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Chapter 7
Introduction to Intuitionistic and Modal Logics

CHAPTER 7 SLIDES

Chapter 7
Introduction to Intuitionistic and Modal Logics

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Introduction to Intuitionistic and Modal Logics

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PART 1: **Intuitionictic Logic:** Philosophical Motivation

Intuitionictic Logic: Philosophical Motivation

Intuitionistic logic has developed as a result of certain philosophical views on the foundation of mathematics, known as intuitionism

Intuitionism was originated by L. E. J. Brouwer in 1908

The first Hilbert style formalization of the intuitionistic logic, formulated as a proof system, is due to A. Heyting (1930)

We **present** a Hilbert style proof system I that is equivalent to the Heyting's original formalization

We also **discuss** the relationship between intuitionistic and classical logic.

Intuitionictic Logic: Philosophical Motivation

There have been several successful attempts at creating **semantics** for the intuitionistic logic
The most **recent** called Kripke models were defined by Kripke in 1964

The **first** intuitionistic semantics was defined in a form of **pseudo-Boolean** algebras by McKinsey and Tarski in years 1944 - 1946
Their **algebraic** approach to intuitionistic and classical semantics was followed by many authors and developed into a new field of **Algebraic Logic**

The pseudo- Boolean algebras are called also Heyting algebras to memorize his **first** accepted formalization of the intuitionistic logic as a proof system

Intuitionictic Logic: Philosophical Motivation

An uniform presentation of **algebraic models** for classical, intuitionistic and modal logics S4, S5 was first given in a now classic **algebraic logic** book:

"Mathematics of Metamathematics", Rasiowa, Sikorski (1964)

The main **goal** of this chapter is to give a presentation of the intuitionistic logic formulated as Hilbert and Gentzen proof systems

We also discuss its **algebraic** semantics and the fundamental theorems that establish the relationship between classical and intuitionistic propositional logics

Intuitionictic Logic: Philosophical Motivation

Intuitionists' view-point on the **meaning** of the basic logical and set theoretical concepts used in mathematics **is different** from that of most mathematicians use in their research

The basic **difference** between the intuitionist and classical mathematician lies in the **interpretation** of the word exists

For example, let $A(x)$ be a statement in the arithmetic of natural numbers. For the mathematicians the sentence $\exists x A(x)$ is **true** if it is a theorem of arithmetic

If a mathematician **proves** sentence $\exists x A(x)$ this **does not** always mean that he is able to indicate a method of construction of a natural number n such that $A(n)$ holds

Intuitionictic Logic: Philosophical Motivation

Moreover, the mathematician often obtains the **proof** of the existential sentence $\exists x A(x)$ by **proving** first a sentence

$$\neg \forall x \, \neg A(x)$$

Next he makes use of a classical tautology

$$(\neg \forall x \, \neg A(x)) \Rightarrow \exists x A(x))$$

By applying Modus Ponens he obtains the **proof** of the existential sentence

$$\exists x A(x)$$

For the intuitionist such method is **not acceptable**, for it **does not** give any method of **constructing** a number n such that $A(n)$ holds

Intuitionictic Logic: Philosophical Motivation

For this reason the intuitionist **do not accept** the classical tautology

$$(\neg \forall x \neg A(x)) \Rightarrow \exists x A(x))$$

as intuitionistic **tautology** or as as an intuitionistically **provable** sentence

Intuitionictic Logic: Philosophical Motivation

We denote by $\vdash_I A$, $\models_I A$ that a formula A is intuitionistically **provable**, and is intuitionistic **tautology**, respectively

The **proof system** I for the intuitionistic logic has to be such that

$$\nvdash_I \ (\neg\forall x \, \neg A(x)) \Rightarrow \exists x A(x))$$

and the **intuitionistic semantics** I has to be such that

$$\not\models_I \ (\neg\forall x \, \neg A(x)) \Rightarrow \exists x A(x))$$

Intuitionictic Logic: Philosophical Motivation

The intuitionists interpret **differently** the meaning of propositional connectives

Intuitionistic implication

The intuitionistic implication $(A \Rightarrow B)$ is considered to be **true** if there exists a method by which a proof of B can be **deduced** from the proof of A

For **example**, in the case of the implication

$$i(\neg\forall x\, \neg A(x)) \Rightarrow \exists x A(x))$$

there is no general method which, from a proof of the sentence

$$(\neg\forall x\, \neg A(x))$$

permits us to **obtain** an intuitionistic proof of the sentence

$$\exists x A(x)$$

Intuitionictic Logic: Philosophical Motivation

Intuitionistic negation

The sentence $\neg A$ is considered intuitionistically **true** only if the **acceptance** of the sentence A leads to **absurdity**

As a result of above understanding of negation and implication we have that in the intuitionistic **proof system** I

$$\vdash_I (A \Rightarrow \neg\neg A) \quad \text{but} \quad \nvdash_I (\neg\neg A \Rightarrow A)$$

Consequently, the intuitionistic **semantics** I has to be such that

$$\models_I (A \Rightarrow \neg\neg A) \quad \text{and} \quad \not\models_I (\neg\neg A \Rightarrow A)$$

Intuitionistic disjunction

The intuitionist regards a **disjunction** $(A \cup B)$ as **true** only if one of the sentences A, B is **true** and there is a method
by which it is possible to find out which of them is **true**

As a consequence a classical law of **excluded middle**

$$(A \cup \neg A)$$

is not acceptable by the intuitionists

This means that the the intuitionistic proof system I must be such that

$$\nvdash_I (A \cup \neg A)$$

and the intuitionistic semantics I has to be such that

$$\not\models_I (A \cup \neg A)$$

Chapter 7
Introduction to Intuitionistic and Modal Logics

PART 2: **Intuitionistic** Proof System I

Algebraic Semantics and Completeness Theorem

Intuitionistic Proof System I

We define now a Hilbert style **proof system** I with a set of axioms that is due to Rasiowa (1959). We adopted this axiomatization for two reasons

First reason is that it is the most natural and appropriate set of axioms to carry the the algebraic proof of the **completeness theorem**

Second reason is that they clearly describe the main **difference** between intuitionistic and classical logic

Namely, by **adding** to I the only one more axiom

$$(A \cup \neg A)$$

we get a **complete** formalization for classical logic

Intuitionistic Proof System I

Here are the components if the proof system I

Language

We adopt a propositional language

$$\mathcal{L} = \mathcal{L}_{\{\cup, \cap, \Rightarrow, \neg\}}$$

with the set of formulas $\mathcal{F}$

Axioms

A1 $((A \Rightarrow B) \Rightarrow ((B \Rightarrow C) \Rightarrow (A \Rightarrow C)))$

A2 $(A \Rightarrow (A \cup B))$

A3 $(B \Rightarrow (A \cup B))$

A4 $((A \Rightarrow C) \Rightarrow ((B \Rightarrow C) \Rightarrow ((A \cup B) \Rightarrow C)))$

A5 $((A \cap B) \Rightarrow A)$

A6 $((A \cap B) \Rightarrow B)$

A7 $((C \Rightarrow A) \Rightarrow ((C \Rightarrow B) \Rightarrow (C \Rightarrow (A \cap B)))$

Intuitionistic Proof System I

A7 $((C \Rightarrow A) \Rightarrow ((C \Rightarrow B) \Rightarrow (C \Rightarrow (A \cap B)))$

A8 $((A \Rightarrow (B \Rightarrow C)) \Rightarrow ((A \cap B) \Rightarrow C))$

A9 $(((A \cap B) \Rightarrow C) \Rightarrow (A \Rightarrow (B \Rightarrow C))$,

A10 $(A \cap \neg A) \Rightarrow B)$,

A11 $((A \Rightarrow (A \cap \neg A)) \Rightarrow \neg A)$,

where A, B, C are any formulas in $\mathcal{L}$

Rules of inference

We adopt the Modus Ponens

$$(MP)\ \frac{A\ ;\ (A \Rightarrow B)}{B}$$

as the **only** rule of inference

Intuitionistic Proof System I

A proof system

$$\mathbf{I} = (\ \mathcal{L}, \mathcal{F}\ A1 - A11,\ (MP)\ )$$

for axioms A1 - A11 defined above is called a Hilbert style **formalization** for intuitionistic propositional logic

We introduce, as usual, the notion of a **formal proof** in I and denote by

$$\vdash_I A$$

the fact that a formula A has a formal **proof** in I or that A is **provable** in I

Algebraic Semantics and Completeness Theorem

Algebraic Semantics

We present now a short version of Tarski, Rasiowa, and Sikorski **psedo-Boolean** algebra semantics

We also discuss the algebraic **completeness theorem** for the intuitionistic propositional logic

We leave the **Kripke semantics** for the reader to **explore** from other, multiple sources

Algebraic Semantics

Here are some **basic** definitions

Relatively Pseudo-Complemented Lattice (Birkhoff, 1935)

A lattice

$$(B, \cap, \cup)$$

is said to be relatively pseudo-complemented if and only if for any elements $a, b \in B$, there exists the **greatest** element c, such that

$$a \cap c \leq b$$

Such greatest element c is denoted by $a \Rightarrow b$ and called the **pseudo-complement** of a **relative** to b

Directly from definition we have that

$$(*)\quad x \leq a \Rightarrow b \quad \text{if and only if} \quad a \cap x \leq b \quad \text{for all} \quad x, a, b \in B$$

This equation $(*)$ can serve as the **definition** of the relative pseudo-complement $a \Rightarrow b$

Fact

Every relatively pseudo-complemented lattice $(B, \cap, \cup)$ has the **greatest** element, called a unit element and denoted by 1

