CSE541 Practice Midterm 2 SOLUTIONS Spring 2015

QUESTION 1

Remark This problem is taken straight from the BOOK and your exercises solutions! I write the solution to spare your time!

Let $S = (\mathcal{L}_{\{\cap,\cup,\Rightarrow,\neg\}}, \ \mathbf{A1}, \mathbf{A2}, \mathbf{A3}, \ MP)$ be a proof system with the following axioms:

A1 $(A \Rightarrow (B \Rightarrow A))$,

A2 $((A \Rightarrow (B \Rightarrow C)) \Rightarrow ((A \Rightarrow B) \Rightarrow (A \Rightarrow C)))$,

A3 $((\neg B \Rightarrow \neg A) \Rightarrow ((\neg B \Rightarrow A) \Rightarrow B))$,

The following Lemma holds in the system S.

LEMMA

For any $A, B, C \in \mathcal{F}$,

(a) $(A \Rightarrow B), (B \Rightarrow C) \vdash_H (A \Rightarrow C)$,

(b) $(A \Rightarrow (B \Rightarrow C)) \vdash_H (B \Rightarrow (A \Rightarrow C))$.

Complete the proof sequence (in S)

$$B_1, ..., B_9$$

by providing comments how each step of the proof was obtained.

Solution

B_1 $(A \Rightarrow B)$
Hypothesis

B_2 $(\neg\neg A \Rightarrow A)$
Already PROVED

B_3 $(\neg\neg A \Rightarrow B)$

Lemma **a** for $A = \neg\neg A, B = A, C = B$

B_4 $(B \Rightarrow \neg\neg B)$
Already PROVED

B_5 $(\neg\neg A \Rightarrow \neg\neg B)$

Lemma **a** for $A = \neg\neg A, B = B, C = \neg\neg B$

B_6 $((\neg\neg A \Rightarrow \neg\neg B) \Rightarrow (\neg B \Rightarrow \neg A))$
Already PROVED
Example 4 ch8 for $B = \neg A, A = \neg B$

B_7 $(\neg B \Rightarrow \neg A)$

B_5, B_6 and MP

B_8 $(A \Rightarrow B) \vdash (\neg B \Rightarrow \neg A)$

$B_1 - B_7$

B_9 $((A \Rightarrow B) \Rightarrow (\neg B \Rightarrow \neg A))$

Deduction Theorem

QUESTION 2

Remark This question is designed to check if you understand the notion of completeness, monotonicity, application of Deduction Theorem and use of some basic tautologies.

Consider any proof system S,

$$S = (\mathcal{L}_{\{\cap,\cup,\Rightarrow,\neg\}},\ LA,\ (MP)\frac{A, (A \Rightarrow B)}{B})$$

that is **complete** under **classical** semantics.

Definition 1 Let $X \subseteq F$ be any subset of the set F of formulas of the language $\mathcal{L}_{\{\cap,\cup,\Rightarrow,\neg\}}$ of S. We define a set $Cn(X)$ of all **consequences** of the set X as follows

$$Cn(X) = \{A \in F : \ X \vdash_S A\},$$

i.e. $Cn(X)$ is the set of all formulas that can be proved in S from the set $(LA \cup X)$.

Part 1

(i) Prove that for any subsets X, Y of the set F of formulas the following **monotonicity property** holds.

If $X \subseteq Y$, then $Cn(X) \subseteq Cn(Y)$

Solution Let $C \in Cn(X)$, it means that $X \vdash_S C$, i.e C has a formal proof from $X \cup AX$, but $X \subseteq Y$, hence also this proof is also a proof from $Y \cup AX$, i.e $Y \vdash_S C$ and $C \in Cn(Y)$. This proves that $Cn(X) \subseteq Cn(Y)$.

(ii) Prove that for any $X \subseteq F$, the set $\mathbf{T}$ of all propositional classical tautologies is a subset of $Cn(X)$, i.e.

$$\mathbf{T} \subseteq Cn(X).$$

Solution The proof system S is complete, i.e $\mathbf{T} = \{A : \ \vdash_S A\}$. By definition of the consequence, $\{A \in F : \ \vdash_S A\} = Cn(\emptyset)$ and hence by completeness, $Cn(\emptyset) = \mathbf{T}$.

For any set X, $\emptyset \subseteq X$ so by monotonicity proved in **(i)**,

$$\mathbf{T} \subseteq Cn(X).$$

Part 2

Prove that for any $A, B \in F$, $X \subseteq F$,

$$(A \cap B) \in Cn(X) \ \text{ iff } \ A \in Cn(X) \text{ and } \ B \in Cn(X)$$

Hint: Use the Monotonicity and Completeness of S i.e. the fact that any tautology you might need is provable in S.

Solution Assume $(A \cap B) \in Cn(X)$, i.e. $X \vdash_S (A \cap B)$.

