C$ hypothesis

$B_3:\quad A$ hypothesis

$B_4:\quad (C \Rightarrow B) \quad B_1,\ B_3$ and (MP)

$B_5:\quad B \quad B_2,\ B_4$ and (MP)

2. (5pts) Explain why **1.** proves that $(\neg a \Rightarrow ((b \Rightarrow \neg a) \Rightarrow b)) \vdash_{H1} ((b \Rightarrow \neg a) \Rightarrow (\neg a \Rightarrow b))$.

Solution

This is **1.** for $A = \neg a$, $C = (b \Rightarrow \neg a)$, and $B = b$.