Proof

Observe that $a \cap x \leq a$ for all $x, a \in B$

By $(*)$ we have that $x \leq a \Rightarrow a$ for all $x \in B$

This means that $a \Rightarrow a$ is the greatest element in the lattice $(B, \cap, \cup)$. We write it as

$$a \Rightarrow a = 1$$

Definition

An abstract algebra

$$\mathcal{B} = (B,\ 1,\ \Rightarrow,\ \cap,\ \cup)$$

is said to be a **relatively pseudo-complemented lattice** if and only if $(B, \cap, \cup)$ is relatively pseudo-complemented lattice with the relative pseudo-complement $\Rightarrow$ defined by the equation

$$(*)\quad x \leq a \Rightarrow b \quad \text{if and only if} \quad a \cap x \leq b \quad \text{for all} \quad x, a, b \in B$$

and with the unit element 1

Relatively Pseudo-complemented Set Lattices

Consider a **topological** space X with an interior operation I
Let $\mathcal{G}(X)$ be the class of all open subsets of X and
$\mathcal{G}^*(X)$ be the class of all both dense and open subsets of X
Then the algebras

$$(\mathcal{G}(X),\ X,\ \cup,\ \cap, \Rightarrow),\quad (\mathcal{G}^*(X),\ X,\ \cup,\ \cap, \Rightarrow)$$

where $\cup,\ \cap$ are set-theoretical operations of union, intersection, and $\Rightarrow$ is defined by

$$Y \Rightarrow Z = I(X - Y) \cup Z$$

are relatively pseudo-complemented lattices
Clearly, all sub algebras of these algebras are also relatively pseudo-complemented lattices They are typical examples of relatively pseudo-complemented lattices

Algebraic Semantics

Pseudo - Boolean Algebra (Heyting Algebra)

An algebra

$$\mathcal{B} = (B,\ 1,\ 0,\ \Rightarrow,\ \cap,\ \cup, \neg)$$

is said to be a **pseudo - Boolean** algebra if and only if

$$(B,\ 1,\ \Rightarrow,\ \cap,\ \cup)$$

is a relatively pseudo-complemented lattice in which a zero element 0 exists and $\neg$ is a one argument operation defined as follows

$$\neg a = a \Rightarrow 0$$

The operation $\neg$ is called a **pseudo-complementation**

The **pseudo - Boolean** algebras are also called **Heyting** algebras to stress their connection to the intuitionistic logic

Algebraic Semantics

Let X be **topological** space with an interior operation I

Let $\mathcal{G}(X)$ be the class of all open subsets of X

Then

$$(\mathcal{G}(X),\ X,\ \emptyset,\ \cup,\ \cap, \Rightarrow,\ \neg)$$

where $\cup$, $\cap$ are set-theoretical operations of union, intersection, and $\Rightarrow$ is defined by

$$Y \Rightarrow Z = I(X - Y) \cup Z$$

and $\neg$ is defined as

$$\neg Y = Y \Rightarrow \emptyset = I(X - Y), \quad \text{for all } \ Y \subseteq X$$

is a **pseudo - Boolean** algebra

Every sub algebra of $\mathcal{G}(X)$ is also a pseudo-Boolean algebra

They are called **pseudo-fields of sets**

Algebraic Semantics

The following theorem states that pseudo-fields are typical **examples** of pseudo - Boolean algebras.

The theorems of this type are often called **Stone Representation Theorems** to remember an American mathematician H. M. Stone

Stone was one of the **first** to initiate the investigations of **relationship** between logic and general topology in the article

"The Theory of Representations for Boolean Algebras", Trans. of the Amer.Math, Soc 40, 1936

Algebraic Semantics

Representation Theorem (McKinsey, Tarski, 1946)

For every **pseudo - Boolean** algebra

$$\mathcal{B} = (B,\ 1,\ 0,\ \Rightarrow,\ \ \cap,\ \cup, \neg)$$

there exists a monomorphism h of $\mathcal{B}$ into a **pseudo-field** $\mathcal{G}(X)$ of all open subsets of a compact topological T_0 space X

Intuitionistic Algebraic Model

We say that a formula A is an **intuitionistic tautology**

if and only if

any **pseudo-Boolean** algebra $\mathcal{B}$ is a **model** for A

This kind of **models** because their connection to abstract algebras are called **algebraic models**

We put it formally as follows.

Intuitionistic Algebraic Model

Let A be a formula of the language $\mathcal{L}_{\{\cup,\cap,\Rightarrow,\neg\}}$ and let

$$\mathcal{B} = (B,\ 1,\ 0,\ \Rightarrow,\ \ \cap,\ \cup, \neg)$$

be a **pseudo - Boolean** algebra

We say that the algebra $\mathcal{B}$ is a **model** for the formula A and denote it by

$$\mathcal{B} \models A$$

if and only if $v^*(A) = 1$ holds for all variables assignments

$$v : VAR \longrightarrow B$$

Intuitionistic Tautology

The formula A is an **intuitionistic tautology** and is denoted by

$$\models_I A$$

if and only if

$$\mathcal{B} \models A \quad \text{for all pseudo-Boolean algebras } \mathcal{B}$$

In **Algebraic Logic** the notion of tautology is often defined using a notion

"a formula A is **valid** in an algebra $\mathcal{B}$"

It is formally defined as follows

Intuitionistic Tautology

Definition

A formula A **is valid** in a pseudo-Boolean algebra

$$\mathcal{B} = (B,\ 1,\ 0,\ \Rightarrow,\ \cap,\ \cup, \neg)$$

if and only if $v^*(A) = 1$ holds for all variables assignments $v : VAR \longrightarrow B$

Directly from definitions we get the following

Fact

For any formula A,

$\models_I A$ if and only if A is **valid**

in all pseudo-Boolean algebras $\mathcal{B}$

The **Fact** is often used as an equivalent **definition** of the intuitionistic tautology

Intuitionistic Completeness

We write now $\vdash_I A$ to denote **any** proof system for the intuitionistic propositional logic, and in particular the Rasiowa (1959) proof system we have defined

Intuitionistic Completeness Theorem (Mostowski 1948)

For any formula A of $\mathcal{L}_{\{\cup, \cap, \Rightarrow, \neg\}}$,

$$\vdash_I A \quad \text{if and only if} \quad \models_I A$$

The intuitionistic completeness theorem follows directly from the general **algebraic completeness theorem** that combines results of of Mostowski (1958), Rasiowa (1951) and Rasiowa-Sikorski (1957)

Algebraic Completeness Theorem

For any formula A he following conditions are equivalent

(i) $\vdash_I A$

(ii) $\models_I A$

(iii) A is **valid** in every pseudo-Boolean algebra

$$(\mathcal{G}(X),\ X,\ \emptyset,\ \cup,\ \cap, \Rightarrow,\ \neg)$$

of open subsets of any topological space X

(iv) A is **valid** in every pseudo-Boolean algebra $\mathcal{B}$ with at most 2^{2^r} elements, where r is the number of all sub formulas of A

Moreover, each of the conditions **(i)** - **(iv)** is equivalent to the following one.

(v) A is **valid** in the pseudo-Boolean algebra $(\mathcal{G}(X),\ X,\ \emptyset,\ \cup,\ \cap, \Rightarrow,\ \neg)$ of open subsets of a dense-in -itself metric space $X \neq \emptyset$ (in particular of an n-dimensional Euclidean space X)

Chapter 7
Introduction to Intuitionistic and Modal Logics

PART 3: **Intuitionistic Tautologies** and Connection with Classical Tautologies

Intuitionistic Tautologies

Here are some important **basic** classical tautologies

that are also intuitionistic **tautologies**

$(A \Rightarrow A)$

$(A \Rightarrow (B \Rightarrow A))$

$(A \Rightarrow (B \Rightarrow (A \cap B)))$

$((A \Rightarrow (B \Rightarrow C)) \Rightarrow ((A \Rightarrow B) \Rightarrow (A \Rightarrow C)))$

$(A \Rightarrow \neg\neg A)$

$\neg(A \cap \neg A)$

$((\neg A \cup B) \Rightarrow (A \Rightarrow B))$

Of course, all of logical axioms A1 - A11 of the proof system **I**

are also classical and intuitionistic tautologies

Intuitionistic Tautologies

Here are some **more** of important classical tautologies that are intuitionistic **tautologies**

$((\neg A \cup B) \Rightarrow (A \Rightarrow B))$

8. $(\neg(A \cup B) \Rightarrow (\neg A \cap \neg B))$

$((\neg A \cap \neg B) \Rightarrow (\neg(A \cup B))$

$((\neg A \cup \neg B) \Rightarrow \neg(A \cap B))$

$((A \Rightarrow B) \Rightarrow (\neg B \Rightarrow \neg A))$

$((A \Rightarrow \neg B) \Rightarrow (B \Rightarrow \neg A))$

$(\neg\neg\neg A \Rightarrow \neg A)$

$(\neg A \Rightarrow \neg\neg\neg A)$

$(\neg\neg(A \Rightarrow B) \Rightarrow (A \Rightarrow \neg\neg B))$

$((C \Rightarrow A) \Rightarrow ((C \Rightarrow (A \Rightarrow B)) \Rightarrow (C \Rightarrow B))$

Intuitionistic Tautologies

Here are some important classical tautologies that **are not** intuitionistic **tautologies**

$(A \cup \neg A)$

$(\neg\neg A \Rightarrow A)$

$((A \Rightarrow B) \Rightarrow (\neg A \cup B))$

$(\neg(A \cap B) \Rightarrow (\neg A \cup \neg B))$

$((\neg A \Rightarrow B) \Rightarrow (\neg B \Rightarrow A))$

$((\neg A \Rightarrow \neg B) \Rightarrow (B \Rightarrow A))$

$(((A \Rightarrow B) \Rightarrow A) \Rightarrow A)$

Connection Between Classical and Intuitionistic Logics

Connection Between Classical and Intuitionistic Logics

The first **connection** is quite obvious

It was proved by Rasiowa, Sikorski in 1964 that by adding the axiom

A12 $(A \cup \neg A)$

to the set of of logical axioms A1 - A11 of the proof system **I** we obtain a proof system **C** that is **complete** with respect to classical semantics

This proves the following

Theorem 1

Every formula that is intuitionistically derivable is also classically derivable, i.e. the implication

$$\text{If} \quad \vdash_I A \quad \text{then} \quad \vdash_C A$$

holds for any $A \in \mathcal{F}$

Classical and Intuitionistic Logics

We write $\models A$ and $\models_I A$ to denote that A is a classical and intuitionistic tautology, respectively.

