Introduction to Predicate Logic

Part 2

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Lecture Notes (2)

Predicate Logic Introduction

Part 2

- Predicate Logic Tautologies;
- Basic Laws of Quantifiers

- Intuitive Semantics for Predicate Logic

Basic Laws of Quantifiers
Predicate Logic Tautologies

De Morgan Laws

$\neg \forall x\, A(x) \equiv \exists x\, \neg A(x)$

$\neg \exists x\, A(x) \equiv \forall x\, \neg A(x)$

where A(x) is any formula with free variable x,

≡ means “logically equivalent”

Definability of Quantifiers

$\forall x\, A(x) \equiv \neg \exists x\, \neg A(x)$

$\exists x\, A(x) \equiv \neg \forall x\, \neg A(x)$

Example

De Morgan and other Laws Application in Mathematical Statements

$$\neg \forall x((x>0 \Rightarrow x+y > 0) \wedge \exists y\, (y<0))$$

$\equiv$ (by De Morgan's Law)

$$\exists x \neg ((x>0 \Rightarrow x+y > 0) \wedge \exists y\, (y < 0))$$

$\equiv$ (by De Morgan's Law and 1., 2., 3., 4.)

$$\exists x((x>0 \wedge x+y \leq 0) \vee \forall y(y \geq 0))$$

We used

1. $\neg (A \Rightarrow B) \equiv (A \wedge \neg B)$, 2. $\neg(A \wedge B) \equiv (\neg A \vee \neg B)$

3. $\neg(x + y) > 0) \equiv x+y \leq 0$

4. $\neg \exists y\, (y <0) \equiv \forall y \neg (y < 0)$

$\equiv \exists y\, (y \geq 0)$

Math Statement--- Logic Formula

Mathematical statement

$\neg \forall x((x>0 \Rightarrow x+y > 0) \wedge \exists y\ (y<0))$

Corresponding Logic Formula is

$\neg \forall x((P(x,c) \Rightarrow R(\ f(x,y),c)\ ) \wedge \exists y\ P(y,c))$

More general; A(x), B(x) any formulas

$\neg \forall x((A(x) \Rightarrow B(\ x,y)) \wedge \exists y\ A(y))$

$\equiv \exists x \neg((A(x) \Rightarrow B(x,y)) \wedge \exists y\ A(y))$

$\equiv \exists x((A(x) \wedge \neg B(\ x,y)\ ) \vee \neg \exists y\ A(y))$

$\equiv \exists x\ ((A(x) \wedge \neg B(\ x,y)) \vee \forall y \neg A\ (y))$

Distributivity Laws

1. $\exists x(A(x) \lor B(x)) \equiv (\exists x\, A(x) \lor \exists x\, B(x))$

   Existential quantifier is distributive over $\lor$

What we write $(\exists x, \lor)$

2. $\forall x\, (A(x) \land B(x)) \equiv (\forall x\, A(x) \land \forall x\, B(x))$

Universal quantifier is distributive over $\land$, $(\forall x\, , \land)$

Existential quantifier is distributive over $\land$ **only in one direction:**

3. $\exists x(A(x) \land B(x)\,) \Rightarrow (\exists x\, A(x) \land \exists x\, B(x))$

Distributivity Laws

We show that **it is not true**, that for any $X \neq \phi$ and any A(x), B(x) the inverse implication

$$(\exists x\, A(x) \wedge \exists x\, B(x)) \Rightarrow \exists x(A(x) \wedge B(x))$$

holds, i.e. that there are $X \neq \phi$ and any A(x), B(x) for which this implication is **FALSE.**

Example: Take: X = R (real numbers),

$A(x) = x > 0$, $B(x) = x<0$ we get

$\exists x\, (x>0) \wedge \exists x(x>0)$ is a **true** statement in R and

$\exists x(x>0 \wedge x< 0)$ is a **false** statement in R.