From Monotonicity (Part 1), completeness, and the fact that $\models ((A \cap B) \Rightarrow A), \models ((A \cap B) \Rightarrow B)$ we get that

$$X \vdash_S ((A \cap B) \Rightarrow A), \quad X \vdash_S ((A \cap B) \Rightarrow B).$$

We have hence the following.

$$X \vdash_S (A \cap B), \quad \text{(assumption)},$$
$$X \vdash_S ((A \cap B) \Rightarrow A) \quad \text{(completeness, monotonicity)},$$
$$X \vdash_S A, \quad \text{MP}.$$

Similarly, $X \vdash_S (A \cap B)$, (assumption), $X \vdash_S ((A \cap B) \Rightarrow B)$ (completeness, monotonicity), and so we get $X \vdash_S B$ by MP.

Hence $A \in Cn(X)$ and $B \in Cn(X)$.

Assume now that $A \in Cn(X)$ and $B \in Cn(X)$. I.e. $X \vdash_S A$, and $X \vdash_S B$.

From Completeness of S and the fact that $\models (A \Rightarrow (B \Rightarrow (A \cap B)))$, and monotonicity (proved in Part 1) we get that

$$X \vdash_S (A \Rightarrow (B \Rightarrow (A \cap B))).$$

From above we have the following:

$$X \vdash_S A \quad \text{(assumption)}, \quad X \vdash_S B \quad \text{(assumption)},$$
$$X \vdash_S (A \Rightarrow (B \Rightarrow (A \cap B))) \quad \text{(completeness, monotonicity)},$$
$$X \vdash_S (B \Rightarrow (A \cap B)) \text{ MP}, \quad X \vdash_S (A \cap B) \text{ MP}.$$

This proves that $(A \cap B) \in Cn(X)$.

QUESTION 3

Let **GL** be the Gentzen style proof system for classical logic.

(1) Prove, by constructing a proper decomposition tree that
$\vdash_{\mathbf{GL}} ((\neg(a \cap b) \Rightarrow b) \Rightarrow (\neg b \Rightarrow (\neg a \cup \neg b)))$.

Solution Consider the following tree.

$$\mathbf{T}_{\rightarrow A}$$

$$\longrightarrow ((\neg(a \cap b) \Rightarrow b) \Rightarrow (\neg b \Rightarrow (\neg a \cup \neg b)))$$
$$\mid (\rightarrow \Rightarrow)$$
$$(\neg(a \cap b) \Rightarrow b) \longrightarrow (\neg b \Rightarrow (\neg a \cup \neg b))$$
$$\mid (\rightarrow \Rightarrow)$$
$$\neg b, (\neg(a \cap b) \Rightarrow b) \longrightarrow (\neg a \cup \neg b)$$
$$\mid (\rightarrow \cup)$$

$$\neg b, (\neg(a \cap b) \Rightarrow b) \longrightarrow \neg a, \neg b$$

$$\mid (\rightarrow \neg)$$

$$b, \neg b, (\neg(a \cap b) \Rightarrow b) \longrightarrow \neg a$$

$$\mid (\rightarrow \neg)$$

$$b, a, \neg b, (\neg(a \cap b) \Rightarrow b) \longrightarrow$$

$$\mid (\neg \rightarrow)$$

$$b, a, (\neg(a \cap b) \Rightarrow b) \longrightarrow b$$

$$\bigwedge(\Rightarrow\rightarrow$$

$$b, a \longrightarrow \neg(a \cap b), b \qquad\qquad b, a, b \longrightarrow b$$

$$\mid (\rightarrow \neg) \qquad\qquad axiom$$

$$b, a, (a \cap b) \longrightarrow b$$

$$\mid (\cap \rightarrow)$$

$$b, a, a, b \longrightarrow b$$

$$axiom$$

All leaves of the decomposition tree are axioms, hence the proof has been found.

(2) Use the **completeness theorem** for **GL** to prove that

$\not\vdash_{\mathbf{GL}}((a \Rightarrow b) \Rightarrow (\neg b \Rightarrow a))$

Solution

By the **Completeness Theorem** we have that

$$\not\vdash_{\mathbf{GL}}((a \Rightarrow b) \Rightarrow (\neg b \Rightarrow a)) \quad \text{if and only if} \quad \not\models ((a \Rightarrow b) \Rightarrow (\neg b \Rightarrow a))$$

Any v, such that $v(a) = v(b) = F$ is a counter-model for $((a \Rightarrow b) \Rightarrow (\neg b \Rightarrow a))$, hence By the **Completeness Theorem** $\not\vdash_{\mathbf{GL}}((a \Rightarrow b) \Rightarrow (\neg b \Rightarrow a))$

QUESTION 4

Let **GL** be the Gentzen style proof system for classical logic.