As both proof systems I and C are **complete** under respective semantics, we can re-write **Theorem 1** as the following **relationship** between classical and intuitionistic tautologies

Theorem 2

For any formula $A \in \mathcal{F}$,

$$\text{If } \models_I A, \text{ then } \models A$$

Classical and Intuitionistic Logics

The next **relationship** shows how to obtain intuitionistic tautologies from the classical tautologies and vice versa

The following has been proved by Glivenko in 1929 and independently by Tarski in 1938

Theorem 3 (Glivenko, Tarski)

For any formula $A \in \mathcal{F}$,

A is classically provable if and only if $\neg\neg A$ is intuitionistically provable, i.e.

$$\vdash A \quad \text{if and only if} \quad \vdash_I \neg\neg A$$

where we use symbol $\vdash$ for classical provability

Classical and Intuitionistic Logics

Theorem 4 (Tarski, 1938)

For any formula $A \in \mathcal{F}$,

A is a classical tautology if and only if $\neg\neg A$ is an intuitionistic tautology, i.e.

$$\models A \quad \text{if and only if} \quad \models_I \neg\neg A$$

Classical and Intuitionistic Logics

Theorem 5 (Gödel, 1931)

For any formulas $A, B \in \mathcal{F}$,

a formula $(A \Rightarrow \neg B)$ is classically provable if and only if it is intuitionistically provable, i.e.

$$\vdash (A \Rightarrow \neg B) \quad \text{if and only if} \quad \vdash_I (A \Rightarrow \neg B)$$

Theorem 6 (Gödel, 1931)

For any formula $A, B \in \mathcal{F}$,

If A contains **no connectives** except $\cap$ and $\neg$,

then A is classically provable if and only if it is intuitionistically provable, i.e

$$\vdash A \quad \text{if and only if} \quad \vdash_I A$$

Classical and Intuitionistic Logics

By the completeness of classical and intuitionisctic logics we get the following **semantic** version of Gödel' s **Theorems 5, 6**

Theorem 7

A formula $(A \Rightarrow \neg B)$ is a classical tautology if and only if it is an intuitionistic tautology, i.e.

$$\models (A \Rightarrow \neg B) \quad \text{if and only if} \quad \models_I (A \Rightarrow \neg B)$$

Theorem 8

If a formula A contains no connectives except $\cap$ and $\neg$, then

$$\models A \quad \text{if and only if} \quad \models_I A$$

On intuitionistically derivable disjunction

In classical logic it is possible for the disjunction

$$(A \cup B)$$

to be a **tautology** when neither A nor B is a **tautology**

The tautology $(A \cup \neg A)$ is the simplest example

This does not hold for the intuitionistic logic

This fact was **stated** without the proof by Gödel in 1931 and **proved** by Gentzen in 1935 via his proof system **LI** which was discussed shortly in chapter 6 and is covered in detail in this chapter and the next set of slides

On intuitionistically derivable disjunction

The following theorem was announced without proof by Gödel in 1931 and proved by Gentzen in 1935

Theorem 9 (Gödel, Gentzen)

A disjunction $(A \cup B)$ is intuitionistically provable if and only if either A or B is intuitionistically provable i.e.

$$\vdash_I (A \cup B) \quad \text{if and only if} \quad \vdash_I A \;\text{ or }\; \vdash_I B$$

We obtain, via the **Completeness Theorems** the following semantic version of the above

Theorem 10

A disjunction $(A \cup B)$ is intuitionistic tautology if and only if either A or B is intuitionistic tautology, i.e.

$$\models_I (A \cup B) \quad \text{if and only if} \quad \models_I A \;\text{ or }\; \models_I B$$

Chapter 7
Introduction to Intuitionistic and Modal Logics

Slides Set 2

PART 4: Gentzen Sequent System **LI**

Gentzen Sequent System **LI**

G. Gentzen formulated in 1935 a first syntactically decidable (in propositional case) proof systems for classical and intuitionistic logics

He proved their equivalence with their well established, respective Hilbert style formalizations

He **named** his classical system **LK** (K for Klassisch) and intuitionistic system **LI** (I for Intuitionistisch)

Gentzen Sequent System **LI**

In order to prove the **completeness** of the system **LK** and to prove the **adequacy** of **LI** he introduced a special inference rule, called **cut** rule that **corresponds** to the Modus Ponens rule in Hilbert style proof systems

Then, as the next step he proved the now famous **Hauptzatz**, called in English the **Cut Elimination Theorem**

Gentzen Sequent System **LI**

Gentzen original proof system **LI** is a particular case of his proof system **LK** for the classical logic

Both of them are presented in chapter 6 together with the original Gentzen's proof of the **Hauptzatz** for both, **LK** and **LI** proof systems

The elimination of the **cut** rule and the **structure** of other rules makes it possible to define effective automatic procedures for **proof** search, what is impossible in a case of the Hilbert style systems

LI Sequents

The Gentzen system **LI** is defined as follows.

Let

$$SQ = \{\ \Gamma \longrightarrow \Delta :\ \Gamma, \Delta \in \mathcal{F}^*\ \}$$

be the set of all Gentzen sequents built out of the formulas of the language

$$\mathcal{L} = \mathcal{L}_{\{\cup, \cap, \Rightarrow, \neg\}}$$

and the additional Gentzen arrow symbol $\longrightarrow$

We assume that all **LI** sequents are elements of a following subset *ISQ* of the set *SQ* of all sequents

$$ISQ = \{\Gamma \longrightarrow \Delta :\ \Delta \text{ consists of at most one formula } \}$$

The set *ISQ* is called the set of all **intuitionistic sequents**; the **LI** sequents

Axioms of **LI**

Logical Axioms of **LI** consist of any sequent from the set *ISQ* which contains a formula that appears on both sides of the sequent arrow $\longrightarrow$, i.e any sequent of the form

$$\Gamma,\ A,\ \Delta\ \longrightarrow\ A$$

for $\Gamma, \Delta \in \mathcal{F}^*$

Rules of Inference of **LI**

The set inference rules of **LI** is divided into two groups : the **structural rules** and the **logical rules**

There are three **Structural Rules** of **LI**: Weakening, Contraction and Exchange

Weakening structural rule

$$(weak \rightarrow) \quad \frac{\Gamma \longrightarrow \Delta}{A, \Gamma \longrightarrow \Delta}$$

$$(\rightarrow weak) \quad \frac{\Gamma \longrightarrow}{\Gamma \longrightarrow A}$$

A is called the weakening formula

Remember that Δ contains at most one formula

Rules of Inference of **LI**

Contraction structural rule

$$(contr \rightarrow) \quad \frac{A, A, \Gamma \longrightarrow \Delta}{A, \Gamma \longrightarrow \Delta}$$

A is called the contraction formula

Remember that Δ contains at most one formula

The rule below is **not VALID** for **LI**; we list it as it is used in the classical case

$$(\rightarrow contr) \quad \frac{\Gamma \longrightarrow \Delta, A, A}{\Gamma \longrightarrow \Delta, A}$$

Rules of Inference of **LI**

Exchange structural rule

$$(exch \to) \quad \frac{\Gamma_1,\ A, B,\ \Gamma_2 \ \longrightarrow\ \Delta}{\Gamma_1,\ B, A,\ \Gamma_2 \ \longrightarrow\ \Delta}$$

Remember that Δ contains at most one formula

The rule below is **not VALID** for **LI**; we list it as it is used in the classical case

$$(\to exch) \quad \frac{\Delta \longrightarrow \Gamma_1,\ A, B,\ \Gamma_2}{\Delta \longrightarrow \Gamma_1,\ B, A,\ \Gamma_2}.$$

Rules of Inference of **LI**

Logical Rules

Conjunction rules

$$(\cap \rightarrow) \quad \frac{A, B, \Gamma \longrightarrow \Delta}{(A \cap B), \Gamma \longrightarrow \Delta},$$

$$(\rightarrow \cap) \quad \frac{\Gamma \longrightarrow A \; ; \; \Gamma \longrightarrow B}{\Gamma \longrightarrow (A \cap B)}$$

Remember that Δ contains at most one formula

Rules of Inference of **LI**

Disjunction rules

$$(\longrightarrow \cup)_1 \quad \frac{\Gamma \longrightarrow A}{\Gamma \longrightarrow (A \cup B)}$$

$$(\longrightarrow \cup)_2 \quad \frac{\Gamma \longrightarrow B}{\Gamma \longrightarrow (A \cup B)}$$

$$(\cup \longrightarrow) \quad \frac{A, \Gamma \longrightarrow \Delta \quad ; \quad B, \Gamma \longrightarrow \Delta}{(A \cup B), \Gamma \longrightarrow \Delta}$$

Remember that Δ contains at most one formula

Rules of Inference of **LI**

Implication rules

$$(\rightarrow\Rightarrow) \quad \frac{A, \Gamma \longrightarrow B}{\Gamma \longrightarrow (A \Rightarrow B)}$$

$$(\Rightarrow\rightarrow) \quad \frac{\Gamma \longrightarrow A \quad ; \quad B, \Gamma \longrightarrow \Delta}{(A \Rightarrow B), \Gamma \longrightarrow \Delta}$$

Remember that Δ contains at most one formula

Gentzen System **LI**

Negation rules

$$(\neg \rightarrow) \quad \frac{\Gamma \longrightarrow A}{\neg A, \; \Gamma \longrightarrow}$$

$$(\rightarrow \neg) \quad \frac{A, \Gamma \longrightarrow}{\Gamma \longrightarrow \neg A}$$

We define the Gentzen system **LI** as

$$\mathbf{LI} = (\mathcal{L}, \; ISQ, \; LA, \; \text{Structural rules}, \; \text{Logical rules} \,)$$

LI Completeness

The completeness of the **cut-free** **LI** follows directly from **LI Hauptzatz** proved in chapter 6 and the **intuitionistic completeness** (Mostowski 1948)

Completeness of LI

For any sequent $\Gamma \longrightarrow \Delta \in ISQ$,

$$\vdash_{LI} \Gamma \longrightarrow \Delta \quad \text{if and only of} \quad \models_{I} \Gamma \longrightarrow \Delta$$

In particular, for any formula A,

$$\vdash_{LI} A \quad \text{if and only of} \quad \models_{I} A$$

Intuitionistic Disjunction

The particular form the following theorem was stated without the proof by Gödel in 1931

The theorem proved by Gentzen in 1935 via **Hauptzatz** and we follow his proof

Intuitionistically Derivable Disjunction

For any formulas $A, B \in \mathcal{F}$,

$$\vdash_{LI} (A \cup B) \quad \text{if and only if} \quad \vdash_{LI} A \quad \text{or} \quad \vdash_{LI} B$$