Distributivity Laws

Universal quantifier is distributive over $\vee$ in only one direction:

4. $((\forall x\, A(x) \vee \forall x\, B(x)) \Rightarrow \forall x(A(x) \vee B(x)))$

Other direction implication counter example:

Take: **X=R** and **A(x) = x < 0** **B(x) = x ≥ 0**

$\forall x\, (x<0 \vee x \geq 0)$ is a **true** statement in **R** (real numbers) and

$\forall x(x<0) \vee \forall x(x \geq 0)$ is **false**

Distributivity Laws

Universal quantifier is distributive over $\Rightarrow$ in one direction only:

5. **$(\forall x(A(x) \Rightarrow B(x)) \Rightarrow (\forall x\, A(x) \Rightarrow \forall x\, B(x)))$**

Other direction implication counter example:

Take: **$X = R$, $A(x) = x < 0$, $B(x) = x+1 > 0$**

$(\forall x(x < 0) \Rightarrow \forall x(x+1 > 0))$ is a **False** statement in set R of Real Numbers

Take $x = -2$, we get $(-2 < 0 \Rightarrow -2+1 > 0)$ False

Introduction and Elimination Laws

B - Formula without free variable x

6. $\forall x(A(x) \Rightarrow B) \equiv (\exists x\, A(x) \Rightarrow B)$
7. $\exists x(A(x) \Rightarrow B) \equiv (\forall x\, A(x) \Rightarrow B)$
8. $\forall x(B \Rightarrow A(x)) \equiv (B \Rightarrow \forall x\, A(x))$
9. $\exists x(B \Rightarrow A(x)) \equiv (B \Rightarrow \exists x\, A(x))$

Introduction and Elimination Laws

B - **Formula without free variable x**

10. $\forall x(A(x) \vee B) \equiv (\forall x\, A(x) \vee B)$

11. $\forall x(A(x) \wedge B) \equiv (\forall x\, A(x) \wedge B)$

12. $\exists x(A(x) \vee B) \equiv (\exists x\, A(x) \vee B)$

13. $\exists x(A(x) \wedge B) \equiv (\exists x\, A(x) \wedge B)$

Remark: we prove 6 -9 from 10 – 13 + de Morgan + definability of implication

TRUTH SETS

We use truth sets for predicates in a set $X \neq \phi$ to define an intuitive semantics for predicate logic.

Given a set $X \neq \phi$ and a predicate P(x),

$\{x \in X: P(x)\}$ is called a truth set for the predicate P(x) in the domain $X \neq \phi$

TRUTH SETS, Interpretations

Example1:

Take P(x) as $x+1 = 3$ – **it is called an interpretation of P(x)** in a set **$X \neq \phi$**

.

Let X={1, 2, 3}, then the **truth set**

$\{x \in X: P(x)\} = \{x \in X: x+1 = 3\} = \{2\}$,

and we say that P(x) is **TRUE** in X **under the interpretation** P(x): $x+1 = 3$

TRUTH SETS, Interpretations

Example2:

Take:

P(x): $x^2 \leq 0$ - Interpretation of P(x) in

$X = N$, the TRUTH Set is

$$\{x \in N: P(x)\} = \{x \in N: x^2 \leq 0\} = \{0\}$$

Take: P(x): $x^2 \leq 0$ - Interpretation of P(x) in

$x = N - \{0\}$, the TRUTH Set is

$$\{x \in N-\{0\}: P(x)\} = \{x \in N-\{0\}: x^2 \leq 0\} = \phi$$

TRUTH SETS semantics for Connectives

We use truth sets for predicates always for $X \neq \phi$

Conjunction:

$\{x \in X: (P(x) \wedge Q(x))\} = \{x: P(x)\} \wedge \{x: Q(x)\}$

Truth set for conjunction $(P(x) \wedge Q(x))$ is the set intersection of truth sets for its components.

Disjunction:

$\{x \in X: (P(x) \vee Q(x))\} = \{x: P(x) \vee \{x: Q(x)\}$

Truth set for disjunction $(P(x) \vee Q(x))$ is the set union of truth sets for its components.

