CSE541 Exercise 2 SOLUTIONS

QUESTION 1 Give a definition and an example of a default reasoning.

Default reasoning is a reasoning in which it is allowed to draw plausible inferences from less-then-conclusive evidence in the absence of information to the contrary.

Example: Consider a statement *Birds fly.* Tweety, we are told, is a bird. From this, and the fact that birds fly, we conclude that Tweety can fly.

This conclusion, however is *defeasible*: Tweety may be an ostrich, a penguin, a bird with a broken wing, or a bird whose feet have been set in concrete. But as long as we don't have the evidence to the contrary (*Tweedy has a broken wing*) we accept the conclusion that *Tweety can fly.*

QUESTION 2 Write the following natural language statement:

From the fact that it is not necessary that an elephant is not a bird we deduce that:
it is not possible that an elephant is a bird or, if it is possible that an elephant is a bird, then it is not necessary that a bird flies.

as a formula

(i) $A_1 \in \mathcal{F}_1$ of a language $\mathcal{L}_{\{\neg, \mathbf{C}, \mathbf{I}, \cap, \cup, \Rightarrow\}}$, and as a formula

(ii) $A_2 \in \mathcal{F}_2$ of a language $\mathcal{L}_{\{\neg, \cap, \cup, \Rightarrow\}}$.

Solution

(i) We translate our statement into a formula

$A_1 \in \mathcal{F}_1$ of a language $\mathcal{L}_{\{\neg, \mathbf{C}, \mathbf{I}, \cap, \cup, \Rightarrow\}}$ as follows.

Propositional Variables: a, b.
a denotes statement: *an elephant is a bird*, b denotes a statement: *a bird flies.*

Propositional Modal Connectives: $\mathbf{C}$, $\mathbf{I}$.
$\mathbf{C}$ denotes statement: *it is possible that*, $\mathbf{I}$ denotes statement: *it is necessary that.*

Translation 1:

$$A_1 = (\neg \mathbf{I} \neg a \Rightarrow (\neg \mathbf{C} a \cup (\mathbf{C} a \Rightarrow \neg \mathbf{I} b))).$$

(ii) Now we translate our statement into a formula

$A_2 \in \mathcal{F}_2$ of a language $\mathcal{L}_{\{\neg, \cap, \cup, \Rightarrow\}}$ as follows.

Propositional Variables: a, b, c.
.

a denotes statement: *it is necessary that an elephant is not a bird* ,
b denotes statement: *it is possible that an elephant is a bird* ,
c denotes a statement: *it is necessary that a bird flies.*

Translation 2:

$$A_2 = (\neg a \Rightarrow (\neg b \cup (b \Rightarrow \neg c))).$$

2. Main connective of the formula A_1 is: $\Rightarrow$, main connective of the formula A_2 is also $\Rightarrow$.

3. Degree of the formula A_1 is: 11, degree of the formula A_2 is: 6.

4. All proper, non-atomic sub-formulas of A_1 are:

$$\neg\mathbf{I}\neg a, (\neg\mathbf{C}a \cup (\mathbf{C}a \Rightarrow \neg\mathbf{I}b)), \mathbf{I}\neg a, \neg a, \neg\mathbf{C}a, (\mathbf{C}a \Rightarrow \neg\mathbf{I}b), \mathbf{C}a, \neg\mathbf{I}b, \mathbf{I}b$$

5. All non-atomic sub-formulas of A_2 are:

$$(\neg a \Rightarrow (\neg b \cup (b \Rightarrow \neg c)), \neg a, (\neg b \cup (b \Rightarrow \neg c)), \neg b, (b \Rightarrow \neg c), \neg c$$

6. Find a model and a counter-model restricted to A_2. Use short-hand notation. Show work.

A restricted model: $a = T, b = T, c = F$

Evaluation: $(\neg a \Rightarrow (\neg b \cup (b \Rightarrow \neg c)) = T$ for, for example $a = T$ and b, c any truth values. ($F \Rightarrow$ *anything* $= T$).

$a = T$ gives 4 models (2^2 values on b and c.)

A Restricted counter-model: $a = F, b = T$ and $c = T$

Evaluation: $(\neg a \Rightarrow (\neg b \cup (b \Rightarrow \neg c)) = F$ if and only if
$\neg a = T$ and $(\neg b \cup (b \Rightarrow \neg c)) = F$, iff
$a = F, \neg b = F$ and $(b \Rightarrow \neg c) = F$, iff
$a = F, b = T$ and $(T \Rightarrow \neg c) = F$, iff
$a = F, b = T$ and $\neg c = F$ iff
$a = F, b = T$ and $c = T$

7. Statement: *There are more then 3 possible restricted counter-models of* A_2. is not true. There is only one possible counter-model restricted to A_2 as shown by above evaluation.

