CSE541 Midterm 2 SOLUTIONS Spring 2015

QUESTION 1

Remark This question is designed to check if you understand the notion of completeness, monotonicity, application of Deduction Theorem and use of some basic tautologies.

Consider any proof system S,

$$S = (\mathcal{L}_{\{\cap,\cup,\Rightarrow,\neg\}},\ AX,\ (MP)\frac{A, (A \Rightarrow B)}{B})$$

that is **complete** under classical classical semantics.

Definition Let $X \subseteq F$ be any subset of the set F of formulas of the language $\mathcal{L}_{\{\cap,\cup,\Rightarrow,\neg\}}$ of S.

We define a set $Cn(X)$ of all **consequences** of the set X as follows

$$Cn(X) = \{A \in F :\ X \vdash_S A\},$$

i.e. $Cn(X)$ is the set of all formulas that can be proved in S from the set $(AX \cup X)$.

Prove that for any $A, B \in F$,

$$Cn(\{A, B\}) = Cn(\{(A \cap B)\})$$

Hint: Use Deduction Theorem and Completeness of S.

Solution Assume $C \in Cn(\{A, B\})$, i.e. $\{A, B\} \vdash_S C$, what we usually write as $A, B \vdash_S C$.

By Deduction Theorem applied twice we get that

$$\vdash_S (A \Rightarrow (B \Rightarrow C)).$$

We use completeness of S and the fact (proof by contradiction) that

$$\models\ (((A \Rightarrow (B \Rightarrow C)) \Rightarrow ((A \cap B) \Rightarrow C)))$$

to construct the following.

$$\vdash_S (A \Rightarrow (B \Rightarrow C)) \ \text{(assumption and Deduction Theorem)},$$
$$\vdash_S (((A \Rightarrow (B \Rightarrow C)) \Rightarrow ((A \cap B) \Rightarrow C))), \text{(completeness,}$$
$$\vdash_S ((A \cap B) \Rightarrow C) \ \text{MP},$$
$$(A \cap B) \vdash_S C, \ \text{(Deduction Theorem)}.$$

i.e. we have proved that $C \in Cn(\{(A \cap B)\}) = Cn(A \cap B)$.

Assume now that $C \in Cn(\{(A \cap B)\})$, i.e. $(A \cap B) \vdash_S C$.

By Deduction Theorem, $\vdash_S ((A \cap B) \Rightarrow C)$.

We want to prove that $C \in Cn(\{A, B\})$, what is equivalent, by the Deduction Theorem applied twice to proving that

$$\vdash_S (A \Rightarrow (B \Rightarrow C)).$$

The proof as in the previous case. We use completeness of S, and the fact (proof by contradiction) that

$$\models (((A \cap B) \Rightarrow C) \Rightarrow (A \Rightarrow (B \Rightarrow C)))$$

to get that $\vdash_S (A \Rightarrow (B \Rightarrow C))$ as follows.

$$\vdash_S ((A \cap B) \Rightarrow C) \quad \text{(assumption and Deduction Theorem)},$$

$$\vdash_S (((A \cap B) \Rightarrow C) \Rightarrow (A \Rightarrow (B \Rightarrow C)), \quad \text{(completeness)},$$

$$\vdash_S (A \Rightarrow (B \Rightarrow C)), \quad \text{MP},$$

what ends the proof.

QUESTION 2

Consider a system **RS3** obtained from **RS** by changing the sequence Γ' into Γ in all of the rules of inference of **RS**.

1. Define SHORTLY Decomposition Tree for any A in **RS3**

Solution

The decomposition tree is a slight modification of definition of **RS** tree; now we can decompose any decomposable formula at the decomposable node.

2. Show an example of a formula and its 2 decomposition trees

Solution

You can use any formula that leads to a node with at least two decomposable formulas. item[3.] Prove Completeness Theorem for **RS3**. We assume that the STRONG soundness has been proved.

Solution

The proof is a gain a modification of **RS** proof.

Assume $\not\vdash_{RS3} A$, i.e. A does not have a proof in RS3 Let $\mathcal{T}_A$ be a set of **all decomposition trees** of A. As $\not\vdash_{RS3} A$, **each** $T \in \mathcal{T}_A$ has a non-axiom leaf.

We choose an arbitrary $T_A \in \mathcal{T}_A$.