Negation:

$\{x \in X: \neg P(x)\} = X - \{x \in X: P(x)\}$

$\neg$ is the negation and $-$ is the set complement

Truth sets semantics for Connectives

Implication:

$\{x \in X: (P(x) \Rightarrow Q(x))\}$ $= X - \{x:P(x)\} \vee \{x : Q(x)\}$

$= -\{x:P(x)\} \vee \{x : Q(x)\}$

$= \{x: \neg P(x)\} \vee \{x : Q(x)\}$

Example:

$\{x \in N: n>0 \Rightarrow n^2 <0\} = \{x \in N\ x \leq 0\} \vee \{\{x \in N : n^2<0\}$

$=\{0\} \vee \phi = \{0\}$

Truth Sets Semantics for Universal Quantifier

Definition:

$$\forall x\, A(x) = T \quad \text{iff} \quad \{x \in X : A(x)\} = X$$

where

$X \neq \phi$ and $A(x)$ is any formula with a free variable x

Definition:

$$\forall x\, A(x) = F \quad \text{iff} \quad \{x \in X : A(x)\} \neq X$$

where

$X \neq \phi$ and $A(x)$ is any formula with a free variable x

Truth Sets semantics for Existential Quantifier

Definition:

$\exists x\, A(x) = T$ (in $x \neq \phi$) iff $\{x \in X : A(x)\} \neq \phi$

Definition:

$\exists x\, A(x) = F$ (in $x \neq \phi$) iff $\{x \in X : A(x)\} = \phi$

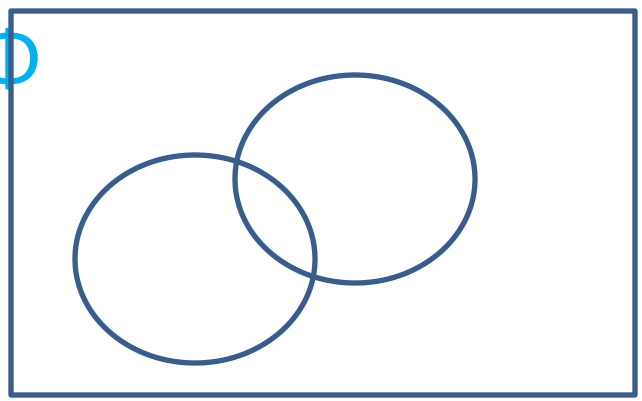
Where $X \neq \phi$ and **A(x) is a formula with a free variable x.**