In particular, a disjunction $(A \cup B)$ is intuitionistically **provable** in any proof system I if and only if either A or B is intuitionistically **provable** in I

Intuitionistic Disjunction

Proof of

$$\vdash_{LI} (A \cup B) \quad \text{if and only if} \quad \vdash_{LI} A \ \text{ or } \ \vdash_{LI} B$$

Assume $\vdash_{LI} (A \cup B)$

This equivalent to $\vdash_{LI} \longrightarrow (A \cup B)$

The last step in the proof of $\longrightarrow (A \cup B)$ in **LI** must be the application of the rule $(\rightarrow \cup)_1$ to the sequent $\longrightarrow A$, or the application of the rule $(\rightarrow \cup)_2$ to the sequent $\longrightarrow B$

There is no other possibilities

We have proved that $\vdash_{LI} (A \cup B)$ implies $\vdash_{LI} A$ or $\vdash_{LI} B$

The inverse implication is obvious by respective applications of rules $(\rightarrow \cup)_1$ or $(\rightarrow \cup)_2$ to the sequents $\longrightarrow A$ or $\longrightarrow B$

Decomposition Trees in **LI**

Decomposition Trees in **LI**

Search for proofs in **LI** is a much more complicated process then the one in classical logic systems **RS** or **GL** defined in chapter 6

Here, as in any other Gentzen style proof system, proof search procedure consists of building the **decomposition** trees

Remark 1

In **RS** the decomposition tree $\mathbf{T}_A$ of any formula A is always unique

Remark 2

In **GL** the "blind search" defines, for any formula A a **finite** number of decomposition trees,

Nevertheless, it can be proved that the search can be reduced to examining only **one** of them, due to the absence of structural rules

Decomposition Trees in **LI**

Remark 3

In **LI** the structural rules play a **vital role** in the proof construction and hence, in the proof search

The fact that a given **decomposition** tree ends with an non-axiom leaf **does not** always imply that the proof does not exist

It might only imply that our search strategy was **not good**

The problem of **deciding** whether a given formula A **does**, or **does not** have a proof in **LI** becomes more complex then in the case of Gentzen system for classical logic

Decomposition Trees in **LI**

Before we define a heuristic method of **searching** for proof and **deciding** whether such a proof exists or not we make some observations

Observation 1

Logical rules of **LI** are similar to those in Gentzen type classical formalizations we already examined in previous chapters in a sense that each of them introduces a logical **connective**

Decomposition Trees in **LI**

Observation 2

The process of searching for a proof is a **decomposition** process in which we use the inverse of logical and structural rules as **decomposition** rules

For **example** the implication rule:

$$(\rightarrow\Rightarrow)\ \frac{A,\Gamma \longrightarrow B}{\Gamma \longrightarrow (A \Rightarrow B)}$$

becomes an implication **decomposition** rule (we use the same name $(\rightarrow\Rightarrow)$ in both cases)

$$(\rightarrow\Rightarrow)\ \frac{\Gamma \longrightarrow (A \Rightarrow B)}{A,\Gamma \longrightarrow B}$$

Decomposition Trees in **LI**

Observation 3

We write proofs as **trees**, so the proof search process is a process of building decomposition trees

To facilitate the process we write the **decomposition** rules in a tree decomposition form as follows

$$\Gamma \longrightarrow (A \Rightarrow B)$$

$$\mid (\rightarrow\Rightarrow)$$

$$A, \Gamma \longrightarrow B$$

Decomposition Trees in **LI**

The two premisses rule $(\Rightarrow\longrightarrow)$ written as the tree decomposition rule becomes

$$(A \Rightarrow B), \Gamma \longrightarrow$$

$$\bigwedge(\Rightarrow\longrightarrow)$$

$$\Gamma \longrightarrow A \qquad\qquad B, \Gamma \longrightarrow$$

Decomposition Trees in **LI**

The structural weakening rule written as the **decomposition** rule is

$$(\rightarrow weak)\ \frac{\Gamma \longrightarrow A}{\Gamma \longrightarrow}$$

We write it in a **tree decomposition** form as

$$\Gamma \longrightarrow A$$

$$\mid (\rightarrow weak)$$

$$\Gamma \longrightarrow$$

Decomposition Trees in **LI**

We define the notion of decomposable and indecomposable formulas and sequents as follows

Decomposable formula is any formula of the degree ≥ 1

Decomposable sequent is any sequent that contains a decomposable formula

Indecomposable formula is any formula of the degree 0 i.e. is any propositional variable

Decomposition Trees in **LI**

Remark

In a case of formulas written with use of capital letters $A, B, C, ..$ etc , we treat these letters as propositional variables , i.e. as **indecomposable formulas**

Indecomposable sequent is a sequent formed from indecomposable formulas only.

Decomposition Trees in **LI**

Decomposition Tree Construction (1)

Given a formula A we construct its **decomposition** tree $\mathbf{T_A}$ as follows

Root of the tree $\mathbf{T_A}$ is the sequent $\longrightarrow A$

Given a **node** n of the tree we identify a **decomposition** rule applicable at this node and write its premisses as the **leaves** of the **node** n

We **stop** the decomposition process when we obtain an **axiom** or all leaves of the tree are **indecomposable**

Decomposition Trees in **LI**

Observation 4

The decomposition tree $\mathbf{T_A}$ obtained by the **Construction (1)** most often is not unique

Observation 5

The fact that we **find** a decomposition tree $\mathbf{T_A}$ with a non-axiom leaf **does not** mean that $\nvdash_{LI} A$

This is due to the role of **structural rules** in **LI** and will be discussed later

Proof Search Examples

Examples

We perform proof search and **decide** the existence of proofs in **LI** for a given formula $A \in \mathcal{F}$ by constructing its **decomposition** trees $\mathbf{T_A}$

We examine here some **examples** to show the complexity of the problem

Reminder

In the following and similar examples when building the decomposition trees for formulas representing general schemas we treat the capital letters $A, B, C, D...$ as propositional variables, i.e. as **indecomposable** formulas

Examples

Example 1

Determine] whether

$$\vdash_{\mathbf{LI}} ((\neg A \cap \neg B) \Rightarrow \neg(A \cup B))$$

Observe that

If we find a decomposition tree of A in **LI** such that all its leaves are axiom, we have a proof, i.e

$$\vdash_{\mathbf{LI}} A$$

If all possible decomposition trees have a non-axiom leaf then the proof of A i n **LI** does not exist, i.e.

$$\nvdash_{\mathbf{LI}} A$$

Examples

Consider the following decomposition tree **T1$_A$**

$$\longrightarrow ((\neg A \cap \neg B) \Rightarrow (\neg(A \cup B))$$

$$\mid (\longrightarrow \Rightarrow)$$

$$(\neg A \cap \neg B) \longrightarrow \neg(A \cup B)$$

$$\mid (\longrightarrow \neg)$$

$$(\neg A \cap \neg B), (A \cup B) \longrightarrow$$

$$\mid (\cap \longrightarrow)$$

$$\neg A, \neg B, (A \cup B) \longrightarrow$$

$$\mid (\neg \longrightarrow)$$

$$\neg B, (A \cup B) \longrightarrow A$$

$$\mid (\longrightarrow weak)$$

$$\neg B, (A \cup B) \longrightarrow$$

$$\mid (\neg \longrightarrow)$$

$$(A \cup B) \longrightarrow B$$

$$\bigwedge (\cup \longrightarrow)$$

$$A \longrightarrow B \qquad\qquad B \longrightarrow B$$

non – axiom   *axiom*

Examples

The tree $\mathbf{T1_A}$ has a non-axiom leaf, so it **does not** constitute a proof in **LI**

Observe that the **decomposition** tree in **LI** is not always unique

Hence the existence of a non-axiom leaf **does not** yet prove that the **proof** of A does not exist

Consider the following decomposition tree $\mathbf{T2_A}$

$$\longrightarrow ((\neg A \cap \neg B) \Rightarrow (\neg(A \cup B))$$

$$\mid (\longrightarrow \Rightarrow)$$

$$(\neg A \cap \neg B) \longrightarrow \neg(A \cup B)$$

$$\mid (\longrightarrow \neg)$$

$$(A \cup B), (\neg A \cap \neg B) \longrightarrow$$

$$\mid (exch \longrightarrow)$$

$$(\neg A \cap \neg B), (A \cup B) \longrightarrow$$

$$\mid (\cap \longrightarrow)$$

$$\neg A, \neg B, (A \cup B) \longrightarrow$$

$$\mid (exch \longrightarrow)$$

$$\neg A, (A \cup B), \neg B \longrightarrow$$

$$\mid (exch \longrightarrow)$$

$$(A \cup B), \neg A, \neg B \longrightarrow$$

$$\bigwedge (\cup \longrightarrow)$$

$A, \neg A, \neg B \longrightarrow$	$B, \neg A, \neg B \longrightarrow$
$\mid (exch \longrightarrow)$	$\mid (exch \longrightarrow)$
$\neg A, A, \neg B \longrightarrow$	$B, \neg B, \neg A \longrightarrow$
$\mid (\neg \longrightarrow)$	$\mid (exch \longrightarrow)$
$A, \neg B \longrightarrow A$	$\neg B, B, \neg A \longrightarrow$
axiom	$\mid (\neg \longrightarrow)$
	$B, \neg A \longrightarrow B$; *axiom*

Examples

All leaves of **$\mathbf{T2_A}$** are axioms

This means that the tree **$\mathbf{T2_A}$** is a **a proof** of A in **LI**

We hence proved that

$$\vdash_{\mathbf{LI}} \; ((\neg A \cap \neg B) \Rightarrow \neg(A \cup B))$$

Examples

Example 2: Show that

1. $\vdash_{\mathbf{LI}} (A \Rightarrow \neg\neg A)$

2. $\nvdash_{\mathbf{LI}} (\neg\neg A \Rightarrow A)$

Solution of **1.**

We construct some, or all **decomposition** trees of

$$\longrightarrow (A \Rightarrow \neg\neg A)$$

A tree $\mathbf{T}_A$ that **ends** with all leaves being axioms is a proof of A in **LI**

Examples

We construct $\mathbf{T}_A$ as follows

$$\longrightarrow (A \Rightarrow \neg\neg A)$$

$$\mid (\longrightarrow\Rightarrow)$$

$$A \longrightarrow \neg\neg A$$

$$\mid (\longrightarrow \neg)$$

$$\neg A, A \longrightarrow$$

$$\mid (\neg \longrightarrow)$$

$$A \longrightarrow A$$

axiom

All leaves of $\mathbf{T}_A$ are axioms so we found the **proof**

We **do not** need to construct any other decomposition trees.

