CSE541 Practice Midterm 1 SOLUTIONS Spring 2015

QUESTION 1

Write the following natural language statement:

One likes to play bridge, or from the fact that the weather is good we conclude the following: one does not like to play bridge or one likes not to play bridge

as a formula of 2 different languages

1. Formula $A_1 \in \mathcal{F}_1$ of a language $\mathcal{L}_{\{\neg,\ \mathbf{L},\ \cup,\ \Rightarrow\}}$, where $\mathbf{L}$ A represents statement "one likes A", "A is liked".

Solution We translate our statement into a formula

$A_1 \in \mathcal{F}_1$ of a language $\mathcal{L}_{\{\neg,\ \mathbf{L},\ \cup,\ \Rightarrow\}}$ as follows.

Propositional Variables: a, b
a denotes statement: *play bridge*,
b denotes a statement: *the weather is good*

Translation 1

$$A_1 = (\mathbf{L}a \cup (b \Rightarrow (\neg \mathbf{I}a \cup \mathbf{L}\neg a)))$$

2. Formula $A_2 \in \mathcal{F}_2$ of a language $\mathcal{L}_{\{\neg,\ \cup,\ \Rightarrow\}}$.

Solution We translate our statement into a formula

$A_2 \in \mathcal{F}_2$ of a language $\mathcal{L}_{\{\neg,\ \cup,\ \Rightarrow\}}$ as follows.

Propositional Variables: a, b, c
a denotes statement: *One likes to play bridge* ,
b denotes a statement: *the weather is good*, and
c denotes a statement: *one likes not to play bridge*

Translation 2:

$$A_2 = (a \cup (b \Rightarrow (\neg a \cup c)))$$

QUESTION 2

Write the formal definition of the language $\mathcal{L}_{\{\neg,\ \mathbf{L},\ \cup,\ \Rightarrow\}}$ and give examples if is formulas of the degrees 0, 1, 2, 3, and 4.

Solution - directly from definition

QUESTION 3

Define formally your OWN 3 valued extensional semantics **M** for the language $\mathcal{L}_{\{\neg,\ \mathbf{L},\ \cup,\ \Rightarrow\}}$ under the following assumptions

1. Assume that the third value is **intermediate** between truth and falsity, i.e. the set of logical values is **ordered** and we have the following

Assumption 1 $F < \perp < T$

Assumption 2 T is the **designated value**

2. Model the situation in which one "likes" only truth; i.e. in which $\mathbf{L}T = T$ and $\mathbf{L} \perp = F$, $\mathbf{L}F = F$

3. The connectives $\neg$, $\cup$, $\Rightarrow$ can be defined as you wish, but you have to define them in such a way to make sure that

$$\models_{\mathbf{M}} (\mathbf{L}A \cup \neg \mathbf{L}A)$$

Solution

Here is MY **M** semantics - yours can be different!

I define the logical connectives by "shorthand" writing functions defining connectives in form of the "truth tables" and skipping other points of the definition - as I have typed it so many times for you before!

L Connective

L	F	⊥	T
	F	*F*	T

Negation :

¬	F	⊥	T
	T	*F*	F

Implication

⇒	F	⊥	T
F	T	T	T
⊥	*T*	⊥	T
T	F	*F*	T

Disjunction :

∪	F	⊥	T
F	F	⊥	T
⊥	⊥	T	T
T	T	*T*	T

Verify $\models_{\mathbf{M}} (\mathbf{L}A \cup \neg \mathbf{L}A)$

$\mathbf{L}T \cup \neg \mathbf{L}T = T \cup F = T$, $\mathbf{L} \perp \cup \neg \mathbf{L} \perp = F \cup \neg F = F \cup T = T$, $\mathbf{L}F \cup \neg \mathbf{L}F = F \cup \neg F = T$

QUESTION 3

1. Verify whether the formulas A_1 and A_2 from the **QUESTION 1** have a model/ counter model under your semantics **M**. You can use **shorthand notation**

Solution

1. $A_1 = (\mathbf{L}a \cup (b \Rightarrow (\neg \mathbf{I}a \cup \mathbf{L}\neg a)))$

2. $A_2 = (a \cup (b \Rightarrow (\neg a \cup c)))$

Any v, such that $v(a) = T$ is a **M model** for A_1 and for A_2 directly from the definition of $\cup$