Venn Diagrams For Existential Quantifier and Conjunction

∃x(A(x) ∧ B(x))=T iff **{x:A(X)} ∧ {x:B(x)} ≠ Φ**

Picture

X≠ Φ

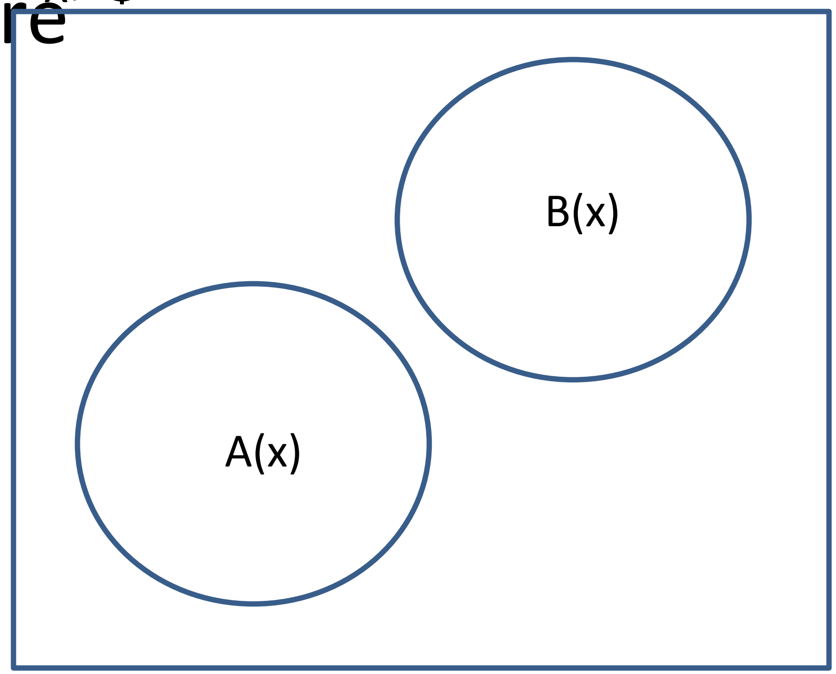


Venn Diagrams For Existential Quantifier and Conjunction

∃x(A(x) ∧ B(x))= F iff {x:A(x) ∧{x:B(x)}= Φ

Picture

X≠ Φ



Remember {x:A(x)}, {x:B(x)}
Can be Φ!

X≠ Φ

Venn Diagrams For Universal Quantifier and Implication

Observe that

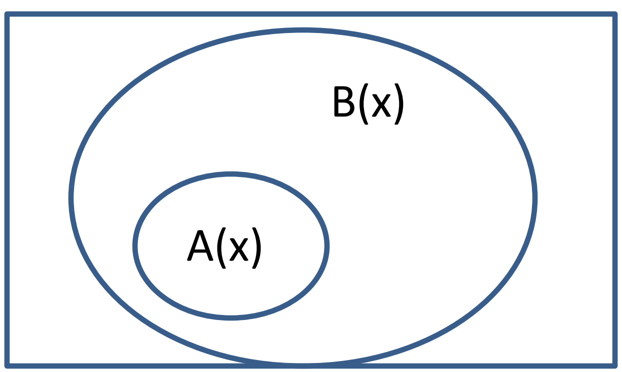
$\forall x\, (A(x) \Rightarrow B(x)) = T$ iff $\{x \in X : A(x) \Rightarrow B(x)\} = X$

Iff

$\{x : A(x)\} \subseteq \{x : B(x)\}$

$X \neq \Phi$

Picture



Venn Diagrams For universal quantifier and Implication

Exercise

Draw a picture for a situation where (in $X \neq \Phi$)

1. **$\exists x\ P(x) = T$,**

2. **$\exists x\ Q(x) = T$,**

3. **$\exists x(P(x) \land Q(x)) = F$ and**

4. **$\forall x\ (P(x) \lor Q(x) = F$**

Exercise Solution

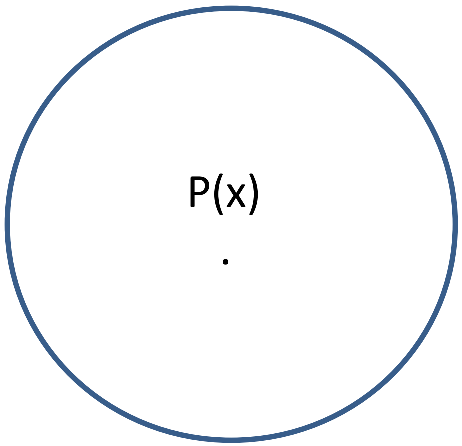
1. $\exists x\, P(x) = T$ iff $\{ x : P(x)\} \neq \Phi$

