CSE371 EXTRA CREDIT Challenge EXERCISES on SETS

SOLVE ALL PROBLEMS as PRACTICE;
They might appear as extra credit problems on Quizzes and Tests

FINITE and INFINITE SETS

Definition 1 A set A is FINITE iff there is a natural number $n \in N$ and there is a $1-1$ function f that maps the set $\{1, 2, ...n\}$ onto A.

Definition 2 A set A is INFINITE iff it is NOT FINITE.

QUESTION 1 Use the above definition to prove the following

FACT 1 A set A is INFINITE iff it contains a countably infinite subset, i.e.prove that one can define a $1-1$ sequence $\{a_n\}_{n \in N}$ of some elements of A.

Definition 3 Two sets A, B have the same CARDINALITY iff there is a function f that maps A one-to-one onto the set B.

We denote it $|A| = |B| = \mathcal{M}$ and $\mathcal{M}$ is called a cardinal number of sets A and B.

QUESTION 2 Use the above definition and FACT 1 from Question 1 to prove the following characterization of infinite sets.

Dedekind Theorem A set A is INFINITE iff there is a set proper subset B of the set A such that $|A| = |B|$.

QUESTION 3 Use technique from DEDEKIND THEOREM to prove the following

Theorem For any infinite set A and its **finite** subset B,

$$|A| = |A - B|.$$

QUESTION 4 Use DEDEKIND THEOREM to prove that the set N of natural numbers is infinite.

QUESTION 5 Use DEDEKIND THEOREM to prove that the set R of real numbers is infinite.

QUESTION 6 Use technique from DEDEKIND THEOREM to prove that the interval $[a, b], a < b$ of real numbers is infinite and that $|[a, b]| = |(a, b)|$.

CARDINALITIES OF SETS

Definition 4 For any sets A, B, let $|A| = \mathcal{N}$ and $|B| = \mathcal{M}$.

We say $\mathcal{N} \leq \mathcal{M}$ iff $|A| = |C|$ for some $C \subseteq B$.

We say $\mathcal{N} < \mathcal{M}$ iff $\mathcal{N} \leq \mathcal{M}$ and $\mathcal{N} \neq \mathcal{M}$.

QUESTION 7 Prove, using the above definitions 3 and 4 that for any cardinal numbers $\mathcal{M}, \mathcal{N}, \mathcal{K}$ the following formulas hold:

$$1. \quad \mathcal{N} \leq \mathcal{N}$$

$$2. \quad If \ \ \mathcal{N} \leq \mathcal{M} \ \ and \ \ \mathcal{M} \leq \mathcal{K}, \ \ then \ \ \mathcal{N} \leq \mathcal{K}.$$

QUESTION 8 Prove, for any sets A, B, C the following holds.

Fact 2

$$If \ \ A \subseteq B \subseteq C \ and \ |A| = |C|, \ \ then \ \ |A| = |B|.$$

To prove $|A| = |B|$ you must use definition 3, i.e to construct a proper function. Use the construction from proofs of Fact 1 and Question 3.

QUESTION 9 Prove the following

Cantor- Berstein Theorem (1898) For any cardinal numbers $\mathcal{M}, \mathcal{N}$

$$\mathcal{N} \leq \mathcal{M} \ \ and \ \ \mathcal{M} \leq \mathcal{N} \ \ then \mathcal{N} = \mathcal{M}.$$

1. Prove first the case when the sets A, B are disjoint.

2. Generalize the construction for 1. to the not-disjoint case.

REMINDER

Definition 5 A set A is INFINITELY COUNTABLE iff A has the same cardinality as Natural numbers N, i.e. $|A| = |N| = \aleph_0$

Definition 6 A set A is COUNTABLE iff A is finite or infinitely countable.

Definition 7 A set A is UNCOUNTABLE iff A is NOT countable.