CSE541 EXERCISE 11 SOLUTIONS

Chapters 10, 11, 12 Read and learn all examples and exercises in the chapters as well!

QUESTION 1 Use the (complete) proof system **GL** from chapter 11 to prove that

$$\models (\neg(a \cap \neg b) \Rightarrow (\neg a \cup b)).$$

Solution By completeness theorem for **GL** we have that $\models A$ is and only if $\vdash_{GL} \longrightarrow A$. We construct a decomposition tree for our formula as follows.

$$\mathbf{T}_{\rightarrow A}$$

$$\longrightarrow (\neg(a \cap \neg b) \Rightarrow (\neg a \cup b))$$

$$\mid (\rightarrow\Rightarrow)$$

$$\neg(a \cap \neg b) \longrightarrow (\neg a \cup b)$$

$$\mid (\rightarrow \cup)$$

$$\neg(a \cap \neg b) \longrightarrow \neg a, b$$

$$\mid (\rightarrow \neg)$$

$$a, \neg(a \cap \neg b) \longrightarrow b$$

$$\mid (\neg \rightarrow)$$

$$a \longrightarrow (a \cap \neg b), b$$

$$\bigwedge(\rightarrow \cap)$$

$a \longrightarrow a, b$	$a \longrightarrow \neg b, b$
axiom	$\mid (\rightarrow \neg)$
	$a, b \longrightarrow b$
	axiom

All leaves are axioms, hence the tree is a proof of A in **GL**.

QUESTION 2

Find a counter-model determined by a decomposition tree $\mathbf{T}_{\rightarrow A}$ in **GL** for a formula A below.

$$A = ((a \cap \neg b) \Rightarrow (\neg a \cup b))$$

Solution: We construct a decomposition tree for $\rightarrow A$ formula as follows.

$$\mathbf{T}_{\rightarrow A}$$

$$\longrightarrow ((a \cap \neg b) \Rightarrow (\neg a \cup b))$$

$$\mid (\rightarrow \Rightarrow)$$

$$(a \cap \neg b) \longrightarrow (\neg a \cup b)$$

$$\mid (\rightarrow \cup)$$

$$(a \cap \neg b) \longrightarrow \neg a, b$$

$$\mid (\rightarrow \neg)$$

$$a, (a \cap \neg b) \longrightarrow b$$

$$\mid (\cap \rightarrow)$$

$$a, a, \neg b \longrightarrow b, b$$

$$\mid (\rightarrow \neg)$$

$$a, a \longrightarrow b, b, b$$

$$non - axiom$$

We have one non-axiom leave: $L = a, a \longrightarrow b, b, b$. It generates a counter-model: $a = T, b = F$.

QUESTION 3 Prove the COMPLETENESS theorem for GL. Assume that the Soundness has been already proved.

Solution

Formula Completeness for GL: for any $A \in \mathcal{F}$,

$$\models A \;\; iff \;\; \vdash_{GL} \rightarrow A$$

Soundness for GL: for any $A \in \mathcal{F}$,

$$If \vdash_{GL} \rightarrow A, \;\; then \;\; \models A$$

Completeness part for GL: for any $A \in \mathcal{F}$,

$$If \;\; \models A, \; then \vdash_{GL} \rightarrow A$$

We prove the logically equivalent form of the Completeness part: for any $A \in \mathcal{F}$,

$$If \not\vdash_{GL} \rightarrow A \; then \;\; \not\models A,$$

proof Assume $\not\vdash_{GL} \rightarrow A$, i.e. $\rightarrow A$ does not have a proof in GL. Let $\mathcal{T}_A$ be a set of all decomposition trees of $\rightarrow A$. As $\not\vdash_{GL} \rightarrow A$, each $T \in \mathcal{T}_A$ has a non-axiom leaf. We choose an arbitrary $T_A \in \mathcal{T}_A$. Let $\Gamma' \rightarrow \Delta', \Gamma'$ be an non-axiom leaf of T_A, for $\Delta' \in VAR^*$ such that $\{\Gamma'\} \cap \{\Delta'\} = \emptyset$.

The non-axiom leaf $\Gamma' \rightarrow \Delta'$ *defines* a truth assignment $v : VAR \rightarrow \{T, F\}$ which falsifies A, as follows:

$$v(a) = \begin{cases} T & \text{if a appears in } \Gamma' \\ F & \text{if a appears in } \Delta' \end{cases}$$

This proves, by soundness of the rules of inference of GL that $\not\models A$.

QUESTION 4

Show that

$$\vdash_{\mathbf{LI}} \ ((\neg A \cup B) \Rightarrow (A \Rightarrow B)).$$

Solution: We construct a decomposition tree for $\rightarrow ((\neg A \cup B) \Rightarrow (A \Rightarrow B))$ formula as follows.

$$\mathbf{T}_{\rightarrow A}$$

$$\longrightarrow ((\neg A \cup B) \Rightarrow (A \Rightarrow B))$$

$$\mid (\rightarrow\Rightarrow)$$

$$(\neg A \cup B) \longrightarrow (A \Rightarrow B)$$

$$\mid (\rightarrow\Rightarrow)$$

$$A, (\neg A \cup B) \longrightarrow B$$

$$\mid ((exch) \rightarrow)$$

$$(\neg A \cup B), A \longrightarrow B$$

$$\bigwedge(\cup \rightarrow)$$

$\neg A, A \longrightarrow B$	$B, A \longrightarrow B$
$\mid (\rightarrow weak)$	*axiom*
$\neg A, A \longrightarrow$	
$\mid (\neg \rightarrow)$	
$A \longrightarrow A$	
axiom	