CSE541 EXAMPLE 1: MIDTERM SOLUTIONS
(75pts)

PART 1: DEFINITIONS     TOTAL 10pts

DEF 1     Given a propositional language \( \mathcal{L}_{CON} \) for \( CON = C_1 \cup C_2 \), where \( C_1 \) is the set of all unary connectives, and \( C_2 \) is the set of all binary connectives

1. (1pts) Write the definition of the set \( \mathcal{F} \) of all formulas of \( \mathcal{L}_{CON} \) for \( C_1 = \{ \neg, \land \} \) and \( C_2 = \{ \lor \} \)

Solution
\( \mathcal{F} \subseteq \mathcal{A}^* \) and \( \mathcal{F} \) is the smallest set for which the following conditions are satisfied

(1) \( \text{VAR} \subseteq \mathcal{F} \) - ATOMIC FORMULAS
(2) If \( A \in \mathcal{F} \), then \( \neg A \in \mathcal{F} \) and \( \land A \in \mathcal{F} \)
(3) If \( A, B \in \mathcal{F} \), then \( (A \lor B) \in \mathcal{F} \)

2. (1pts) Write an example of 4 formulas, each of a different degree, of the language \( \mathcal{L}_{\{ \neg, \land, \lor \}} \)

Solution
Here are, for example, formulas of the degree 0, 1, 2, 3, respectively
\( a, \neg a, \land a, \lor (a \land \neg b) \)

DEF 2     Given the language \( \mathcal{L}_{\{ \neg, \land, \lor \}} \) and a \( M \) truth assignment \( \nu : \text{VAR} \rightarrow LV \), where \( LV \neq \emptyset \) is the set of logical values on the extensional semantics \( M \). Let \( T \in LV \) be its distinguished logical value.

1. (1pts) We say that a function \( \nu^* \) is the \( M \) extension of \( \nu \) to the set \( \mathcal{F} \) of the language \( \mathcal{L}_{\{ \neg, \land, \lor \}} \) if and only if the following conditions hold.

Solution
(i) for any \( a \in \text{VAR} \), \( \nu^*(a) = \nu(a) \); and
(ii) for any formulas \( A, B \in \mathcal{F} \),

\[ \nu^*(\neg A) = \neg \nu^*(A), \quad \nu^*(\land A) = \land \nu^*(A), \quad \text{and} \quad \nu^*((A \lor B)) = \lor(\nu^*(A), \nu^*(B)) \]

We also use standard notation \( \nu^*((A \lor B)) = \nu^*(A) \lor \nu^*(B) \)

2. (1pts) We say that \( \vdash_M A \) if and only if

Solution
\( \nu^*(A) = T \) for all truth assignments \( \nu : \text{VAR} \rightarrow LV \)

DEF 3     Given a language \( \mathcal{L}_{\{ \neg, \Rightarrow, \land, \lor, \cap \}} \) and its extensional semantics \( M \)

1. (2pts) A formula \( A \in \mathcal{F} \) is called \( M \) independent from a set \( G \subseteq \mathcal{F} \) if and only if

Solution
the sets \( G \cup \{A\} \) and \( G \cup \{\neg A\} \) are both \( M \) consistent.

I.e. when there are truth assignments \( \nu_1, \nu_2 \) such that \( \nu_1 \vdash_M G \cup \{A\} \) and \( \nu_2 \vdash_M G \cup \{\neg A\} \).

2. (2pts) Give example of a set \( G \subseteq \mathcal{F} \) and a formula \( A \in \mathcal{F} \) that is classically independent from \( G \)

Solution
Here is a very simple example: $G = \{a\}$ and $A = b$

Let $v_1$, $v_2$ be any truth assignments such that $v_1(a) = T$, $v_1(b) = T$ and $v_2(a) = T$, $v_2(b) = F$

Obviously, $G \cup \{A\} = \{a, b\}$ and $G \cup \{\neg A\} = \{a, \neg b\}

\[ v_1 \models \{a, b\} \quad \text{and} \quad v_2 \models \{a, \neg b\} \]

