CSE371  Practice Midterm SOLUTIONS  Fall 2011

**L semantics for L{¬,⇒,∩,∪} is defined as follows**

<table>
<thead>
<tr>
<th>L Negation</th>
<th>L Disjunction</th>
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<tbody>
<tr>
<td>( \neg )</td>
<td>( \cup )</td>
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<tr>
<td>( \neg F )</td>
<td>( F )</td>
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<td>( \neg \top )</td>
<td>( \top )</td>
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<tr>
<td>( \neg \bot )</td>
<td>( \bot )</td>
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<tr>
<td>( \bot )</td>
<td>( T )</td>
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<td>( T )</td>
<td>( F )</td>
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<tr>
<th>L Conjunction</th>
<th>L-Implication</th>
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<td>( \cap )</td>
<td>( \Rightarrow )</td>
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<td>( F )</td>
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<td>( \bot )</td>
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<td>( \bot )</td>
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<td>( T )</td>
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**QUESTION 1**

(1) **Use the fact** that \( v : VAR \rightarrow \{ F, \bot, T \} \) be such that

\[ v^*((a \cap b) \Rightarrow \neg b) = \bot \]

under L semantics **to evaluate** \( v^*((b \Rightarrow \neg a) \Rightarrow (a \Rightarrow \neg b)) \cup (a \Rightarrow b) \).

Use shorthand notation.

(1) **Solution**: \((a \cap b) \Rightarrow \neg b) = \bot \) in two cases.

- **C1** \((a \cap b) = \bot \) and \( \neg b = F \).
- **C2** \((a \cap b) = T \) and \( \neg b = \bot \).

**Case C1**: \( \neg b = F \), i.e. \( b = T \), and hence \( (a \cap T) = \bot \) iff \( a = \bot \). We get that \( v \) is such that \( v(a) = \bot \) and \( v(b) = T \).

**We evaluate**: \( v^*((b \Rightarrow \neg a) \Rightarrow (a \Rightarrow \neg b)) \cup (a \Rightarrow b) = (((T \Rightarrow \neg \bot) \Rightarrow (\bot \Rightarrow 
\neg T)) \cup (\bot \Rightarrow T)) = ((\bot \Rightarrow \bot) \cup T) = T \).

**Case C2**: \( \neg b = \bot \), i.e. \( b = \bot \), and hence \( (a \cap \bot) = T \) what is impossible, hence \( v \) from case C1 is the only one.

(2) **Prove that in classical semantics** \( L_{\{\neg,\Rightarrow\}} \equiv L_{\{\neg,\Rightarrow,\cup\}} \).
We define the EQUIVALENCE of LANGUAGES as follows:

Given two languages:
\[ L_1 = L_{CON_1} \] and \[ L_2 = L_{CON_2} \], for \( CON_1 \neq CON_2 \).

We say that they are logically equivalent, i.e.
\[ L_1 \equiv L_2 \]

if and only if the following conditions \( \text{C1, C2} \) hold.

\( \text{C1:} \) For every formula \( A \) of \( L_1 \), there is a formula \( B \) of \( L_2 \), such that
\[ A \equiv B, \]

\( \text{C2:} \) For every formula \( C \) of \( L_2 \), there is a formula \( D \) of \( L_1 \), such that
\[ C \equiv D. \]

(2) Solution: (Classical case)

\( \text{C1} \) holds because any formula of \( L_{\{\neg, \Rightarrow\}} \) is a formula of \( L_{\{\neg, \Rightarrow, \cup\}} \).

\( \text{C2} \) holds due to the following definability of connectives equivalence
\[ (A \cup B) \equiv (\neg A \Rightarrow B). \]

(3) Prove that the equivalence defining \( \cup \) in classical logic does not hold under \( L \) semantics, but nevertheless \( L_{\{\neg, \Rightarrow\}} \equiv_L L_{\{\neg, \Rightarrow, \cup\}} \).

