Introduction to Predicate Logic

Part 1

cse371/ math371/cse541

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Predicate Logic Language

Symbols:

1. P, Q, R… **predicates symbols**, denote relations in "real life", countably infinite set
2. x,y,z…. **variables,** countably infinite set
3. c1, c2, … **constants**, countably infinite set
4. f, g, h … **functional symbols**, may be empty, denote functions in "real life"
5. **Propositional connectives:**

   $\vee, \wedge, \Rightarrow, \neg, \Leftrightarrow$

6. **Symbols for quantifiers**

   $\forall$ **x** – universal quantifier reads: For all x…

   $\exists$ **x** – existential quantifier reads: There is x…

Formulas of Predicate Logic

We use symbols **1** - **6** to build **formulas** of predicate logic as follows

1. P(x), Q(x,f(y)), R(x)… R(c1), Q(x, c3), Q(g(x,y), c), …
   are called **atomic formulas** for any variables x, y,… ,

functions f, g …. and constants c, c1, c2, …

2. **All atomic formulas are formulas ;**
3. If **A,B** are **formulas** then (like in propositional logic):
   $(A \vee B)$, $(A \wedge B)$, $(A \Rightarrow B)$, $(A \Leftrightarrow B)$, $\neg A$
   are **formulas**
4. **$\forall x\, A$, $\exists y\, A$** are **formulas,** for any variables x, y
5. The set **F** of **all formulas** is the **smallest** set that fulfills the conditions 1 -4.

Examples

For example: let

$P(y)$, $Q(x,c)$, $R(z)$, $P_1(g(x, y), z)$ be **atomic** formulas, i.e.

$\mathbf{P(x),\ Q(x,c),\ R(z),\ P_1(g(x, y), z) \in F}$

Then we form **some** other formulas out of them as follows:

$\mathbf{(P(y) \vee \neg Q(x, c)) \in F}$

It is a **formula** **A** with two **free** **variables** x, y

We denote it as a formula **A(x,y)**

$\mathbf{\exists x\, (P(y) \vee \neg Q(x, c)) \in F}$ – y is a free variable

We denote it as a formula **B(y)**

$\mathbf{\forall y\, (P(y) \vee \neg Q(x, c)) \in F}$ – x is a free variable

We denote it as a formula **C(x)**

$\mathbf{\forall y\, \exists x (P(y) \vee \neg Q(x, c)) \in F}$ – no free variables

Free and Bound Variables

Quantifiers **bound** variables within formulas

For example: A is a formula:

$\exists \mathbf{x}\, (P(\mathbf{x}) \Rightarrow \neg Q(\mathbf{x}, y))$

all the **x**'s in **A** are **bounded** by $\exists \mathbf{x}$

y is a **free variable** in **A** and we write **A=A(y)**

A(y) can be bounded by a quantifier, for example

$\forall \mathbf{y}\ \exists x\, (P(x) \Rightarrow \neg Q(x, \mathbf{y}))$

y got **bounded** and there **are no free** variables in **A** now

A **formula without free variables** is called a **sentence**

Logic and Mathematical Formulas

We often use **logic symbols** while writing mathematical statements in a more **symbolic** way

Example of a Mathematical Statement:

$\forall x \in N\ (x > 0 \land \exists y \in N\ (y = 1))$

1. Quantifier $\forall x \in N$ is a quantifier with **restricted domain**
2. **Logic uses only $\forall x$, $\exists y$**
3. **$x > 0$** and **$y = 1$** are mathematical statements about "real relations" **>** and =
4. Logic **uses symbols** P, Q, R… for relations
5. **For example**

   $R(y, c_1)$ for **$y = 1$** and **$P(x, c_2)$** for **$x > 0$** where **c_1** and **c_2** are **constants** representing **numbers 1 and 0**, respectively

Translation of Mathematical Statements to Logic Formulas

Consider a Mathematical Statement written with logical symbols

$$\forall x \in N\ (x > 0 \wedge \exists y \in N\ (y = 1))$$

$x \in N$ – we translate it as **one** argument predicate $Q(x)$
$x > 0$ – we translate as $P(x, c_1)$, and $y = 1$ as $R(y, c_2)$ and get

$$\forall Q(x)\ (P(x, c_1) \wedge \exists Q(y)\ R(y, c_2))$$

↑ Logic formula with **restricted domain** quantifiers

But this is **not yet a proper formula** since **we cannot** have quantifiers $\forall Q(x)$, $\exists Q(y)$ **in LOGIC,** but only quantifiers $\forall x$, $\exists x$

$\forall Q(x)$, $\exists Q(y)$ are called **quantifiers with restricted domain**

Logic Formula Corresponding Mathematical Statement

We need to "**get rid**" of **quantifiers with restricted domain** i.e. to **translate them** into **logic quantifiers:** **$\forall x$, $\exists y$**

$\exists x \in N$, $\exists y \in N$ **are restricted quantifiers**

↑ certain **predicate** $P(x)$

General: restricted domain quantifiers are :

$$\forall_{A(x)}, \quad \exists_{B(x)}$$

for $A(x)$, $B(x)$ any formulas, in particular atomic formulas (predicates) $P(x)$, $Q(x)$

Restricted Domain Existential Quantifiers

Translation for existential l quantifier

$$\exists_{A(x)} B(x) \equiv \exists x(A(x) \wedge B(x))$$

↑ **restricted** ↑**logic, not restricted**

Example (mathematical formulas):

$\exists\ x\neq 1\ (x>0 \Rightarrow x + y > 5)$ - restricted

$\exists\ x\ ((x\neq 1\ ) \wedge (x > 0 \Rightarrow x + y > 5))$ - not restricted

↑ B(x, y)

English statement:

Some students are good.

