QUESTION 1

1. Given a formula \( A = ((B \cap \neg C) \Rightarrow (\neg A \cup B)) \) of a language \( L_1 = L_{\{\neg, \cap, \cup, \Rightarrow\}} \). FIND a formula \( B \) of a language \( L_2 = L_{\{\neg, \Rightarrow\}} \), such that \( A \equiv B \).

   LIST all proper logical equivalences used at each step.

Solution:

\[
A = ((B \cap \neg C) \Rightarrow (\neg A \cup B)) \equiv ((B \cap \neg C) \Rightarrow (A \Rightarrow B)) \equiv (\neg (B \Rightarrow \neg C) \Rightarrow (A \Rightarrow B)) \equiv (\neg (B \Rightarrow C) \Rightarrow (A \Rightarrow B)) = B
\]

Equivalences used:
1. \((\neg A \cup B) \equiv (A \Rightarrow B)\)
2. \((A \cap B) \equiv \neg (A \Rightarrow \neg B)\)
3. \(\neg \neg A \equiv A\)

2. Prove that \( L_1 \equiv L_2 \).

We define the EQUIVALENCE of LANGUAGES as follows:

Given two languages:
\( L_1 = L_{CON_1} \) and \( L_2 = L_{CON_2} \), for \( CON_1 \neq CON_2 \).

We say that they are logically equivalent, i.e.

\[
L_1 \equiv L_2
\]

if and only if the following conditions C1, C2 hold.

C1: For every formula \( A \) of \( L_1 \), there is a formula \( B \) of \( L_2 \), such that

\[
A \equiv B,
\]

C2: For every formula \( C \) of \( L_2 \), there is a formula \( D \) of \( L_1 \), such that

\[
C \equiv D.
\]

Solution: We have to prove that

\[
L_{\{\neg, \Rightarrow\}} \equiv L_{\{\neg, \cap, \cup, \Rightarrow\}}.
\]

Condition C1 holds because \( \{\neg, \Rightarrow\} \subseteq \{\neg, \cap, \cup, \Rightarrow\} \).

Condition C1 holds because

\[(A \cap B) \equiv \neg (A \Rightarrow \neg B) \text{ and } (A \cup B) \equiv (\neg A \Rightarrow B).\]
QUESTION 2

S is the following proof system:

\[ S = (\mathcal{L}_{\Rightarrow, \neg}, \mathcal{F}, AX = \{A_1, A_2, A_3, A_4\}, MP) \]

A1 \hspace{1em} (A \Rightarrow (B \Rightarrow A)),
A2 \hspace{1em} ((A \Rightarrow (B \Rightarrow C)) \Rightarrow ((A \Rightarrow B) \Rightarrow (A \Rightarrow C))),
A3 \hspace{1em} ((\neg B \Rightarrow \neg A) \Rightarrow ((\neg B \Rightarrow A) \Rightarrow B))
A4 \hspace{1em} (((A \Rightarrow B) \Rightarrow B) \Rightarrow A)
MP \hspace{1em} (\text{Rule of inference})

(1) Does Deduction Theorem holds for S? Justify shortly your answer.

Solution : Yes, it does as only axioms A1 and A2 were used in its proof.

(2) Is S COMPLETE with respect to classical semantics? JUSTIFY your answer.

Solution : NO, is NOT Complete as it is not SOUND. Axiom A4 is not a tautology.

QUESTION 3

Let S be from QUESTION 2.

The following Lemma holds in the system S.

LEMMA For any \( A, B, C \in \mathcal{F} \),

(a) \((A \Rightarrow B), (B \Rightarrow C) \vdash_S (A \Rightarrow C)\),
(b) \((A \Rightarrow (B \Rightarrow C)) \vdash_S (B \Rightarrow (A \Rightarrow C))\).

Complete the proof sequence (in S) \( B_1, ..., B_9 \) by providing comments how each step of the proof was obtained.

Solution

\( B_1 = (A \Rightarrow B) \)
Hypothesis
\( B_2 = (\neg \neg A \Rightarrow A) \)
Already Proven
\( B_3 = (\neg \neg A \Rightarrow B) \)
Lemma a for \( A = \neg \neg A, B = A, C = B \), in \( B_1, B_2 \) i.e.

\((\neg \neg A \Rightarrow A), (A \Rightarrow B) \vdash (\neg \neg A \Rightarrow B)\)
\[ B_4 = (B \Rightarrow \neg\neg B) \]
Formula 5

\[ B_5 = (\neg\neg A \Rightarrow \neg\neg B) \]
Lemma a on \( B_3, B_4 \) for \( A = \neg\neg A, B = B, C = \neg\neg B \)

\[ (\neg\neg A \Rightarrow B), (B \Rightarrow \neg\neg B) \vdash (\neg\neg A \Rightarrow \neg\neg B) \]

\[ B_6 = ((\neg\neg A \Rightarrow \neg\neg B) \Rightarrow (\neg B \Rightarrow \neg A)) \]
ALREADY PROVED

\[ B_7 = (\neg B \Rightarrow \neg A) \]
\( B_5, B_6 \) and MP on \( B_5, B_6 \)

\[ \frac{\neg\neg A \Rightarrow \neg\neg B; (\neg\neg A \Rightarrow \neg\neg B) \Rightarrow (\neg B \Rightarrow \neg A)}{(\neg B \Rightarrow \neg A)} \]

\[ B_8 = (A \Rightarrow B) \vdash (\neg B \Rightarrow \neg A) \]
\( B_1 - B_7 \)

\[ B_9 = ((A \Rightarrow B) \Rightarrow (\neg B \Rightarrow \neg A)) \]
Deduction Theorem on \( B_8 \)

**Extra Credit** Prove the above LEMMA (b), i.e prove that

(b) \((A \Rightarrow (B \Rightarrow C)) \vdash_S (B \Rightarrow (A \Rightarrow C)).\)

**Solution** : Deduction Theorem Holds for \( S \). By Deduction Theorem applied twice we have that \((A \Rightarrow (B \Rightarrow C)) \vdash_S (B \Rightarrow (A \Rightarrow C))\) iff

\[ (A \Rightarrow (B \Rightarrow C)), B \vdash_S (A \Rightarrow C) \text{ iff} \]

\[ (A \Rightarrow (B \Rightarrow C)), B, A \vdash_S C \]

The proof of \( C \) from \((A \Rightarrow (B \Rightarrow C)), B, A\) is the following.

\[ B_1 = (A \Rightarrow (B \Rightarrow C)) \]
Hypothesis

\[ B_2 = A \]
Hypothesis

\[ B_3 = (B \Rightarrow C) \]
\( B_1, B_2 \) and MP

\[ B_4 = B \]
Hypothesis

\[ B_5 = C \]
\( BB_3, B_4 \) and MP