Solution \( (A \cup B) \neq_L (\neg A \Rightarrow B) \) Take \( A = B = \bot \). We get \( \bot \cup \bot = \bot \) and \( \neg \bot \Rightarrow \bot = \bot \Rightarrow \bot = T. \)

Proof that \( L_{\{\neg, \Rightarrow\}} \equiv_L L_{\{\neg, \Rightarrow, \cup\}} \) holds for \( L \) semantics.

\( \text{C1:} \) holds because any formula of \( L_{\{\neg, \Rightarrow\}} \) is a formula of \( L_{\{\neg, \Rightarrow, \cup\}} \).

\( \text{C2:} \) holds because the definability of connectives equivalence
\[ (A \cup B) \equiv ((A \Rightarrow B) \Rightarrow B) \]
holds for \( L \). Easy to check by verification.

Observe that the equivalence \( (A \cup B) \equiv (\neg A \Rightarrow B) \) defining \( \cup \) in terms of \( \neg \) and \( \Rightarrow \) is a valuable candidate for \( L \) semantics definability as the definition of all connectives restricted to \( T, F \) is the same as in the classical case. Unfortunately it is not a good one for \( L \) semantics. It does not prove that other definability equivalence does not exist! Observe that the equivalence \( (A \cup B) \equiv (A \Rightarrow B) \Rightarrow B \) provides and alternative proof of \( \text{C2} \) in classical case.
QUESTION 2  Let $H$ be the following proof system:

$$H = (\mathcal{L}_{\rightarrow, \neg}, \mathcal{F}, AX = \{A_1, A_2, A_3, A_4\}, MP)$$

A1  $(A \rightarrow (B \rightarrow A))$,

A2  $((A \rightarrow (B \rightarrow C)) \rightarrow ((A \rightarrow B) \rightarrow (A \rightarrow C)))$,

A3  $((\neg B \rightarrow \neg A) \rightarrow ((\neg B \rightarrow A) \rightarrow B))$,

A4  $((A \rightarrow B) \rightarrow A)$

MP  (Rule of inference)

$$(MP) \quad \frac{A : (A \rightarrow B)}{B}$$

(1) Justify that $H$ is SOUND under classical semantics.

Solution  Axioms A1-A3 are axioms of a sound system $H_2$, with the same rule MP. So we need only to check if A4 is sound, i.e $\models (((A \rightarrow B) \rightarrow A) \rightarrow A)$. Assume not, i.e. $((A \rightarrow B) \rightarrow A) = T$ and $A = F$. We get $((F \rightarrow B) \rightarrow F) = T$. This is impossible, as $(F \rightarrow B) = T$ for all values of $B$ and $T \rightarrow F = F$.

(2) Does Deduction Theorem holds for $H$? Justify shortly your answer.

Solution  Axioms A1-A2 are axioms of system $H_1$ for which we proved the Deduction Theorem. System $H$ is a (sound) extension of $H_1$ and hence the Deduction Theorem holds for it as well.

(3) Justify the fact that $H$ is COMPLETE with respect to all classical semantics tautologies.

Solution  Axioms A1-A3 are axioms of system $H_2$ for which we proved the Completeness Theorem. System $H$ is a (sound) extension of $H_2$ and hence the Completeness Theorem holds for it as well.

(4) Prove that the system $H$ in NOT COMPLETE under the Lukasiewicz semantics $L$.

Solution  System $H$ is not sound under $L$ semantics. For example axiom A2 is not $L$ tautology. $A = \bot, B = \bot, C = F$ evaluates it to $\bot$. System that is not sound can’t be complete.

(5) All classical tautologies include for example de Morgan Laws

$$\neg(A \cup B) \Rightarrow (\neg A \cap \neg B), \quad (A \cap B) \Rightarrow (\neg A \cup \neg B)$$
Explain what does it mean that they are provable in $H$.

**Solution** Obviously $\mathcal{L}_{\{\Rightarrow, \neg\}}$ does not contain connectives $\cup, \cap$ and hence de Morgan Laws as written above are not formulas in our language. But we proved that $\mathcal{L}_{\{\Rightarrow, \neg\}} \equiv \mathcal{L}_{\{\Rightarrow, \neg, \cup, \cap\}}$ so the Morgan Laws (as any other formula) expressed in the language $\mathcal{L}_{\{\Rightarrow, \neg\}}$ as an equivalent formula, and then proved.

