QUESTION 1 Use the above definition to prove the following

FACT 1 A set $A$ is INFINITE iff it contains a countably infinite subset, i.e. one can define a $1-1$ sequence $\{a_n\}_{n \in \mathbb{N}}$ of some elements of $A$.

SOLUTION 1. Implication $\rightarrow$
If $A$ is infinite, then we can define a $1-1$ sequence of elements of $A$.
Let $A$ be infinite,
We define a sequence

$$a_1, a_2, \ldots, a_n, \ldots$$

as follows.
1. Observe that $A \neq \emptyset$, because if $A = \emptyset$, $A$ would be finite. Contradiction.
So there is an element of $a \in A$.
We define

$$a_1 = a$$

2. Consider a set $A - \{a\} = A_1$. $A_1 \neq \emptyset$ because if $A = \emptyset$, then $A - \{a\} = \emptyset$ and $A$ is finite. Contradiction.
So there is an element $a_2 \in A - \{a\}$ and $a_1 \neq a_2$.
We defined

$$a_1, a_2$$

3. Assume now that we have defined an $n$-elements and sequence

$$a_1, a_2, \ldots, a_n \text{ for } a_1 \neq a_2 \neq \ldots \neq a_n$$

Consider a set $A_n = A - \{a_1, \ldots, a_n\}$.
The set $A_n \neq \emptyset$ because if $A - \{a_1, \ldots, a_n\} = \emptyset$, then $A$ is finite. Contradiction.
So there is an element

$$a_{n+1} \in A - \{a_1, \ldots, a_n\}$$

and $a_{n+1} \neq a_n \neq \ldots \neq a_1$.
By mathematical induction,
we have defined a $1-1$ sequence

$$a_1, a_2, \ldots, a_n, \ldots$$

elements of $A$.

2. Implication $\leftarrow$
If $A$ contain a $1-1$ sequence, then $A$ is infinite.
Assume $A$ is not infinite; i.e $A$ is finite. Every subset of finite set is finite, so we can’t have a $1-1$ infinite sequence of elements of $A$. Contradiction.
QUESTION 2 Use the above definition and FACT 1 from Question 1 to prove the following characterization of infinite sets.

Dedekind Theorem A set $A$ is INFINITE iff there is a set proper subset $B$ of the set $A$ such that $|A| = |B|$. 

SOLUTION Part 1. If $A$ is infinite, then there is $B \subsetneq A$ and

$$f : A \xrightarrow{1-1} B$$

$A$ is infinite, by Q1, we have a 1-1 sequence

$$a_1, a_2, \ldots, a_n, \ldots$$

of elements $A$.
We take $B = A - \{a_1\}$, $B \subsetneq A$ and we define a function

$$f : A \xrightarrow{1-1} B$$

as follows

$$f(a_1) = a_2$$

$$f(a_2) = a_3$$

$$\vdots$$

$$f(a_n) = a_{n+1}$$

$$f(a) = a, \text{ for all other } a \in A$$
obviously, $f$ is 1-1, onto
Observe: we have other choises of B!

Part 2. Assume that we have $B \subset A$ are

$$f : A \xrightarrow{\text{1-1}} B$$

We use Q1 to show that A is infinite; i.e we construct an 1-1 sequence $a_1 \ldots a_n$ of elements of $A_n$ as follows.

$B \subset A$, so $A - B \neq \emptyset$ and we have $b \in A - B$. This is our first element of the sequence.

Observe: $f : A \xrightarrow{\text{1-1}} B$, so $f(b) \in B$ and $b \in A - B$, hence $f(b) \neq b$ and $f(b)$ is our second element of the sequence.

We have now,

$b, f(b) \neq b, b \in A - B, f(b) \leftarrow B$

Take new,

$ff(b)$. As $f$ is 1-1 and $f(b) \neq b$, we get $ff(b) \neq f(b) \neq b$, $ff(b) \in B$ and the sequence $b, f(b), ff(b)$ is 1-1.

We create $ff(b) = f^2(b)$

We continue the construction by mathematical induction.

