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Anita Wasilewska
Chapter 5
Hilbert Proof Systems
Completeness of Classical Propositional Logic

CHAPTER 5 SLIDES
Chapter 5
Hilbert Proof Systems
Completeness of Classical Propositional Logic

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Slides Set 1

PART 1: Hilbert Proof Systems: Proof System $H_1$
Hilbert Proof Systems

Hilbert proof systems are based on a language with implication and contain Modus Ponens as a rule of inference.

Modus Ponens is probably the oldest of all known rules of inference as it was already known to the Stoics (3 B.C.). It is also considered as the most natural to our intuitive thinking and the proof systems containing Modus Ponens as the inference rule play a special role in logic.

Hilbert systems put major emphasis on logical axioms, keeping the rules of inference to minimum often admitting Modus Ponens as the sole rule of inference.
Hilbert Proof Systems

There are many proof systems that describe classical propositional logic, i.e. that are complete with respect to the classical semantics.

We present a Hilbert proof system for the classical propositional logic and discuss two ways of proving the Completeness Theorem for it.

The first proof is based on the one included in Elliott Mendelson’s book Introduction to Mathematical Logic. It is is a constructive proof that shows how one can use the assumption that a formula $A$ is a tautology in order to construct its formal proof.
Hilbert Proof Systems

The second proof is non-constructive.

Its importance lies in a fact that the methods it uses can be applied to the proof of completeness theorem for classical predicate logic as we present it in (chapter 9).

It also generalizes to some non-classical logics.
Hilbert Proof Systems

We prove completeness part of the Completeness Theorem by proving the converse implication to it.

We show how one can deduce that a formula $A$ is not a tautology from the fact that it does not have a proof.

It is hence called a counter-model construction proof.

Both proofs relay on the Deduction Theorem and so this is the theorem we are now going to prove.
Hilbert Proof System $H_1$

We consider now a Hilbert proof system $H_1$ based on a language with implication as the only connective.

The proof system $H_1$ has only two logical axioms and has the Modus Ponens as a sole rule of inference.
Hilbert Proof System  $H_1$

**Definition**

**Hilbert system**  $H_1$ is defined as follows

$$H_1 = (\mathcal{L}_{\Rightarrow}, \mathcal{F}, \{A_1, A_2\}, MP)$$

**A1** (Law of simplification)

$$(A \Rightarrow (B \Rightarrow A))$$

**A2** (Frege’s Law)

$$(((A \Rightarrow (B \Rightarrow C)) \Rightarrow ((A \Rightarrow B) \Rightarrow (A \Rightarrow C)))$$

**MP** is the Modus Ponens rule

$$MP \quad \frac{A \ ; \ (A \Rightarrow B)}{B}$$

where  $A, B, C$ are any formulas from $\mathcal{F}$
Formal Proofs in $H_1$

The **formal proof** of

$$(A \Rightarrow A)$$

in $H_1$ is a sequence

$$B_1, B_2, B_3, B_4, B_5$$

as defined below

1. $B_1$ \(((A \Rightarrow ((A \Rightarrow A) \Rightarrow A))) \Rightarrow ((A \Rightarrow (A \Rightarrow A)) \Rightarrow (A \Rightarrow A)))\)
2. Axiom A2 for $A = A$, $B = (A \Rightarrow A)$, and $C = A$
3. $B_2$ \((A \Rightarrow ((A \Rightarrow A) \Rightarrow A))\)
4. Axiom A1 for $A = A$, $B = (A \Rightarrow A)$
5. $B_3$ \((A \Rightarrow (A \Rightarrow A)) \Rightarrow (A \Rightarrow A))\)
6. MP application to $B_1$ and $B_2$
7. $B_4$ \((A \Rightarrow (A \Rightarrow A))\),
   axiom A1 for $A = A$, $B = A$
8. $B_5$ \((A \Rightarrow A)\)
9. MP application to $B_3$ and $B_4$
Formal Proofs in $H_1$

We have hence proved the following

**Fact**

For any $A \in F$, $\vdash_{H_1} (A \Rightarrow A)$

It is easy to see that the proof of $(A \Rightarrow A)$ wasn’t constructed automatically.

The main step in its construction was the choice of a proper form (substitution) of logical axioms to start with, and to continue the proof with.

This choice is far from obvious for un-experienced human and impossible for a machine, as the number of possible substitutions is infinite.
In Chapter 4 we gave some examples of simple proof systems with inference rules such that it was possible to "reverse" the usual way they were used. We could use them in a reverse manner in order to search for proofs.

Moreover and we were able to do so in an effective and fully automatic way.

We called such proof systems syntactically decidable and we defined them formally as follows.
Syntactically Decidable Proof Systems

Definition
A proof system \( S = (\mathcal{L}, \mathcal{E}, \mathcal{L}_A, \mathcal{R}) \) for which there is an effective mechanical procedure that finds (generates) a formal proof of any expression \( E \in \mathcal{E} \), if it exists, is called a syntactically semi-decidable system.