Examples

Solution of **2.**

In order to prove that

$$\nvdash_{\mathbf{LI}} \quad (\neg\neg A \Rightarrow A)$$

we have to construct all **decomposition** trees of

$$\longrightarrow (\neg\neg A \Rightarrow A)$$

and show that **each** of them has a non-axiom leaf

Examples

Here is **$\mathbf{T1}_A$**

$$\longrightarrow (\neg\neg A \Rightarrow A)$$

$$\mid (\longrightarrow\Rightarrow)$$

one of 2 choices

$$\neg\neg A \longrightarrow A$$

$$\mid (\longrightarrow weak)$$

one of 3 choices

$$\neg\neg A \longrightarrow$$

$$\mid (\neg \longrightarrow)$$

one of 3 choices

$$\longrightarrow \neg A$$

$$\mid (\longrightarrow \neg)$$

one of 2 choices

$$A \longrightarrow$$

non – axiom

Here is $\mathbf{T2}_A$

$$\longrightarrow (\neg\neg A \Rightarrow A)$$

$\mid (\longrightarrow\Rightarrow)$ *one of 2 choices*

$$\neg\neg A \longrightarrow A$$

$\mid (contr \longrightarrow)$ *second of 2 choices*

$$\neg\neg A, \neg\neg A \longrightarrow A$$

$\mid (\longrightarrow weak)$ *first of 2 choices*

$$\neg\neg A, \neg\neg A \longrightarrow$$

$\mid (\neg \longrightarrow)$ *first of 2 choices*

$$\neg\neg A \longrightarrow \neg A$$

$\mid (\longrightarrow \neg)$ *one of 2 choices*

$$A, \neg\neg A \longrightarrow$$

$\mid (exch \longrightarrow)$ *one of 2 choices*

$$\neg\neg A, A \longrightarrow$$

$\mid (\neg \longrightarrow)$*one of 2 choices*

$$A \longrightarrow \neg A$$

$\mid (\longrightarrow \neg)$ *first of 2 choices*

$$A, A \longrightarrow$$

non – axiom

Structural Rules

We can see from the above **decomposition** trees that the "blind" construction of all possible trees only leads to more complicated trees

This is due to the presence of structural rules

The "blind" application of the rule $(contr \rightarrow)$ gives always an infinite number of **decomposition** trees

In order to decide that none of them will produce a proof we need some **extra knowledge** about patterns of their construction, or just simply about the number o useful of application of **structural rules**

Structural Rules

In this case we can just make an ”external” **observation** that the our first tree $\mathbf{T1}_A$ is in a sense a minimal one

It means that all other trees would only **complicate** this one in an inessential way, i.e. the we will never produce a tree with all axioms leaves

One can formulate a deterministic procedure giving a finite number of trees, but the proof of its **correctness** is needed and that requires some extra knowledge

Within the scope of this book we accept the ”external explanation as a **sufficient solution**

Structural Rules

As we can see from the above examples the structural rules and especially the $(contr \longrightarrow)$ rule **complicates** the proof searching task.

Both Gentzen type proof systems **RS** and **GL** from the previous chapter don't contain the structural rules

They also are as we have proved, **complete** with respect to classical semantics.

The original Gentzen system **LK** which does contain the structural rules is also, as proved by Gentzen, **complete**

Structural Rules

Hence all three classical proof system **RS, GL, LK** are equivalent

This proves that the structural rules can be **eliminated** from the system **LK**

A natural question of elimination of **structural rules** from the system **LI** arises

The following **example** illustrates the negative answer

Examples

Example 3

We know that for any formula $A \in \mathcal{F}$,

$$\models A \quad \text{if and only if} \quad \vdash_I \neg\neg A$$

where $\models A$ means that A is classical tautology

$\vdash_I A$ means that A is Intutionistically **provable** in any intuitionistically **complete** proof system **I**

The system **LI** is intuitionistically **complete** so have that for any formula $A \in \mathcal{F}$,

$$\models A \quad \text{if and only if} \quad \vdash_{\mathbf{LI}} \neg\neg A$$

Examples

Obviously $\models (\neg\neg A \Rightarrow A)$, so we must have that

$$\vdash_{\mathbf{LI}} \neg\neg(\neg\neg A \Rightarrow A)$$

We are going to prove now that the rule $(contr \longrightarrow)$ is **essential** to the existence of the proof $\neg\neg(\neg\neg A \Rightarrow A)$

It means that $\neg\neg(\neg\neg A \Rightarrow A)$ **is not provable** without the rule $(contr \longrightarrow)$

The following decomposition tree $\mathbf{T}_A$ is a proof of $\neg\neg(\neg\neg A \Rightarrow A)$ **with use** of the rule $(contr \longrightarrow)$

Examples

$$\longrightarrow \neg\neg(\neg\neg A \Rightarrow A)$$

$$\mid (\longrightarrow \neg)$$

$$\neg(\neg\neg A \Rightarrow A) \longrightarrow$$

$$\mid (contr \longrightarrow)$$

$$\neg(\neg\neg A \Rightarrow A), \neg(\neg\neg A \Rightarrow A) \longrightarrow$$

$$\mid (\neg \longrightarrow)$$

$$\neg(\neg\neg A \Rightarrow A) \longrightarrow (\neg\neg A \Rightarrow A)$$

$$\mid (\longrightarrow\Rightarrow)$$

$$\neg\neg A, \neg(\neg\neg A \Rightarrow A) \longrightarrow A$$

$$\mid (\longrightarrow weak)$$

$$\neg\neg A, \neg(\neg\neg A \Rightarrow A) \longrightarrow$$

$$\mid (\neg \longrightarrow)$$

$$\neg(\neg\neg A \Rightarrow A) \longrightarrow \neg A$$

$$\mid (\longrightarrow \neg)$$

$$A, \neg(\neg\neg A \Rightarrow A) \longrightarrow$$

$$\mid (exch \longrightarrow)$$

$$\neg(\neg\neg A \Rightarrow A), A \longrightarrow$$

$$\mid (\neg \longrightarrow)$$

$$A \longrightarrow (\neg\neg A \Rightarrow A)$$

$$\mid (\longrightarrow\Rightarrow)$$

$$\neg\neg A, A \longrightarrow A$$

axiom

Contraction Rule

Assume now that the rule $(contr \longrightarrow)$ **is not** available. All possible decomposition trees are as follows

Tree $\mathbf{T1}_A$

$$\longrightarrow \neg\neg(\neg\neg A \Rightarrow A)$$

$$\mid (\longrightarrow \neg)$$

$$\neg(\neg\neg A \Rightarrow A) \longrightarrow$$

$$\mid (\neg \longrightarrow)$$

$$\longrightarrow (\neg\neg A \Rightarrow A)$$

$$\mid (\longrightarrow \Rightarrow)$$

$$\neg\neg A \longrightarrow A$$

$$\mid (\longrightarrow weak)$$

$$\neg\neg A \longrightarrow$$

$$\mid (\neg \longrightarrow)$$

$$\longrightarrow \neg A$$

$$\mid (\longrightarrow \neg)$$

$$A \longrightarrow$$

$$non-axiom$$

Contraction Rule

The next is **$\mathbf{T2}_A$**

$$\longrightarrow \neg\neg(\neg\neg A \Rightarrow A)$$

$$\mid (\longrightarrow \neg)$$

$$\neg(\neg\neg A \Rightarrow A) \longrightarrow$$

$$\mid (\neg \longrightarrow)$$

$$\longrightarrow (\neg\neg A \Rightarrow A)$$

$$\mid (\longrightarrow weak)$$

$$\longrightarrow$$

$$non - axiom$$

Contraction Rule

The next is $\mathbf{T3}_A$

$$\longrightarrow \neg\neg(\neg\neg A \Rightarrow A)$$

$$\mid (\longrightarrow weak)$$

$$\longrightarrow$$

$$non - axiom$$

Contraction Rule

The last one is $\mathbf{T4}_A$

$$\longrightarrow \neg\neg(\neg\neg A \Rightarrow A)$$

$$\mid (\longrightarrow \neg)$$

$$\neg(\neg\neg A \Rightarrow A) \longrightarrow$$

$$\mid (\neg \longrightarrow)$$

$$\longrightarrow (\neg\neg A \Rightarrow A)$$

$$\mid (\longrightarrow \Rightarrow)$$

$$]$$

$$\neg\neg A \longrightarrow A$$

$$\mid (\longrightarrow weak)$$

$$\neg\neg A \longrightarrow$$

$$\mid (\neg \longrightarrow)$$

$$\longrightarrow \neg A$$

$$\mid (\longrightarrow weak)$$

$$\longrightarrow$$

non – axiom

Contraction Rule

We have considered all possible decomposition trees that **do not** involve the contraction rule $(contr \longrightarrow)$ and **none** of them was a proof

This shows that the formula

$$\neg\neg(\neg\neg A \Rightarrow A)$$

is not provable in **LI** without $(contr \longrightarrow)$ rule, i.e. that we proved the following

Fact

The contraction rule $(contr \longrightarrow)$ **can not** be eliminated from **LI**

Proof Search Heuristic Method

Proof Search Heuristic Method

Before we define a heuristic method of searching for proof in **LI** let's make some additional observations to the already made **observations 1-5**

Observation 6

The goal of constructing the decomposition tree is to **obtain** axioms or indecomposable leaves

With respect to this goal the use **logical** decomposition rules has a priority over the use of the **structural** rules

We use this information while describing the proof search **heuristic**

Proof Search Heuristic Method

Observation 7

All logical decomposition rules $(\circ \rightarrow)$, where $\circ$ denotes any connective, must have a formula we want to decompose as the first formula at the decomposition node

It means that if we want to **decompose** a formula $\circ A$ the node must have a form $\circ A, \Gamma \longrightarrow \Delta$

Remember: order of decomposition is important

Also sometimes it is necessary to decompose a **formula within the sequence** Γ **first**, before decomposing $\circ A$ in order to **find** a proof