Assume that we have constructed a 1-1 sequence

$$b, f(b), f^2(b), f^3(b), \ldots, f^n(b)$$

Observe that $ff^n(b) = f^{n+1}(b) \neq f^n(b)$ as $f$ is 1-1.

By mathematical induction, we have that $\{f^n(b)\}_{n \in N}$ is a 1-1 sequence of elements of $A$ and hence $A$ is infinite.
QUESTION 3  Use technique from DEDEKIND THEOREM to prove the following

**Theorem** For any infinite set \( A \) and its finite subset \( B \), \(|A| = |A - B|\).

**SOLUTION** A is infinite, then by Q1 there is a 1-1 sequence:

\[ a_1, a_2, \ldots, a_n, \ldots \]

of elements of \( A \).

Let \(|B| = k\). We choose \( k \) 1-1 sequence \( \{C^k_n\}_{n \in N} \) of the sequence \( \{a_n\}_{n \in N} \), such that \( C^k_i \neq C^i_n \) for all \( j \neq i, 1 \leq i, j \leq k \) and all \( n \in N \).

Let \( B = \{b_1, \ldots, b_k\} \). We construct a function \( f : A \xrightarrow{1-1} A - \{b_1, \ldots, b_k\} \) as follows

\[
\begin{align*}
    f(b_1) &= c^1_1, & f(c^1_1) &= c^1_2, \ldots, f(c^1_n) &= c^1_{n+1} \\
    f(b_2) &= c^2_1, & f(c^2_1) &= c^2_2, \ldots, f(c^2_n) &= c^2_{n+1} \\
    & \vdots \\
    f(b_k) &= c^k_1, & f(c^k_1) &= c^k_2, \ldots, f(c^k_n) &= c^k_{n+1} \\
    f(a) &= a \quad \text{all } a \in A - B
\end{align*}
\]

As all sequences \( \{C^m_n\}_{n \in N, m=1, \ldots, k} \) are 1-1, and different, the function \( f \) is 1-1 and obviously ONTO \( A - B \).

QUESTION 4 Use DEDEKIND THEOREM to prove that the set \( N \) of natural numbers is infinite.

**SOLUTION** We use Dedekind theorem i.e we must define \( f : N \xrightarrow{1-1} B \subset N \). There are many such function for example \( f(n) = n + 1, f : N \xrightarrow{1-1} N - \{0\} \).

One can also use Q1 and define any 1-1 sequences in \( N \).
QUESTION 5 Use DEDEKIND THEOREM to prove that the set $R$ of real numbers is infinite.

SOLUTION We use Dedekind theorem

$$f(x) = 2^x \quad x \in R$$

$$f : R \xrightarrow{1-1} R^+$$

One can also use Q1 and define any 1-1 sequences in $R$.

QUESTION 6 Use technique from DEDEKIND THEOREM to prove that the interval $[a, b], a < b$ of real numbers is infinite and that $|[a, b]| = |(a, b)|$.

SOLUTION1 Use construction in the proof of Q3.

$$f : [a, b] \xrightarrow{1-1} [a, b] - \{a, b\} = (a, b)$$

This is the solution I had in mind!

SOLUTION2 Use Q3 $(a, b) = [a, b] - B, B$ :finite

QUESTION 7 Prove, using the above definitions 3 and 4 that for any cardinal numbers $\mathcal{M}, \mathcal{N}, \mathcal{K}$ the following formulas hold:

1. $\mathcal{N} \leq \mathcal{N}$
2. If $\mathcal{N} \leq \mathcal{M}$ and $\mathcal{M} \leq \mathcal{K}$, then $\mathcal{N} \leq \mathcal{K}$.

SOLUTION 1. $\mathcal{N} \leq \mathcal{N}$ means that for any set $A$, $|A| \leq |A|

f(a) = a$ establishes $f : A \xrightarrow{1-1} A$

2. $\mathcal{N} \leq \mathcal{M}$ and $\mathcal{M} \leq \mathcal{K}$, then $\mathcal{N} \leq \mathcal{K}$.