The non-axiom leaf L_A **defines** a truth assignment v which falsifies A, as follows:

$$v(a) = \begin{cases} F & \text{if } a \text{ appears in } L_A \\ T & \text{if } \neg a \text{ appears in } L_A \\ \text{any value} & \text{if } a \text{ does not appear in } L_A \end{cases}$$

QUESTION 3

We know that a classical tautology $(\neg(a \cap b) \Rightarrow (\neg a \cup \neg b))$ is NOT Intuitionistic tautology and we know by **Tarski Theorem** that $\neg\neg(\neg(a \cap b) \Rightarrow (\neg a \cup \neg b))$ is intuitionistically PROVABLE

FIND the proof of the formula

$$\neg\neg(\neg(a \cap b) \Rightarrow (\neg a \cup \neg b))$$

in the Gentzen system **LI** for Intuitionistic Logic.

Solution

$$\mathbf{T}_{\rightarrow A}$$

$$\longrightarrow \neg\neg(\neg(a \cap b) \Rightarrow (\neg a \cup \neg b))$$

$$\mid (\rightarrow \neg)$$

$$\neg(\neg(a \cap b) \Rightarrow (\neg a \cup \neg b)) \longrightarrow$$

$$\mid (contr \rightarrow)$$

$$\neg(\neg(a \cap b) \Rightarrow (\neg a \cup \neg b)), \neg(\neg(a \cap b) \Rightarrow (\neg a \cup \neg b)) \longrightarrow$$

$$\mid (\neg \rightarrow)$$

$$\neg(\neg(a \cap b) \Rightarrow (\neg a \cup \neg b)) \longrightarrow (\neg(a \cap b) \Rightarrow (\neg a \cup \neg b))$$

$$\mid (\rightarrow \Rightarrow)$$

$$\neg(a \cap b), \neg(\neg(a \cap b) \Rightarrow (\neg a \cup \neg b)) \longrightarrow (\neg a \cup \neg b)$$

$$\mid (\rightarrow \cup)_1$$

$$\neg(a \cap b), \neg(\neg(a \cap b) \Rightarrow (\neg a \cup \neg b)) \longrightarrow \neg a$$

$$\mid (\rightarrow \neg)$$

$$a, \neg(a \cap b), \neg(\neg(a \cap b) \Rightarrow (\neg a \cup \neg b)) \longrightarrow$$

$$\mid (exch \rightarrow)$$

$$\neg(a \cap b), a, \neg(\neg(a \cap b) \Rightarrow (\neg a \cup \neg b)) \longrightarrow$$

$$\mid (\neg \rightarrow)$$

$$a, \neg(\neg(a \cap b) \Rightarrow (\neg a \cup \neg b)) \longrightarrow (a \cap b)$$

$$\bigwedge(\rightarrow \cap$$

$$a, \neg(\neg(a \cap b) \Rightarrow (\neg a \cup \neg b)) \longrightarrow a)$$

axiom

$$a, \neg(\neg(a \cap b) \Rightarrow (\neg a \cup \neg b)) \longrightarrow b$$

$$\mid (\rightarrow weak)$$

$$a, \neg(\neg(a \cap b) \Rightarrow (\neg a \cup \neg b)) \longrightarrow$$

$$\mid (exch \rightarrow)$$

$$\neg(\neg(a \cap b) \Rightarrow (\neg a \cup \neg b)), a \longrightarrow$$

$$\mid (\neg \rightarrow)$$

$$a \longrightarrow (\neg(a \cap b) \Rightarrow (\neg a \cup \neg b))$$

$$\mid (\rightarrow\Rightarrow)$$

$$\neg(a \cap b), a \longrightarrow (\neg a \cup \neg b)$$

$$\mid (\rightarrow \cup)_2$$

$$\neg(a \cap b), a \longrightarrow \neg b$$

$$\mid (\rightarrow \neg)$$

$$b, \neg(a \cap b), a \longrightarrow$$

$$\mid (exch \rightarrow)$$

$$\neg(a \cap b), b, a \longrightarrow$$

$$\mid (\neg \rightarrow)$$

$$b, a \longrightarrow (a \cap b)$$

$$\bigwedge (\rightarrow \cap$$

$$b, a \longrightarrow a \qquad b, a \longrightarrow b$$

axiom *axiom*

All leaves are axioms, the tree is a proof of A in **LI**.