1. Define SHORTLY Decomposition Tree for any A in **GL**.

Solution —item[] Here is my short definition.

Decomposition Tree $\mathbf{T}_A$

For each formula $A \in \mathcal{F}$, a decomposition tree $\mathbf{T}_A$ is a tree build as follows.

Step 1. The sequent $\longrightarrow A$ is the **root** of $\mathbf{T}_A$ and for any node $\Gamma \longrightarrow \Delta$ of the tree we follow the steps below.

Step 2. If $\Gamma \longrightarrow \Delta$ is indecomposable, then $\Gamma \longrightarrow \Delta$ becomes a **leaf** of the tree.

Step 3. If $\Gamma \longrightarrow \Delta$ is decomposable, then we pick one rule that applies by matching the sequent of the current node with the domain of the rules. Then we apply this rule as decomposition rule and put its left and right premises as the left and right leaves, or as one leaf in case of one premiss rule.

Step 4. We repeat steps 2 and 3 until we obtain only indecomposable leaves.

2. Prove Completeness Theorem for **GL**. We assume that the STRONG soundness has been proved.

Solution

Theorem 0.1 (Formula Completeness Theorem)
For any formula $A \in \mathcal{F}$,

$$\vdash_{\mathbf{GL}} A \ \ iff \ \ \models A.$$

We prove the logically equivalent form of the Completeness part: for any $A \in \mathcal{F}$

$$\text{If} \ \ \nvdash_{\mathbf{GL}} \longrightarrow A \ \ \text{then} \ \ \not\models A.$$

Assume $\nvdash_{\mathbf{GL}} \longrightarrow A$, i.e. $\longrightarrow A$ does not have a proof in **GL**. Let $\mathcal{T}_A$ be a set of all decomposition trees of $\longrightarrow A$. As $\nvdash_{\mathbf{GL}} \longrightarrow A$, each $\mathcal{T} \in \mathcal{T}_A$ has a non-axiom leaf. We choose an arbitrary $T_A \in \mathcal{T}_A$. Let $\Gamma^{'} \longrightarrow \Delta^{'}$, $\Gamma^{'}$ be an non-axiom leaf of T_A, for $\Delta^{'} \in VAR^*$ such that $\{\Gamma^{'}\} \cap \{\Delta^{'}\} = \emptyset$. The non-axiom leaf $L = \Gamma^{'} \longrightarrow \Delta^{'}$ *defines* a truth assignment $v : VAR \leftarrow \{T, F\}$ which falsifies A as follows:

$$v(a) = \begin{cases} T & \text{if a appears in } \Gamma^{'} \\ F & \text{if } a \text{ appears in} \Delta^{'} \\ \textit{any value} & \text{if a does not appear in } L \end{cases}$$

This proves, by **strong soundness** of the rules of inference of **GL** that $\not\models A$.

QUESTION 5

We know that a classical tautology $(\neg(a \cap b) \cup (a \cap b))$ is NOT Intuitionistic tautology and we know by **Tarski Theorem** that $\neg\neg(\neg(a \cap b) \cup (a \cap b))$ is intuitionistically PROVABLE

FIND the proof of the formula

$$\neg\neg(\neg(a \cap b) \cup (a \cap b))$$

in the Gentzen system **LI** for Intuitionistic Logic.

Solution

$$\longrightarrow \neg\neg(\neg(a \cap b) \cup (a \cap b))$$

$$\mid (\longrightarrow \neg)$$

$$\neg(\neg(a \cap b) \cup (a \cap b)) \longrightarrow$$

$$\mid (contr \longrightarrow)$$

$$\neg(\neg(a \cap b) \cup (a \cap b)), \ \neg(\neg(a \cap b) \cup (a \cap b)) \longrightarrow$$

$$\mid (\neg \longrightarrow)$$

$$\neg(\neg(a \cap b) \cup (a \cap b)) \longrightarrow \; (\neg(a \cap b) \cup (a \cap b))$$

$$\mid (\longrightarrow \cup_1)$$

$$\neg(\neg(a \cap b) \cup (a \cap b)) \longrightarrow \; \neg(a \cap b)$$

$$\mid (\longrightarrow \neg)$$

$$(a \cap b), \; \neg(\neg(a \cap b) \cup (a \cap b)) \longrightarrow$$

$$\mid (exch \longrightarrow)$$

$$\neg(\neg(a \cap b) \cup (a \cap b)), \; (a \cap b), \; \longrightarrow$$

$$\mid (\neg \longrightarrow)$$

$$(a \cap b) \; \longrightarrow \; (\neg(a \cap b) \cup (a \cap b))$$

$$\mid (\longrightarrow \cup_2)$$

$$(a \cap b) \; \longrightarrow \; (a \cap b)$$

$$axiom$$