(6) Let $H'$ be a proof system obtained from $H$ by adding an additional axiom

A5 $((A \Rightarrow B) \Rightarrow \neg A)$

Is the system $H'$ complete under classical semantics? Justify your answer.

**Solution** $H'$ is not SOUND (axiom A5 is not a tautology!) hence can’t be complete.

We consider a sound proof system (under classical semantics)

$$S = (\mathcal{L}_{\{\Rightarrow, \neg\}}, \ AX, \ MP),$$

such that the formulas listed below are provable in $S$.

1. $(A \Rightarrow (B \Rightarrow A))$,
2. $((A \Rightarrow (B \Rightarrow C)) \Rightarrow ((A \Rightarrow B) \Rightarrow (A \Rightarrow C)))$,
3. $((\neg B \Rightarrow \neg A) \Rightarrow ((\neg B \Rightarrow A) \Rightarrow B))$,
4. $(A \Rightarrow A)$,
5. $(B \Rightarrow \neg B)$,
6. $(\neg A \Rightarrow (A \Rightarrow B))$,
7. $(A \Rightarrow (\neg B \Rightarrow \neg (A \Rightarrow B)))$,
8. $((A \Rightarrow B) \Rightarrow ((\neg A \Rightarrow B) \Rightarrow B))$,
9. $((\neg A \Rightarrow A) \Rightarrow A)$.

The following Lemma holds in $S$

**LEMMA** For any $A, B, C \in \mathcal{F}$,

(a) $(A \Rightarrow B), (B \Rightarrow C) \vdash_H (A \Rightarrow C)$,
(b) $(A \Rightarrow (B \Rightarrow C)) \vdash_H (B \Rightarrow (A \Rightarrow C))$. 

4
QUESTION 3

Complete the proof sequence (in $S$)

$B_1, \ldots, B_9$

of

$((A \Rightarrow B) \Rightarrow (\neg B \Rightarrow \neg A))$

by providing comments how each step of the proof was obtained.

Solution

$B_1 = (A \Rightarrow B)$
Hypothesis

$B_2 = (\neg \neg A \Rightarrow A)$
Already Proven

$B_3 = (\neg \neg A \Rightarrow B)$
Lemma a for $A = \neg \neg A, B = A, C = B$, in $B_1, B_2$ i.e.

$(\neg \neg A \Rightarrow A), (A \Rightarrow B) \vdash (\neg \neg A \Rightarrow B)$

$B_4 = (B \Rightarrow \neg \neg B)$
Formula 5

$B_5 = (\neg \neg A \Rightarrow \neg \neg B)$
Lemma a on $B_3, B_4$ for $A = \neg \neg A, B = B, C = \neg \neg B$

$(\neg \neg A \Rightarrow B), (B \Rightarrow \neg \neg B) \vdash (\neg \neg A \Rightarrow \neg \neg B)$

$B_6 = ((\neg \neg A \Rightarrow \neg \neg B) \Rightarrow (\neg B \Rightarrow \neg A))$
ALREADY PROVED

$B_7 = (\neg B \Rightarrow \neg A)$
$B_5, B_6$ and MP on $B_5, B_6$

$\frac{(\neg \neg A \Rightarrow \neg \neg B); ((\neg \neg A \Rightarrow \neg \neg B) \Rightarrow (\neg B \Rightarrow \neg A))}{(\neg B \Rightarrow \neg A)}$

$B_8 = (A \Rightarrow B) \vdash (\neg B \Rightarrow \neg A)$
$B_1 - B_7$

$B_9 = ((A \Rightarrow B) \Rightarrow (\neg B \Rightarrow \neg A))$
Deduction Theorem on $B_8$
HERE IS the Main Definition and Main Lemma needed for the PROOF 1 of the Completeness Theorem for the system $S$.