Proof Search Heuristic Method

For example, consider two nodes

$$n_1 = \neg\neg A,\ (A \cap B) \longrightarrow B$$

and

$$n_2 = (A \cap B),\ \neg\neg A \longrightarrow B$$

We are going to see that the results of decomposing n_1 and n_2 **differ dramatically**

Let's decompose the node n_1

Observe that the only way to be able to decompose the formula $\neg\neg A$ is to use the rule $(\rightarrow weak)$ as a **first step**

The **two possible** decomposition trees that starts at the node n_1 are as follows

Proof Search Heuristic Method

First Tree

T1$_{n_1}$

$$\neg\neg A, (A \cap B) \longrightarrow B$$

$$\mid (\to \textit{weak})$$

$$\neg\neg A, (A \cap B) \longrightarrow$$

$$\mid (\neg \to)$$

$$(A \cap B) \longrightarrow \neg A$$

$$\mid (\cap \to)$$

$$A, B \longrightarrow \neg A$$

$$\mid (\to \neg)$$

$$A, A, B \longrightarrow$$

non – axiom

Proof Search Heuristic Method

Second Tree

$\mathbf{T2}_{n_1}$

$$\neg\neg A, (A \cap B) \longrightarrow B$$

$$\mid (\rightarrow weak)$$

$$\neg\neg A, (A \cap B) \longrightarrow$$

$$\mid (\neg \rightarrow)$$

$$(A \cap B) \longrightarrow \neg A$$

$$\mid (\rightarrow \neg)$$

$$A, (A \cap B) \longrightarrow$$

$$\mid (\cap \rightarrow)$$

$$A, A, B \longrightarrow$$

$$non - axiom$$

Proof Search Heuristic Method

Let's now decompose the node n_2

Observe that following our **Observation 6** we start by decomposing the formula $(A \cap B)$ by the use of the rule $(\cap \rightarrow)$ as the **first step**

A decomposition tree that starts at the node n_2 is as follows

$$\mathbf{T}_{n_2}$$

$$(A \cap B), \neg\neg A \longrightarrow B$$

$$\mid (\cap \rightarrow)$$

$$A, B, \neg\neg A \longrightarrow B$$

$$axiom$$

This proves that the node n_2 is **provable** in **LI**, i.e.

$$\vdash_{\mathbf{LI}} \quad (A \cap B), \neg\neg A \longrightarrow B$$

Proof Search Heuristic Method

Observation 8

The use of structural rules is **important** and **necessary** while we search for proofs

Nevertheless we have to **use them** on the ”must” basis and set up some **guidelines** and **priorities** for their use

For example, the use of weakening rule **discharges** the weakening formula, and hence we might **loose an information** that may be essential to finding the **proof**

We should use the weakening rule only when it is **absolutely necessary** for the next decomposition steps

Proof Search Heuristic Method

Hence, the use of weakening rule $(\rightarrow \mathit{weak})$ **can**, and **should be restricted** to the cases when it leads to **possibility** of the future use of the **negation rule** $(\neg \rightarrow)$

This was the case of the decomposition tree $\mathbf{T1}_{n_1}$

We used the rule $(\rightarrow \mathit{weak})$ as an necessary step, but it **discharged** too much information and we didn't get a proof, when **proof on this node existed**

Proof Search Heuristic Method

Here is such a proof

T3$_{n_1}$

$$\neg\neg A, (A \cap B) \longrightarrow B$$

$$\mid (exch \longrightarrow)$$

$$(A \cap B), \neg\neg A \longrightarrow B$$

$$\mid (\cap \rightarrow)$$

$$A, B, \neg\neg A \longrightarrow B$$

axiom

Method

For any $A \in \mathcal{F}$ we construct the set of decomposition trees $\mathbf{T}_{\to A}$ following the rules below.

1. Use first logical rules where applicable.

2. Use $(exch \to)$ rule to decompose, via logical rules, as many formulas on the left side of $\longrightarrow$ as possible

Remember that the order of decomposition matters! so you have to cover different choices

3. Use $(\to weak)$ only on a "must" basis and in connection with the **possibility** of the future use of the $(\neg \to)$ rule

4. Use $(contr \to)$ rule as the **last recourse** and only to formulas that contain $\neg$ or $\Rightarrow$ as a main connective

5. Let's call a formula A to which we apply $(contr \to)$ rule a **a contraction formula**

6. The only contraction formulas are formulas containing $\neg$ or $\Rightarrow$ between theirs logical connectives

Proof Search Heuristic Method

7. Within the process of construction of all possible trees use $(contr \rightarrow)$ rule **only** to **contraction formulas**

8. Let C be a **contraction formula** appearing on a node n of the decomposition tree of $\mathbf{T}_{\rightarrow A}$

For any **contraction formula** C, any node n, we apply $(contr \rightarrow)$ rule to the the formula C at the node n **at most** as many times as the number of sub-formulas of C

If we **find** a tree with all axiom leaves we have a **proof**, i.e.

$$\vdash_{LI} A$$

If **all trees** (finite number) have a non-axiom leaf we have proved that proof of A **does not exist**, i.e.

$$\nvdash_{LI} A$$

Chapter 7

Introduction to Intuitionistic and Modal Logics

Slides Set 3

PART 5: Introduction to Modal Logics

Algebraic Semantics for modal S4 and S5

Introduction to Modal Logics

The **non-classical** logics can be divided in two groups: those that **rival** classical logic and those which **extend it**

The Lukasiewicz, Kleene, and intuitionistic logics are in the **first** group

The modal logics are in the **second** group

The **rival** logics **do not** differ from classical logic in terms of the language employed

The **rival** logics differ in that certain theorems or tautologies of classical logic are rendered **false**, or **not provable** in them

Introduction to Modal Logics

The most notorious example of the **rival** difference of logics based on the same language is the law of excluded middle

$$(A \cup \neg A)$$

This is **provable** in, and is a **tautology** of classical logic

But **is not** provable in, and **is not** tautology of the intuitionistic logic

It also **is not** a tautology under any of the extensional logics semantics we have discussed

Introduction to Modal Logics

Logics which **extend classical** logic sanction all the theorems of classical logic but, generally, supplement it in **two** ways

Firstly, the languages of these non-classical logics are **extensions** of those of classical logic

Secondly, the theorems of these non-classical logics **supplement** those of classical logic

Introduction to Modal Logics

Modal logics are enriched by the addition of two new **connectives** that represent the meaning of expressions "it is necessary that" and " it is possible that"

We use the notation:

I for " it is necessary that" and

C for " it is possible that"

Other notations commonly used are:

∇, N, L for " it is necessary that" and

$\diamond$, P, M for " it is possible that"

Introduction to Modal Logics

The symbols N, L, P, M or alike, are often used in computer science

The symbols ∇ and $\diamond$ were **first** to be used in modal logic literature

The symbols **I, C** come from **algebraic** and **topological** interpretation of modal logics

I corresponds to the topological **interior** of the set and **C** to its **closure**

Introduction to Modal Logics

The idea of a modal logic was **first** formulated by an American philosopher, C.I. Lewis in 1918

Lewis has proposed yet another interpretation of lasting consequences, of the logical **implication**

He created a notion of a **modal truth**, which lead to the notion of modal logic

He did it in an attempt to avoid, what some felt, the **paradoxes** of semantics for classical implication which accepts as **true** that a false sentence **implies** any sentence

Introduction to Modal Logics

Lewis' notions appeal to **epistemic** considerations and the whole area of modal logics bristles with **philosophical** difficulties and hence the numbers of modal logics have been **created**

Unlike the classical connectives, the modal connectives **do not** admit of truth-functional interpretation, i.e. **do not** accept the extensional semantics

This was the **reason** for which modal logics were **first** developed as proof systems, with intuitive notion of **semantics** expressed by the set of adopted axioms

Introduction to Modal Logics

The **first** definition of modal semantics, and hence the proofs of the **completeness** theorems came some 20 years later

It took yet another 25 years for discovery and development of the **second** and more general approach to the modal semantics

These are the two **established** ways of interpret modal connectives, i.e. to define the modal semantics

The historically, the **first** modal **semantics** is due to Mc Kinsey and Tarski (1944, 1946)
It is a **topological** semantics that provides a powerful mathematical interpretation of some of modal logics, namely modal S4 and S5

It connects the **modal** notion of necessity with the **topological** notion of the interior of a set, and
the **modal** notion of possibility with the notion of the closure of a set

Our **choice** of symbols **I** and **C** for necessity and possibility **connectives** comes from this interpretation

The **topological** interpretation mathematically powerful as it is, is less universal in providing models for **other** modal logics

Introduction to Modal Logics

The most recent and the most **general** semantics is due to Kripke (1964). It uses the notion of possible worlds.

Roughly, we say that the formula **C***A* is **true** if *A* is **true** in **some** possible world, called actual world

The formula **I***A* is **true** if *A* is **true** in **every** possible world

We present here a short version of the topological semantics in a form of **algebraic models**

We leave the **Kripke semantics** for the reader to **explore** from other, multiple sources

As we have already mentioned, modal logics were first **developed**, as was the intuitionistic logic, in a **form** of proof systems only

First Hilbert style **modal** proof system was published by Lewis and Langford in 1932

They presented a formalization for **two** modal logics, which they called S1 and S2

They also outlined **three** other proof systems, called S3, S4, and S5

Since then **hundreds** of modal logics have been **created**

There are some **standard** books in the subject

These are, **between** the others:

Hughes and Cresswell (1969) for **philosophical** motivation for various modal logics and intuitionistic logic,

Bowen (1979) for a detailed and uniform study of **Kripke models** for modal logics,

Segeberg (1971) for excellent modal logics **classification**,

Fitting (1983), for extended and uniform studies of **automated** proof methods for **classes** of modal logics

Hilbert Style Modal Proof Systems

Hilbert Style Modal Proof Systems

We present now Hilbert style formalization forS4 and S5 logics due to Mc Kinsey and Tarski (1948) and Rasiowa and Sikorski (1964)

We also discuss the **relationship** between S4 and S5 , and between the intuitionistic logic and S4 modal logic, as first observed by Gödel

The formalizations stress the **connection** between S4, S5 and topological spaces which constitute **models** for them