2. Verify whether the following set **G** is **M**-consistent. You can use **shorthand notation**

$$\mathbf{G} = \{ \ \mathbf{L}a, \quad (a \cup \neg \mathbf{L}b), \quad (a \Rightarrow b), \ b \ \}$$

Solution

Any v, such that $v(a) = T, \ v(b) = T$ is a **M model** for **G** as

$\mathbf{L}T = T, \quad (T \cup \neg \mathbf{L}T) = T, \quad (T \Rightarrow T) = T, \quad b = T$

3. Give an **example** on an infinite, **M**-consistent set of formulas of the language $\mathcal{L}_{\{\neg, \ \mathbf{L}, \ \cup, \ \Rightarrow\}}$

Solution

Take **G** be a set of all propositional variables, i.e. $\mathbf{G} = \mathrm{VAR}$

v such that v(a) = T for all $a \in VAR$ is obviously a **M** model for **G** and it proves that **G** is **M**-consistent

QUESTION 4

Let S be the following **proof system**

$$S = (\ \mathcal{L}_{\{\neg, \mathbf{L}, \cup, \Rightarrow\}}, \ \mathcal{F}, \quad \{\mathbf{A1}, \mathbf{A2}\}, \quad \{r1, \ r2\} \ )$$

for the logical axioms and rules of inference defined for any formulas $A, B \in \mathcal{F}$ as follows

Logical Axioms

A1 $(\mathbf{L}A \cup \neg \mathbf{L}A)$

A2 $(A \Rightarrow \mathbf{L}A)$

Rules of inference:

$$(r1) \ \frac{A \ ; B}{(A \cup B)}, \qquad (r2) \ \frac{A}{\mathbf{L}(A \Rightarrow B)}$$

1. Write a proof in S with 2 applications of rule (r1) and one application of rule (r2)

You must write comments how each step pot the proof was obtained

2. Show, by constructing a formal proof that

$$\vdash_S ((\mathbf{L}b \cup \neg\mathbf{L}b) \cup \mathbf{L}((\mathbf{L}a \cup \neg\mathbf{L}a) \Rightarrow b)))$$

Solution Here is the proof $B_1, B_2,\ \ B_3,\ \ B_4$

B_1: $(\mathbf{L}a \cup \neg\mathbf{L}a)$ Axiom A_1 for A= a

B_2: $\mathbf{L}((\mathbf{L}a \cup \neg\mathbf{L}a) \Rightarrow b)$ rule r2 for B=b applied to B_1

B_3: $(\mathbf{L}b \cup \neg\mathbf{L}Ab)$ Axiom A_1 for A=b

B_4: $((\mathbf{L}b \cup \neg\mathbf{L}b) \cup \mathbf{L}((\mathbf{L}a \cup \neg\mathbf{L}a) \Rightarrow b))$ r1 applied to B_3 and B_2

3. Verify whether the inference rules r1, r2 are **M**-sound. You can use **shorthand notation**

Solution Rule r1 is sound because when $A = T$ and $B = T$ we get $A \cup B = T \cup T = T$

Rule 2 is not sound because when $A = T$ and $B = F$ (or $B = \perp$) we get $\mathbf{L}(A \Rightarrow B) = \mathbf{L}(T \Rightarrow F) = \mathbf{L}F = F$ or $\mathbf{L}(T \Rightarrow \perp) = \mathbf{L} \perp = F$

Observe that both logical axioms of S are **M** tautologies

4. Verify whether the system S is **M**-sound. You can use **shorthand notation**

Solution S is not sound as r2 is not sound

EXTRA QUESTION

If the system S **is not sound** under your semantics **M** then **re-define the connectives** in a way that such obtained new semantics **N** would make S sound.

You can use **shorthand notation**

Solution To make rule $r2$ sound while preserving the "soundness off axioms we have to modify ONLY the definition of implication. Here is the **N** semantics implication

N- Implication

$\Rightarrow$	F	$\perp$	T
F	T	T	T
$\perp$	T	$\perp$	T
T	T	T	T

Remark that it would be hard to convince anybody to use our sound proof system it as it would be hard to convince anybody to adopt our **N** semantics!