Logic Translation (restricted domain):

$$\exists_{S(x)} G(x)$$

Predicates are :

S(x) – x is a student

G(x) – x is good

TRANSLATION:

$$\exists x(S(X) \wedge G(x))$$

Restricted Quantifiers and Logic Quantifiers

Translation for universal quantifier

Restricted Logic (non-restricted)

$\forall_{A(x)} B(x) \quad \equiv \quad \forall x\, (A(x) \Rightarrow B(x))$

Example (mathematical statement)

$\forall x \in N\, (x = 1 \lor x < 0)$ restricted domain

$\equiv \forall x\, (x \in N \Rightarrow (x = 1 \lor x < 0)\,)$ – non-restricted

Translation of Mathematic statements to Logic formulas

Mathematical statement:

$$\forall x\, (x \in N \Rightarrow (\, x = 1 \vee x < 0)\,)$$

$x \in N$ – translates to $N(x)$

$x < 0$ – translates to $P(x, c_1)$

$x < y$ – $<$ is a 2 argument relation - two argument predicate $P(x, y)$, x, y are variables

0 – is a constant – denote by c_1

$x=1$ - $=$ is a two argument predicate $Q(x,y)$

$x =1$ - 1 is constant denoted by c_2

$x=1$ translates to $Q(x, c_2)$

Corresponding logic formula:

$$\forall x\, (N(x) \Rightarrow (Q(x, c_2) \vee P(x, c_1))\,)$$

Remark

Mathematical statement: **$x + y = 5$**

We re-write it as

$$= (+ (x, y), 5)$$

Given $x = 2$, $x = 1$, we get **$+(2,1) = 3$** and the statement:

$= (3,5)$ is FALSE (F)

Predicates always returns F or T

We really need also **function** symbols (like +, etc..) to translate mathematical statements to logic, even if we could use only relations as functions are special relations

This is why in **formal** definition of the predicate language we often we have **2 sets of symbols**

1. **Predicates** symbols which can be true or false in proper domains
2. **Functions** symbols (formally called **terms**)

Translations to Logic

Rules:

1. **Identify** the domain: always a set $X \neq \phi$
2. **Identify** predicates (simple: atomic)
3. **Identify** functions (if needed)

4. **Identify** the connectives $\vee$, $\wedge$, $\Rightarrow$, $\neg$, $\Leftrightarrow$
5. **Identify** the quantifiers $\forall x$, $\exists x$

   Write a formula using only symbols for 2 ,3, 4

6. **Use restricted domain quantifier translation rules,** where needed

Translations Examples

Translate:

For every bird there are some birds that are white

Predicates:

B(x) – x is a bird

W(x) – x is white

Restricted:

$$\forall_{B(x)} \exists_{B(x)} W(x)$$

Logic

$$\forall x(B(x) \Rightarrow \exists x\ (B(x) \land W(x)))$$

Re-name variables

$$\forall x(B(x) \Rightarrow \exists y(B(y) \land W(y)))$$

By Laws of Quantifiers - we will study the laws later, we can re-write it as

$$\forall x \exists y\ (B(x) \Rightarrow (B(y) \land W(y)))$$

Example

For every student there is a student that is an elephant

B(x)- x is a student

W(x) – x is an elephant

$\forall_{B(x)} \exists_{B(x)} W(x)$ - restricted

$\forall_{B(x)} \exists x(B(x) \wedge W(x))$

$\boldsymbol{\forall x(B(x) \Rightarrow \exists x(B(x) \wedge W(x)))}$ (logic formula)

Translations Example

Translate: **Some patients like all doctors**

Predicates:

$P(x)$ – x is a patient

$D(x)$ – x is a doctor

$L(x,y)$ – x likes y

$\exists_{P(x)} \forall_{D(y)} L(x,y)$

There is a **patient(x)**, such that for all **doctors(y)**, **x** likes **y**

$\exists x(P(x) \land \forall y(D(y) \Rightarrow L(x,y)))$

(by **law of quantifiers** to be studied later we can "pull out $\forall y$")

$\exists x \forall y(P(x) \land (D(y) \Rightarrow L(x,y)))$

Translations Exercise

- Here is a mathematical statement S:
- *For all natural numbers n the following hold:*

IF $n < 0$, then there is a natural number m, such that $m + n < 0$

1. Re-write S as a "formula" SF that only uses mathematical and logical symbols
2. Translate your SF to a correct logic formula LF
3. Argue whether the statement S it true of false
4. Give an interpretation of the logic formula LF (in a non-empty set X) under which LF is false