Modal Language

We **add** two extra one argument connectives **I** and **C** to the propositional language $\mathcal{L}_{\{\cup,\cap,\Rightarrow,\neg\}}$, i.e. we adopt

$$\mathcal{L} = \mathcal{L}_{\{\cup,\cap,\Rightarrow,\neg,\mathbf{I},\mathbf{C}\}}$$

as the modal language. We read a formulas $\mathbf{I}A$, $\mathbf{C}A$ as necessary A and possible A, respectively

The language $\mathcal{L}_{\{\cup,\cap,\Rightarrow,\neg,\mathbf{I},\mathbf{C}\}}$ is **common** to all modal logics

Modal logics differ on a **choice** of axioms and rules of inference, when studied as proof systems and on a **choice** of respective semantics

McKinsey, Tarski Proof Systems

As modal logics extend the classical logic, any modal logic contains **two groups** of axioms: classical and modal

McKinsey, Tarski (1948)

AG1 classical axioms

We **adopt** as classical axioms any **complete** set of axioms under classical semantics

AG2 modal axioms

M1 $(\mathbf{I}A \Rightarrow A)$

M2 $(\mathbf{I}(A \Rightarrow B) \Rightarrow (\mathbf{I}A \Rightarrow \mathbf{I}B))$

M3 $(\mathbf{I}A \Rightarrow \mathbf{II}A)$

M4 $(\mathbf{C}A \Rightarrow \mathbf{IC}A)$

Modal S4 and S5

Rules of inference

$$(MP)\ \frac{A\ ;\ (A \Rightarrow B)}{B}, \quad \text{and} \quad (\mathbf{I})\ \frac{A}{\mathbf{I}A}$$

The modal rule $(\mathbf{I})$ was introduced by Gödel and is referred to as a necessitation rule

We define **modal** proof systems S4 and S5 as follows

$$S4 = (\ \mathcal{L},\ \mathcal{F},\ \text{classical axioms},\ M1 - M3,\ (MP),\ (\mathbf{I})\ )$$

$$S5 = (\ \mathcal{L},\ \mathcal{F},\ \text{classical axioms},\ M1 - M4,\ (MP),\ (\mathbf{I})\ )$$

Modal S4 and S5

Observe that the axioms of S5 **extend** the axioms of S4 and both system **share** the same inference rules, hence we have immediately the following

Fact For any formula $A \in \mathcal{F}$,

$$\text{if} \quad \vdash_{S4} A, \quad \text{then} \quad \vdash_{S5} A$$

It is often the case, as it is for S4 and S5, that **modal connectives** are definable by each other

We define them as follows

$$\mathbf{I}A = \neg\mathbf{C}\neg A, \quad \text{and} \quad \mathbf{C}A = \neg\mathbf{I}\neg A$$

Language

We hence assume now that the language $\mathcal{L}$ of Rasiowa, Sikorski modal proof systems contains only **one** modal connective

We **choose** it to be $\mathbf{I}$ and adopt the following language

$$\mathcal{L} = \mathcal{L}_{\{\cap,\cup,\Rightarrow,\neg,\mathbf{I}\}}$$

There are, as before, **two groups** of axioms: classical and modal

Rasiowa, Sikorski Proof Systems

Rasiowa, Sikorski (1964)

AG1 classical axioms

We **adopt** as classical axioms any **complete** set of axioms under classical semantics

AG2 modal axioms

R1 $((\mathbf{I}A \cap \mathbf{I}B) \Rightarrow \mathbf{I}(A \cap B))$

R2 $(\mathbf{I}A \Rightarrow A)$

R3 $(\mathbf{I}A \Rightarrow \mathbf{I}\mathbf{I}A)$

R4 $\mathbf{I}(A \cup \neg A)$

R5 $(\neg\mathbf{I}\neg A \Rightarrow \mathbf{I}\neg\mathbf{I}\neg A)$

Rules of inference

We adopt the Modus Ponens and an additional rule (R$\mathbf{I}$)

$$(MP)\ \frac{A\ ;\ (A \Rightarrow B)}{B} \quad \text{and} \quad (R\mathbf{I})\ \frac{(A \Rightarrow B)}{(\mathbf{I}A \Rightarrow \mathbf{I}B)}$$

We define modal proof systems $RS4$ and $RS5$ as follows

$$RS4 = (\ \mathcal{L},\ \mathcal{F},\ \text{classical axioms},\ R1 - R4,\ (MP),\ (R\mathbf{I})\ )$$

$$RS5 = (\ \mathcal{L},\ \mathcal{F},\ \text{classical axioms},\ R1 - R5,\ (MP),\ (R\mathbf{I})\ )$$

Modal RS4 and RS5

Observe that the axioms of RS5 **extend** the axioms of RS4 and both systems **share** the same inference rules, hence we have immediately the following

Fact For any formula $A \in \mathcal{F}$,

$$\text{if} \quad \vdash_{RS4} A, \quad \text{then} \quad \vdash_{RS5} A$$

Algebraic Semantics for S4 and S5

Algebraic Semantics for S4 and S5

The McKinsey, Tarski proof systems S4, S5 and Rasiowa, Sikorski proof systems RS4, RS5 are **complete** with the respect to **both** topological semantics, and Kripke semantics

We shortly discuss the topological semantics, and **algebraic completeness** theorems

We leave the Kripke semantics for the reader to **explore** from other, multiple sources

Algebraic Semantics for S4 and S5

The **topological semantics** was initiated by McKinsey and Tarski in 1946, 1948 and consequently developed into a field of **Algebraic Logic**

The **algebraic** approach to logic is presented in detail in now classic algebraic logic books:

"Mathematics of Metamathematics", Rasiowa, Sikorski (1964),

"An Algebraic Approach to Non-Classical Logics", Rasiowa (1974)

We want to point out that the **first idea** of a connection between **modal** propositional logic and **topology** is due to Tang Tsao -Chen, (1938) and Dugunji (1940)

Algebraic Semantics for S4 and S5

Here are some basic definitions

Boolean Algebra

An abstract algebra $\mathcal{B} = (B,\ 1,\ 0,\ \Rightarrow,\ \cap,\ \cup, \neg)$ is said to be a **Boolean algebra** if it is a distributive lattice and every element $a \in B$ has a complement $\neg a \in B$

Topological Boolean algebra

By a topological Boolean algebra we mean an abstract algebra

$$\mathcal{B} = (B,\ 1,\ 0,\ \Rightarrow,\ \cap, \cup, \neg,\ I)$$

where $(B,\ 1,\ 0,\ \Rightarrow,\ \cap,\ \cup, \neg)$ is a **Boolean algebra** and, moreover, the following conditions hold for any $a, b \in B$

$$I(a \cap b) = Ia \cap Ib, \quad Ia \cap a = Ia, \quad IIa = Ia, \text{ and } I1 = 1$$

Algebraic Semantics for S4 and S5

The element Ia is called a **interior** of a

The element $\neg I \neg a$ is called a **closure** of a and will be **denoted** by Ca

Thus the operations I and C are such that

$$Ca = \neg I \neg a \quad \text{and} \quad Ia = \neg C \neg a$$

In this case we write the **topological Boolean algebra** as

$$\mathcal{B} = (B,\ 1,\ 0,\ \Rightarrow,\ \cap,\ \cup, \neg,\ I,\ C\ )$$

It is easy to prove that in in any topological Boolean algebra the following conditions hold for any $a, b \in B$

$$C(a \cup b) = Ca \cup Cb, \quad Ca \cup a = Ca, \quad CCa = Ca \ \text{and} \ C0 = 0$$

Algebraic Semantics for S4 and S5

Example

Let X be a topological space with an interior operation I

Then the family $\mathcal{P}(X)$ of all subsets of X is a **topological Boolean algebra** with $1 = X$, with

the operation $\Rightarrow$ defined by the formula

$$Y \Rightarrow Z = (X - Y) \cup Z \quad \text{for all subsets } Y, Z \text{ of } X$$

and with set-theoretical operations of union, intersection, complementation, and the interior operation I

Every sub algebra of this algebra is a **topological Boolean** algebra, called a topological field of sets or, more precisely, a **topological field** of subsets of X

Algebraic Semantics for S4 and S5

Given a topological Boolean algebra

$$(B,\ 1,\ 0,\ \Rightarrow,\ \cap,\ \cup, \neg)$$

The element $a \in B$ is said to be **open** (**closed**)
if $a = Ia$ $(a = Ca)$

Clopen Topological Boolean Algebra

A topological Boolean algebra

$$\mathcal{B} = (B,\ 1,\ 0,\ \Rightarrow,\ \cap,\ \cup, \neg,\ I,\ C\ )$$

such that every **open** element is **closed** and every **closed** element is **open**, i.e. such that for any $a \in B$

$$CIa = Ia \quad \text{and} \quad ICa = Ca$$

is called a **clopen** topological Boolean algebra

S4, S5 Tautology

We loosely say that a formula A is a modal $S4$ **tautology** if and only if any topological Boolean algebra is a **model** for A

We say that A is a modal $S5$ **tautology** if and only if any **clopen** topological Boolean algebra is a **model** for A

We put it formally as follows

Modal Algebraic Model

For any formula A of a modal language $\mathcal{L}_{\{\cup,\cap,\Rightarrow,\neg,\mathbf{I},\mathbf{C}\}}$ and for any topological Boolean algebra

$$\mathcal{B} = (B,\ 1,\ 0,\ \Rightarrow,\ \cap,\ \cup, \neg,\ I,\ C)$$

the algebra $\mathcal{B}$ is a **model** for the formula A and denote it by

$$\mathcal{B} \models A$$

if and only if $v^*(A) = 1$ holds for all variables assignments $v : VAR \longrightarrow B$

Definition of $S4$ **Tautology**

A formula A is a modal $S4$ **tautology** and is denoted by

$$\models_{S4} A$$

if and only if for all **topological Boolean** algebras $\mathcal{B}$ we have that

$$\mathcal{B} \models A$$

Definition of $S5$ **Tautology**

A formula A is a modal $S5$ **tautology** and is denoted by

$$\models_{S5} A$$

if and only if for all **clopen** topological Boolean algebras $\mathcal{B}$ we have that

$$\mathcal{B} \models A$$

S4, S5 Completeness Theorem

We write $\vdash_{S4}\ A$ and $\vdash_{S5}\ A$ do denote **provability any** proof system for modal S4, S5 logics and in particular the proof systems defined here