8. Statement: *There are more then 2 possible restricted models of* A_2. is true. There are 7 possible restricted models of A_2. Justification: $2^3 - 1 = 7$.

9. List 3 models and 3 counter-models for A_2 by extending the model and the counter-model you have found in **5.** to the VAR of all variables.

A model for A_2 is, by definition, any function

$$w : VAR \longrightarrow \{T, F\},$$

such that $w(A_2) = T$.

A restricted model for A_2 is, as defined in **7.** is a function

$$v : \{a, b, c\} \longrightarrow \{T, F\},$$

such that $v(A_2) = T$, i.e. for example:
$A = T, b = T, c = F$.

We extend v to the set of all propositional variables VAR to obtain a (non restricted) model. Here are three of such extensions.

Model w_1

$$w_1 : VAR \longrightarrow \{T, F\}$$

$$w_1(a) = v(a) = T,\ \ w_1(b) = v(b) = T, w_1(c) = v(c) = F,\ \ and\, w_1(x) = T,\ \ for\ \ all\ \ x \in VAR - \{a, b, c\}.$$

Model w_2:

$$w_2(a) = v(a) = T,\ \ \ w_2(b) = v(b) = T, w_2(c) = v(c) = F,\ \ and\ w_2(x) = F,$$

$$for\ \ \ all\ \ \ x \in VAR - \{a, b, c\}.$$

Model w_3:

$$w_3(a) = v(a) = T,\ \ w_3(b) = v(b) = T,\ \ w_3(c) = v(c) = F, w_3(d) = Fand\ w_3(x) = T,$$

$$for\ \ \ all\ \ \ x \in VAR - \{a, b, c, d\}.$$

There is an many of such models, as extensions of v to the set VAR, i.e. as many as real numbers.

A counter-model for A_2, by definition, is any function

$$w : VAR \longrightarrow \{T, F\},$$

such that $w(A_2) = F$.

A restricted counter-model for A_2 is, as defined in **6.** a function

$$v : \{a, b\} \longrightarrow \{T, F\},$$

such that $v(A) = F$, i.e. (only one) such that $v(a) = F,\ v(b) = T,\ v(c) = T$.

There is only one **restricted counter-model** v for A_2.

We extend v to the set of all propositional variables VAR to obtain a (non restricted) counter-models. Here are three of such extensions.

Counter- model w_1:

$$w_1(a) = v(a) = F,\ \ w_1(b) = v(b) = T, w_1(c) = v(c) = T,\ \ and\, w_1(x) = F,\ \ for\ \ all\ \ x \in VAR - \{a, b, c\}.$$

Counter- model w_2:

$$w_2(a)=v(a)=T,\ w_2(b)=v(b)=T,\ w_2(c)=v(c)=T,\ and\ w_2(x)=T,$$

$$for\ \ all\ \ x\in VAR-\{a,b,c\}.$$

There is an many of such counter- models, as extensions of v to the set VAR, i.e. as many as real numbers.

9. There are $2^{\aleph_0}=\mathcal{C}$ possible models for A_2. There are $2^{\aleph_0}=\mathcal{C}$ possible counter-models for A_2.

item[QUESTION 3] Show that $v\models((\neg a\cup b)\Rightarrow(((c\cap d)\Rightarrow\neg d)\Rightarrow(\neg a\cup b)))$ for all $v:VAR\longrightarrow\{T,F\}$, i.e. that

$$\models((\neg a\cup b)\Rightarrow(((c\cap d)\Rightarrow\neg d)\Rightarrow(\neg a\cup b))).$$

Solution: we apply first the substitution method. We substitute:

$$A=(\neg a\cup b),\quad B=((c\cap d)\Rightarrow\neg d)$$

and out initial formula becomes $(A\Rightarrow(B\Rightarrow A))$.

As the second step we show that

$$\models(A\Rightarrow(B\Rightarrow A))$$

using "proof by contradiction" method. We use short hand notation.
Assume that $(A\Rightarrow(B\Rightarrow A)))=F$. This is possible only if $A=T$ and $(B\Rightarrow A)=F$. Substituting $A=T$ in $(B\Rightarrow A)=F$ we get $(B\Rightarrow T)=F$ what is impossible. This proves that $\models(A\Rightarrow(B\Rightarrow A)$ and by substitution theorem, also $\models((\neg a\cup b)\Rightarrow(((c\cap d)\Rightarrow\neg d)\Rightarrow(\neg a\cup b)))$.