2. $\exists x\, Q(x) = T$ iff $\{x : Q(x)\} \neq \Phi$

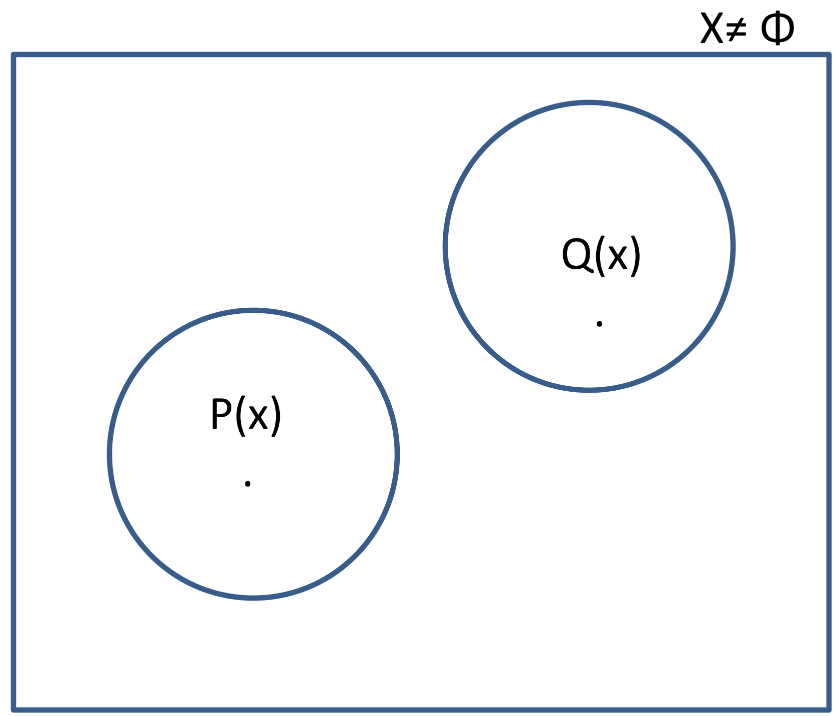
3. $\exists x(P(x) \wedge Q(x)) = F$ iff $\{x: P(x)\} \wedge \{x: Q(x)\} = \Phi$

4. $\forall x\, (P(x) \vee Q(x) = F$ iff $\{x:P(x)\} \vee \{ x:Q(x)\} \neq X$

Picture:



Denotes P(x) ≠ Φ



Proving Predicate Tautologies with TRUTH Sets

Prove that

$$\models (\forall x\, A(x) \Rightarrow \exists x\, A(x))$$

Proof:

Assume that not True

(Proof by contradiction) i.e. that there are X≠ Φ,A(x) such that.

$(\forall x\, A(x) \Rightarrow \exists x\, A(x)) =$ **F**

iff **$\forall x\, A(x)$=T** and **$\exists x\, A(x)$=F** (A ⇒B) = F

iff **X ≠ ϕ** and

{x ∈ X : A(x)} = X and **{x ∈ X : A(x)} = ϕ**

iff X= ϕ

Contradiction with **X ≠ ϕ,** hence proved.

Proving Predicate Tautologies with TRUTH Sets

Prove:

$$\neg \forall x\, A(x) \equiv \exists x\, \neg A(x)$$

Case1: $\exists x\, \neg A(x) = T$ in $X \neq \phi$ **iff** $\{x: \neg A(x)\} \neq \phi$ **iff** $X - \{x: A(x)\} \neq \phi$ **iff** $\{x: A(x)\} \neq X$ **iff** $\forall x\, A(x) = F$ **iff** $\neg \forall x\, A(x) = T$

Case1: $\exists x\, \neg A(x) = F$ in $X \neq \phi$ **iff** $\{x: \neg A(x)\} = \phi$ **iff** $X - \{x: A(x)\} = \phi$ **iff** $\{x: A(x)\} = X$ **iff** $\forall x\, A(x) = T$ **iff** $\neg \forall x\, A(x) = F$

Prove

$\exists x(A(x) \lor B(x)) \equiv \exists x\, A(x) \lor \exists x\, B(x)$

Case 1: $\exists x(A(x) \lor B(x)\} = T$ iff

$\{x: (A(x) \lor B(x)\} \neq \phi$ (definition)

$= \{x: (A(x)\} \lor \{x: (B(x)\} \neq \phi$ iff

$\{x: A(x)\} \neq \phi$ or $\{x: B(x)\} \neq \phi$ iff

$= \exists x\, A(x)=T$ or $\exists x\, B(x)=T$

We used: for any sets, $A \lor B \neq \phi$ iff
$A \neq \phi$ and $B \neq \phi$

Case2 – similar

We assume that for any A(x), the TRUTH set $\{x \in X: A(x)\}$ exists .

Russell Antinomy showed that that technique of TRUTH sets is not sufficient.

This is why we need a proper semantics!