Completeness Theorem

For any formula A of the modal language $\mathcal{L}_{\{\cup,\cap,\ \Rightarrow,\ \neg,\ \mathbf{I},\mathbf{C}\}}$

$$\vdash_{S4}\ A \quad \text{if and only if} \quad \models_{S4} A$$

$$\vdash_{S5}\ A \quad \text{if and only if} \quad \models_{S5} A$$

The completeness for $S4$, $S4$ follows directly from the following general Algebraic Completeness Theorems

S4 Algebraic Completeness Theorem

For any formula A of the modal language $\mathcal{L}_{\{\cup,\cap,\Rightarrow,\neg,\mathbf{I},\mathbf{C}\}}$ the following conditions are equivalent

(i) $\vdash_{S4} A$

(ii) $\models_{S4} A$

(iii) A is valid in every topological field of sets $\mathcal{B}(X)$

(iv) A is valid in every topological Boolean algebra $\mathcal{B}$ with at most 2^{2^r} elements, where r is the number of all sub formulas of A

(iv) $v^*(A) = X$ for every variable assignment v in the topological field of sets $\mathcal{B}(X)$ of all subsets of a dense-in -itself metric space $X \neq \emptyset$ (in particular of an n-dimensional Euclidean space X)

S5 Algebraic Completeness Theorem

For any formula A of the modal language $\mathcal{L}_{\{\cup,\cap,\Rightarrow,\neg,\mathbf{I},\mathbf{C}\}}$ the following conditions are equivalent

(i) $\vdash_{S5} A$

(ii) $\models_{S5} A$

(iii) A is valid in every **clopen** topological field of sets $\mathcal{B}(X)$

(iv) A is valid in every **clopen** topological Boolean algebra $\mathcal{B}$ with at most 2^{2^r} elements, where r is the number of all sub formulas of A

S4 and S5 Decidability

The equivalence of conditions **(i)** and **(iv)** of the Algebraic Completeness Theorems proves the **semantical** decidability of modal S4 and S5

S4, S5 Decidability

Any complete S4, S5 proof system is **semantically decidable**, i.e. the following holds

$$\vdash_{S4} A \quad \text{if and only if} \quad \mathcal{B} \models A$$

for every topological Boolean algebra $\mathcal{B}$ with at most 2^{2^r} elements, where r is the number of all sub formulas of A

Similarly, we also have

$$\vdash_{S5} A \quad \text{if and only if} \quad \mathcal{B} \models A$$

for every **clopen** topological Boolean algebra $\mathcal{B}$ with at most 2^{2^r} elements, where r is the number of all sub formulas of A

S4 and S5 Syntactic Decidability

S4, S5 Syntactic Decidability (Wasilewska 1967,1971)

Rasiowa stated in 1950 an **an open problem** of providing a cut-free RS type formalization for modal propositional S4 calculus

Wasilewska solved this open problem in 1967 and presented the result at the ASL Summer School and Colloquium in Mathematical Logic, Manchester, August 1969

It appeared in print as *A Formalization of the Modal Propositional S4-Calculus*, Studia Logica, North Holland, XXVII (1971)

S4 and S5 Syntactic Decidability

The paper also contained an algebraic proof of **completeness** theorem followed by Gentzen cut-elimination theorem, the **Hauptzatz**

The resulting implementation written in LISP-ALGOL was the **first** modal logic **theorem prover** created

It was done with collaboration with B. Waligorski and the authors didn't think it to be worth a separate publication

Its existence was only mentioned in the published paper

The S5 Syntactic Decidability follows from the one for S4 and the following **Embedding Theorems**

Modal S4 and Modal S5

The relationship between S4 and S5 was **first** established by Ohnishi and Matsumoto in 1957-59 and is as follows .

Embedding 1

For any formula $A \in \mathcal{F}$,

$$\models_{S4} A \quad \text{if and only if} \quad \models_{S5} \mathbf{IC}A$$

$$\vdash_{S4} A \quad \text{if and only if} \quad \vdash_{S5} \mathbf{IC}A$$

Embedding 2

For any formula $A \in \mathcal{F}$

$$\models_{S5} A \quad \text{if and only if} \quad \models_{S4} \mathbf{ICI}A$$

$$\vdash_{S5} A \quad \text{if and only if} \quad \vdash_{S4} \mathbf{ICI}A$$

On S4 derivable disjunction

In a classical logic it is possible for the disjunction $(A \cup B)$ to be a tautology when **neither** A **nor** B is a tautology

This does not hold for the intuitionistic logic. We have a following theorem similar to the intuitionistic case to the for modal S4

Theorem McKinsey, Tarski (1948)

A disjunction $(IA \cup IB)$ is S4 **provable** if and only if either A or B S4 **provable**, i.e.

$$\vdash_{S4} (IA \cup IB) \quad \text{if and only if} \quad \vdash_{S4} A \quad \text{or} \quad \vdash_{S4} B$$

S4 and Intuitionistic Logic, S5 and Classical Logic

S4 and Intuitionistic Logic

As we have said in the introduction, Gödel was the first to consider the **connection** between the intuitionistic logic and a logic which was named later S4

Gödel's proof was purely **syntactic** in its nature, as the semantics for neither intuitionistic logic nor modal logicS4 had not been invented yet

The **algebraic** proof of this fact, was first published by McKinsey and Tarski in 1948

S4 and Intuitionistic Logic

We define here the Gödel-Tarski **mapping** establishing the S4 and intuitionistic logic connection

We refer the reader to Rasiowa, Sikorski book "Mathematics of Metamathematics" (i965) for the algebraic proofs of its properties and respective theorems

S4 and Intuitionistic Logic

Let $\mathcal{L}$ be a propositional language of modal logic i.e the language

$$\mathcal{L} = \mathcal{L}_{\{\cap, \cup, \Rightarrow, \neg, \mathbf{I}\}}$$

Let $\mathcal{L}_0$ be a language obtained from $\mathcal{L}$ by elimination of the connective $\mathbf{I}$ and by the replacement the classical negation connective $\neg$ by the intuitionistic negation, which we will **denote** here by a symbol $\sim$

Such obtained language

$$\mathcal{L}_0 = \mathcal{L}_{\{\cap, \cup, \Rightarrow, \sim\}}$$

is a propositional language of the intuitionistic logic

S4 and Intuitionistic Logic

In order to establish the connection between the languages

$$\mathcal{L} \quad \text{and} \quad \mathcal{L}_0$$

and hence between modal and intuitionistic logic, we consider a **mapping** f which to every formula $A \in \mathcal{F}_0$ of $\mathcal{L}_0$ **assigns** a formula $f(A) \in \mathcal{F}$ of $\mathcal{L}$

We define the **mapping** f as follows

Gödel - Tarski Mapping

Definition of Gödel-Tarski **mapping**

A function

$$f: \mathcal{F}_0 \rightarrow \mathcal{F}$$

such that

$$f(a) = \mathbf{I}a \quad \textit{for any} \quad a \in VAR$$

$$f((A \Rightarrow B)) = \mathbf{I}(f(A) \Rightarrow f(B))$$

$$f((A \cup B)) = (f(A) \cup f(B))$$

$$f((A \cap B)) = (f(A) \cap f(B))$$

$$f(\sim A) = \mathbf{I}\neg f(A)$$

where A, B are any formulas in $\mathcal{L}_0$ is called a Gödel-Tarski **mapping**

Example

Example

Let A be a formula

$$((\sim A \cap \sim B) \Rightarrow \sim (A \cup B))$$

and f be the Gödel-Tarski mapping. We evaluate $f(A)$ as follows

$$f((\sim A \cap \sim B) \Rightarrow \sim (A \cup B)) =$$

$$\mathbf{I}(f(\sim A \cap \sim B) \Rightarrow f(\sim (A \cup B)) =$$

$$\mathbf{I}((f(\sim A) \cap f(\sim B)) \Rightarrow f(\sim (A \cup B)) =$$

$$\mathbf{I}((\mathbf{I}\neg fA \cap \mathbf{I}\neg fB) \Rightarrow \mathbf{I}\neg f(A \cup B)) =$$

$$\mathbf{I}((\mathbf{I}\neg A \cap \mathbf{I}\neg B) \Rightarrow \mathbf{I}\neg(fA \cup fB)) =$$

$$\mathbf{I}((\mathbf{I}\neg A \cap \mathbf{I}\neg B) \Rightarrow \mathbf{I}\neg(A \cup B))$$

S4 and Intuitionistic Logic

The following theorem established relationship between intuitionistic and modal $S4$ logics

Theorem

Let f be the Gödel-Tarski **mapping**

For any formula A of intuitionistic language $\mathcal{L}_0$,

$$\vdash_I A \quad \text{if and only if} \quad \vdash_{S4} f(A)$$

where I, $S4$ denote any proof systems for intuitionistic and and S4 logic, respectively

Classical Logic and Modal $S5$

In order to establish the connection between the modal S5 and classical logics we consider the following G΄odel-Tarski **mapping** between the modal language $\mathcal{L}_{\{\cap,\cup,\Rightarrow,\neg,\mathbf{I}\}}$ and its classical sub-language $\mathcal{L}_{\{\neg,\cap,\cup,\Rightarrow\}}$

With every **classical** formula A we associate a **modal** formula $g(A)$ defined by induction on the length of A as follows:

$$g(a) = \mathbf{I}a, \quad g((A \Rightarrow B)) = \mathbf{I}(g(A) \Rightarrow g(B),)$$

$$g((A \cup B)) = (g(A) \cup g(B)), \; g((A \cap B)) = (g(A) \cap g(B)),$$

$$g(\neg A) = \mathbf{I}\neg g(A)$$

Classical Logic and Modal $S5$

The following theorem establishes **relationship** between classical and $S5$ logics

Theorem

Let g be the Gödel-Tarski **mapping** between

$$\mathcal{L}_{\{\neg,\cap,\cup,\Rightarrow\}} \quad \text{and} \quad \mathcal{L}_{\{\cap,\cup,\Rightarrow,\neg,\mathbf{I}\}}$$

For any formula A of $\mathcal{L}_{\{\neg,\cap,\cup,\Rightarrow\}}$,

$$\vdash_H A \quad \text{if and only if} \quad \vdash_{S5} g(A)$$

where $H, S5$ denote any proof systems for classical and and $S5$ modal logic, respectively