**DEF 4 (2pts)** Given a proof system $S$ = $(\mathcal{L}_{\neg, \cup, \Rightarrow}, \mathcal{F}, \mathcal{L}_{A}, \mathcal{R})$. We write $P_S = \{A \in \mathcal{F} : \vdash_S A\}$

The proof system $S$ is **complete** under a semantics $M$ if and only if the following condition holds.

\[ P_S = T_M \quad \text{for} \quad T_M = \{A \in \mathcal{F} : \models_M A\} \]

**PART 2: PROBLEMS (65 pts)**

**QUESTION 1 (25pts)** By a m-valued semantics $S_m$, for a propositional language $\mathcal{L} = \mathcal{L}_{\neg, \cap, \cup, \Rightarrow}$ we understand any definition of connectives $\neg, \cap, \cup, \Rightarrow$ as operations on a set $L_m = \{l_1, l_2, \ldots, l_m\}$ of logical values (for $m \geq 2$).

We assume that $l_1 \leq l_2 \leq \ldots \leq l_m$, i.e. the set $L_m = \{l_1 \leq l_2 \leq \ldots \leq l_m\}$ is totally ordered by a certain relation $\leq$ with $l_1, l_m$ being smallest and greatest elements, respectively. We denote $l_1 = F$, $l_m = T$ and call them (total) False and Truth, respectively. For example, when $m = 2, L_2 = \{F, T\}$, $F \leq T$. Semantics $S_2$ is called a classical semantics if the connectives are defined as $x \cup y = \max(x, y)$, $x \cap y = \min(x, y)$, $\neg T = F$, $\neg F = T$, and $x \Rightarrow y = \neg x \cup y$, for any $x, y \in L_2$.

Let $VAR$ be a set of propositional variables of $\mathcal{L}$ and let $S_m$ be any m-valued semantics for $\mathcal{L}$. A truth assignment $v : VAR \rightarrow L_m$ is called a $S_m$ model for a formula $A$ of $\mathcal{L}$ if and only if $v(A) = T$ and logical value $v'(A)$ is evaluated accordingly to the semantics $S_m$. We note $S_m$ is symbolically as $v \models S_m A$.

1. Let $S_3$ be a 3-valued semantics for $L_{\neg, \cap, \cup, \Rightarrow}$ defined as follows.

\[ L_3 = \{F, U, T\} \]

For $F \leq U \leq T$, and for any $x, y \in L_3$ we put $x \cap y = \min(x, y)$, $x \Rightarrow y = \neg x \cup y$, where

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Consider the following classical tautologies: $A_1 = (A \cup \neg A)$, $A_2 = (A \Rightarrow (B \Rightarrow A))$

(a) (5pts) Find $S_3$ counter-models for $A_1, A_2$, if exist. Use shorthand notation.

**Solution**

Any $v$ such that $v'(A) = v'(B) = U$ is a counter-model for both, $A_1$ and $A_2$.

(b) (5pts) Define a 2-valued semantics $S_2$ for $\mathcal{L}$, such that none of $A_1, A_2$ is a $S_2$ tautology. Verify your results. Use shorthand notation.

**Solution**

This is not the only solution, but it is the simplest and most obvious. Here it is.

We define $\neg x = F$, $x \Rightarrow y = F$ for all $x, y \in \{F, T\}$, and $x \cap y$, $x \cup y$ can be defined in the same, or another way,
as these connectives do not appear in our formulas.

2. Let \( S = \langle L, \{\Rightarrow\}, F, A1, A2, A3, MP \rangle \) be a proof system with axioms:

- **A1** \((A \Rightarrow (B \Rightarrow A))\),
- **A2** \(((A \Rightarrow (B \Rightarrow C)) \Rightarrow ((A \Rightarrow B) \Rightarrow (A \Rightarrow C)))\),
- **A3** \(((\neg B \Rightarrow \neg A) \Rightarrow ((\neg B \Rightarrow A) \Rightarrow B))\).

(a) (5pts) The system \( S \) is **complete** with respect to classical semantics.

Verify whether \( S \) is complete with respect to 3-valued semantics \( S_3 \), defined in 1.

**Solution**

(This is a kind of a "free points" problem.)

System \( S \) is **not complete** because it is **not sound**, as we have shown in 1.(a)

(b) (5pts) Define your own \( S_2 \) semantics under which \( S \) is not sound.