We have $|A| = \mathcal{N}, |b| = \mathcal{M}, |C| = \mathcal{K}$ and $f : A \xrightarrow{1-1} B$ and $g : B \xrightarrow{1-1} C$, then we have to construct $h : A \xrightarrow{1-1} C$. 5
h is a composition of f and g. i.e \( h(a) = g(f(a)) \), all \( a \in A \)

**QUESTION 8** Prove, for any sets \( A, B, C \) the following holds.

**Fact 2**

If \( C \subseteq B \subseteq A \) and \( |A| = |C| \), then \( |A| = |B| = |C| \).

To prove \( |A| = |B| \) you must use definition 3, i.e to construct a proper function. Use the construction from proofs of Fact 1 and Question 3

**SOLUTION**

1. \( A, B, C \) are finite and \( |A| = |C| \), and \( C \subseteq B \subseteq A \), so \( A = B = C \), and have \( |A| = |B| = |C| \)

2. \( A, B, C \) are infinite sets, we have \( |A| = |C| \) i.e we have \( f : A \xrightarrow{1\rightarrow} C \).

We want to construct a function\( g : A \xrightarrow{1\rightarrow} B \), where \( A \subseteq B \subseteq C \)

Take \( A - B \). We assume that \( A - B \neq \emptyset \), if not, \( A = B \), and \( |A| = |C| \) given \( |A| = |B| = |C| \).

We consider case \( C \subseteq B \subseteq A \). Take any \( a \in (A - B) \), as \( f : A \xrightarrow{1\rightarrow} C \), \( f(a) \in C \), \( f \) is 1-1 so \( f f(a) \neq f(a) \)

... in general \( f^n(a) \neq f^{n+1}(a) \) and we have a sequence for any \( a \in A - B \)

\( f(a), f^2(a), \ldots, f^n(a) \ldots \) of elements of \( C \).

We construct a function \( g : A \xrightarrow{1\rightarrow} B \)

\[
\begin{align*}
g(a) &= f(a) \\
g(f(a)) &= f^2(a) \\
g(f^2(a)) &= f^3(a) \\
&
\vdots \\
g(f^n(a)) &= f^{n+1}(a) \\
g(x) &= x \quad \text{for all other } x \in A
\end{align*}
\]

Figure 5: problem 8: Figure of function \( g : A \xrightarrow{1\rightarrow} B \). a,b represent any two element of \( A \)
QUESTION 9 Prove the following

Berstein Theorem (1898) For any cardinal numbers $\mathcal{M}, \mathcal{N}$

$$\mathcal{N} \leq \mathcal{M} \quad \text{and} \quad \mathcal{M} \leq \mathcal{N} \quad \text{then} \mathcal{N} = \mathcal{M}.$$ 

SOLUTION Let $A,B$ be two sets such that $|A| = \mathcal{N}, |B| = \mathcal{M}$, we rewrite on theorem as

Berstein Theorem For any sets $A,B$

If $|A| \leq |B|$ and $|B| \leq |A|$, then $|A| = |B|$

case1. The sets $A,B$ are disjoint.

As $|A| \leq |B|$, we have a function $f : A \to B$, i.e $f : A \rightarrowonto fA \subseteq B$ and $|A| = |fA|$ where $fA$ denotes the image of $A$ under $f$.

As $|B| \leq |A|$, we have a function $g : B \to A$, $gB \subseteq A$ and $|B| = |gB|$

We picture it as follow.

Figure 6: problem 9

As $f : A \rightarrowonto B$ and $gB \subseteq A$, we get $fgB \subseteq fA$ and hence

$$fgB \subseteq fA \subseteq B \quad (1)$$

Also, $gB \subseteq A$ and $g : B \rightarrowonto B$. Hence $fg : B \rightarrowonto fgB$ and

$$|B| = |fgB| \quad (2)$$

We have a following picture.

Figure 7: problem 9

By eq.2, $|B| = |fgB|$ and by eq.1, $fgB \subseteq fA \subseteq B$ and $|B| = |fA|$

By Q8, we get

$$|fA| = |B|$$
Hence, $|B| = |A|$

**case 2.** the set $A, B$ are NOT disjoint.
Repeat the same(or Google the proof) for the following picture.

![Diagram of sets A and B with overlap](image)

Figure 8: problem 9