**Main Definition**

Let $A$ be a formula and $b_1, b_2, ..., b_n$ be all propositional variables that occur in $A$. Let $v$ be variable assignment $v: VAR \rightarrow \{T, F\}$. We define, for any $A, b_1, b_2, ..., b_n$ and $v$ a corresponding formulas $A', B_1, B_2, ..., B_n$ as follows:

$$
A' = \begin{cases} 
   A & \text{if } v^*(A) = T \\
   \neg A & \text{if } v^*(A) = F 
\end{cases}
$$

$$
B_i = \begin{cases} 
   b_i & \text{if } v(b_i) = T \\
   \neg b_i & \text{if } v(b_i) = F 
\end{cases}
$$

for $i = 1, 2, ..., n$.

**Main Lemma** For any formula $A$ and a variable assignment $v$, if $A', B_1, B_2, ..., B_n$ are corresponding formulas defined by the definition stated above, then

$$B_1, B_2, ..., B_n \vdash A'.$$

We write $\vdash A$ for $\vdash_S A$ as the system $S$ is fixed.

**QUESTION 4**

We know that the formula

$$
A = ((a \Rightarrow b) \Rightarrow (\neg b \Rightarrow \neg a))
$$

is a tautology; i.e. we know that

$$\models ((a \Rightarrow b) \Rightarrow (\neg b \Rightarrow \neg a)).$$

Use this information and the method developed in the Proof 1 of Completeness Theorem to show the

$$\vdash ((a \Rightarrow b) \Rightarrow (\neg b \Rightarrow \neg a))$$

**Solution** This is a shorter solution than one in the Example in the book; it follows directly the proof 1.

We know that $A$ is a tautology ($\models A$), so $v^*(A) = T$ for all $v$ and $A' = A$ for all $v$. $A = A(a,b)$, so by the Main Lemma $B_1, B_2 \vdash A$, for $B_1, B_2$ defined accordingly to $v$, and $v(a), v(b)$.
Step 1: $B_2$ elimination. $B_2 = b$ if $v(b) = T$ and $B_2 = \neg b$ if $v(b) = F$.

For any $v$ such that $v(b) = T$ we get

$$B_1, b \vdash A$$

and for any $v$ such that $v(b) = F$ we get

$$B_1, \neg b \vdash A.$$  

By Deduction Theorem we get

(1) $B_1 \vdash (b \Rightarrow A)$ and $B_1 \vdash (\neg b \Rightarrow A).$ We have assumed about the proof system $S$ that for ant formulas $A, B,$

$$\vdash ((A \Rightarrow B) \Rightarrow (\neg A \Rightarrow B) \Rightarrow B))$$

so in particular

$$\vdash ((b \Rightarrow A) \Rightarrow (\neg b \Rightarrow A) \Rightarrow A))$$

and by monotonicity

$$B_1 \vdash ((b \Rightarrow A) \Rightarrow (\neg b \Rightarrow A) \Rightarrow A).$$

We apply MP twice to (1) and $B_1 \vdash ((b \Rightarrow A) \Rightarrow (\neg b \Rightarrow A) \Rightarrow A))$ we get that

$$B_1 \vdash A.$$

Step 2: $B_1$ elimination. $B_1 = a$ if $v(a) = T$ and $B_1 = \neg a$ if $v(a) = F$.

For any $v$ such that $v(a) = T$ we get

$$a \vdash A$$

and for any $v$ such that $v(a) = F$ we get

$$\neg a \vdash A.$$  

By Deduction Theorem we get

(2) $\vdash (a \Rightarrow A)$ and $\vdash (\neg a \Rightarrow A).$ We have assumed about the proof system $S$ that for ant formulas $A, B,$

$$\vdash ((A \Rightarrow B) \Rightarrow (\neg A \Rightarrow B) \Rightarrow B))$$

so in particular

$$\vdash ((a \Rightarrow A) \Rightarrow (\neg a \Rightarrow A) \Rightarrow A))$$

We apply MP twice to (2) and $\vdash ((a \Rightarrow A) \Rightarrow (\neg a \Rightarrow A) \Rightarrow A))$ we get that

$$\vdash A.$$