**Solution**

This is not the only solution, but it is the simplest and most obvious. Here it is.

We define \(x \Rightarrow y = F \) for all \( x, y \in \{F, T\} \), and \(\neg\) in anyway, as for such defined \(\Rightarrow\) any \(v\) is obviously a counter-model for \(A1\) and in fact, for all axioms.

(c) (5pts) Define your own \( S_2 \) semantics such that \( S \) is sound for all for \( 2 \leq n \leq m \)

**Solution**

This is not, again the only solution, but it is the simplest and most obvious. Here it is.

Consider \( n \in N \), such that \( 2 \leq n \leq m \). We define a semantics \( S_2 \) as follows.

Let \( L_n = \{l_1, l_2, ..., l_n\} \) be the set of logical values of \( S_2 \). We put \(\neg x = T \) and \( x \Rightarrow y = T \) for all \( x \in L_n \)

**QUESTION 2 (15pts)**

\( S \) is the following proof system:

\[ S = (L_{\{\Rightarrow, \cup, \neg\}}, F, LA, \{(r1), (r2)\}) \]

**Logical Axioms**

\[ LA; (a \Rightarrow (a \cup b)), \ \text{where} \ a, b \in \text{VAR} \]

**Rules** of inference:

\[ (r1) \frac{A ; B}{(A \cup \neg B)}, \ \quad (r2) \frac{A ; (A \cup B)}{B} \ \text{where} \ A, B \in F \]

1. (10pts) Find a formal proof of \(\neg(c \Rightarrow (c \cup a)) \) in \( S \), i.e. show that

\[ \vdash_S \neg(c \Rightarrow (c \cup a)) \]

**Solution**

The formal proof \( B_1, B_2, B_3, B_4 \) of \(\neg(c \Rightarrow (c \cup a)) \) in \( S \) is as follows

\[ B_1: \ (c \Rightarrow (c \cup a)) \]

Axiom LA for \( a = c, b = a \)
\[ B_2: \quad (c \Rightarrow (c \cup a)) \]

Axiom LA for \( a = c, b = a \)

\[ B_3: \quad ((c \Rightarrow (c \cup a)) \cup \neg(c \Rightarrow (c \cup a))) \]

Rule \((r1)\) application to \( B_1 \) and \( B_2 \)

\[ B_4: \quad \neg(c \Rightarrow (c \cup a)) \]

Rule \((r2)\) application to \( B_1 \) and \( B_3 \)

2. (5pts) Does above point 1. prove that \( \models \neg(c \Rightarrow (c \cup a)) \)? Justify your answer

Solution

The system \( S \) is not sound. Consider rule \((r2)\).

Take any \( v \), such that it evaluates \( A = T \) and \( B = F \).

The premiss \((A \cup B)\) of \((r2)\) is \( T \) and the conclusion \( B \) is \( F \).

Moreover, the proof \( B_1, B_2, B_3, B_4 \) of \(((c \Rightarrow (c \cup a)) \cup \neg(c \Rightarrow (c \cup a))) \) used the rule \((r2)\) that is not sound.

**QUESTION 3 (10pts)**

Consider any Hilbert proof system \( S = (\mathcal{L}_{\land, \lor, \Rightarrow, \neg}, F, \text{LA}, \{MP\}) \) that is complete under classical classical semantics.

We define a set \( Cn(X) \) of all consequences of the set \( X \subseteq F \) as follows

\[ Cn(X) = \{ A \in F : \ X \vdash_S A \} \]

i.e. \( Cn(X) \) is the set of all formulas that can be proved in \( S \) from the set \( (LA \cup X) \) using the Modus Ponens rule MP as the only rule of inference

1. (5pts) Prove that for any \( A, B \in F \) and any \( X \subseteq F \),

\[ \text{if } A \in Cn(X) \text{ or } B \in Cn(X), \text{ then } (A \cup B) \in Cn(X) \]

Solution

Assume that \( A \in Cn(X) \) or \( B \in Cn(X) \).

We have, by the \( Cn(X) \) definition, that

\[ (*) \quad X \vdash_S A \text{ or } X \vdash_S B \]

From Completeness of \( S \), the fact that

\[ \models (A \Rightarrow (A \cup B)), \quad \models (B \Rightarrow (A \cup B)) \]

and from monotonicity of the \( Cn \) operation we get that

\[ (***) \quad X \vdash_S (A \Rightarrow (A \cup B)) \text{ and } X \vdash_S (B \Rightarrow (A \cup B)) \]

Applying the Modus Ponens rule MP to \((*)\) and \((***)\) we get \( X \vdash_S (A \cup B) \).

This proves

\[ X \vdash_S (A \Rightarrow (A \cup B)) \]
2. (5pts) Prove that the inverse implication to 2. does not hold, i.e. prove that it is not true that for any \( A, B \in F \) and any \( X \subseteq F \),

\[
\text{if } (A \cup B) \in \text{Cn}(X), \quad \text{then } A \in \text{Cn}(X) \text{ or } B \in \text{Cn}(X)
\]

**Solution**

We have to show that there are \( A, B \in F \) and \( X \subseteq F \), such that

\[
(A \cup B) \in \text{Cn}(X), \quad \text{and } A \notin \text{Cn}(X) \text{ and } B \notin \text{Cn}(X)
\]

This is not the only solution, but it is the simplest and most obvious counter-example. Here it is.

Take \( A = a, \ B = \neg a \) and \( X = \emptyset \)

Obviously, in classical semantics \( \models (a \cup \neg a) \) and \( \not\models a \) and \( \not\models \neg a \)

By the completeness of \( S \) we get

\[
\vdash_S (a \cup \neg a) \quad \text{and} \quad \not\vdash_S a \quad \text{and} \quad \not\vdash_S \neg a
\]

This proves that

\[
(a \cup \neg a) \in \text{Cn}(\emptyset), \quad \text{and} \quad a \notin \text{Cn}(\emptyset) \text{ and } \neg a \notin \text{Cn}(\emptyset)
\]

**Remark**

By the monotonicity argument we get that \( A = a, \ B = \neg a \) and \( X \) any subset of \( F \) is also a counter-example.

**QUESTION 4 (15pts)**

Consider the Hilbert system \( H1 = (\mathcal{L}_{\rightarrow}, \quad \mathcal{F}, \quad \{A1,A2\}, \quad (MP) \quad \frac{\Delta : (A \Rightarrow B)}{B} ) \) where for any \( A, B \in \mathcal{F} \)

\( A1: \quad (A \Rightarrow (B \Rightarrow A)), \quad A2: \quad ((A \Rightarrow (B \Rightarrow C)) \Rightarrow ((A \Rightarrow B) \Rightarrow (A \Rightarrow C))). \)

1. (10pts) The Deduction Theorem holds for \( H1 \).

Use the Deduction Theorem to show that \( (A \Rightarrow (C \Rightarrow B)) \vdash_{H1} (C \Rightarrow (A \Rightarrow B)) \)

**Solution**

We apply the Deduction Theorem twice, i.e. we get

\( (A \Rightarrow (C \Rightarrow B)) \vdash_H (C \Rightarrow (A \Rightarrow B)) \) if and only if

\( (A \Rightarrow (C \Rightarrow B)), \ C \vdash_H (A \Rightarrow B) \) if and only if

\( (A \Rightarrow (C \Rightarrow B)), \ C, \ A \vdash_H B \)

We now construct a proof of \( (A \Rightarrow (C \Rightarrow B)), C, A \vdash_H B \) as follows

\( B_1 : \quad (A \Rightarrow (C \Rightarrow B)) \) hypothesis

\( B_2 : \quad C \) hypothesis

\( B_3 : \quad A \) hypothesis

\( B_4 : \quad (C \Rightarrow B) \) \( B_1, B_3 \) and (MP)

\( B_5 : \quad B \) \( B_2, B_4 \) and (MP)
2. (5pts) Explain why 1. proves that \((\neg a \Rightarrow ((b \Rightarrow \neg a) \Rightarrow b)) \vdash H_1 ((b \Rightarrow \neg a) \Rightarrow (\neg a \Rightarrow b))\).

Solution
This is 1. for \(A = \neg a\), \(C = (b \Rightarrow \neg a)\), and \(B = b\).