If additionally there is an effective method of deciding that if a proof of \( E \) is not found that it does not exist, the system \( S \) is called syntactically decidable.

Otherwise \( S \) is syntactically undecidable.
Searching for Proofs in a Proof Systems

We will argue now, that the presence of Modus Ponens inference rule in Hilbert systems makes them syntactically undecidable.

A general procedure for automated search for proofs in a proof system $S$ can be stated as follows. Let $B$ be an expression of the system $S$ that is not an axiom. If $B$ has a proof in $S$, $B$ must be the conclusion of one of the inference rules. Let’s say it is a rule $r$. We find all its premisses, i.e. we evaluate $r^{-1}(B)$. If all premisses are axioms, the proof is found. Otherwise we repeat the procedure for any non-axiom premiss.
Search for Proof by the Means of MP

Search for proofs in any Hilbert System $S$ must involve, between other rules, if any, the Modus Ponens inference rule.

Let's analyze a search for proofs by the means of Modus Ponens rule MP.

The MP rule says: given two formulas $A$ and $(A \Rightarrow B)$ we conclude a formula $B$.

Assume now that we have a certain formula, we name it for convenience $B$.

We want to find a proof of $B$.

If $B$ is an axiom, we have the proof; the formula itself.
Search for Proof by the Means of MP

If $B$ is not an axiom, it was obtained by the application of the Modus Ponens rule, to certain two formulas $A$ and $(A \Rightarrow B)$.

But there is infinitely many of formulas $A$, $(A \Rightarrow B)$, as $A$ is any formula. It means that in for any $B$, $MP^{-1}(B)$ is countably infinite.

Obviously, we have the following Fact.

Every Hilbert System $S$ is not syntactically decidable.

In particular, the system $H_1$ is not syntactically decidable.
Semantic Links

Semantic Link 1

System $H_1$ is sound under classical, Ł, H semantics and not sound under K semantics

We leave the proof of the following theorem (by induction with respect of the length of the formal proof) as an easy exercise

Soundness Theorem for $H_1$

For any $A \in \mathcal{F}$, if $\vdash_{H_1} A$, then $\models A$
The system $H_1$ is not complete under classical semantics.
It means that we have to show that not all classical tautologies have a proof in $H_1$.
We have proved in Chapter 3 that one needs $\neg$ and one of the other connectives $\cup, \cap, \Rightarrow$ to express all classical connectives, and hence all classical tautologies.

For example we can’t express negation in term of implication alone and so a tautology $(\neg\neg A \Rightarrow A)$ is not definable in the language of $H_1$, hence

$$\kappa_{H_1} (\neg\neg A \Rightarrow A)$$
Proof from Hypothesis

We have constructed a **formal proof** of

\[(A \Rightarrow A)\]

in \(H_1\) on a base of logical axioms, as an **example** of complexity of finding proofs in Hilbert systems. In order to make the **construction** of formal proofs **easier** by the use of **previously proved** formulas we use the notion of a formal proof from some **hypotheses** (and logical axioms) in any proof system

\[S = (\mathcal{L}, \mathcal{E}, LA, \mathcal{R})\]

as defined as follows in **chapter 4**
Proof from Hypothesis

Given a proof system \( S = (L, \mathcal{E}, LA, R) \)

While proving expressions we often use some extra information available, besides the axioms of the proof system. This extra information is called hypothesis in the proof.

Let \( \Gamma \subseteq \mathcal{E} \) be a set expressions called hypothesis.

**Definition**

A proof of \( E \in \mathcal{E} \) from the set of hypothesis \( \Gamma \) in \( S \) is a formal proof in \( S \), where the expressions from \( \Gamma \) are treated as additional hypothesis added to the set \( LA \) of the logical axioms of the system \( S \).

**Notation:** \( \Gamma \vdash_S E \)

We read it: \( E \) has a proof in \( S \) from the set \( \Gamma \) (and the logical axioms \( LA \)).
Formal Definition

Definition
We say that $E \in \mathcal{E}$ has a formal proof in $S$ from the set $\Gamma$ and the logical axioms $\mathcal{L}A$ and denote it as $\Gamma \vdash_{S} E$ if and only if there is a sequence $A_1, \ldots, A_n$ of expressions from $\mathcal{E}$, such that

$$A_1 \in \mathcal{L}A \cup \Gamma, \quad A_n = E$$

and for each $1 < i \leq n$, either $A_i \in \mathcal{L}A \cup \Gamma$ or $A_i$ is a direct consequence of some of the preceding expressions by virtue of one of the rules of inference of $S$. 
Special Cases

Case 1: $\Gamma \subseteq \mathcal{E}$ is a finite set and $\Gamma = \{B_1, B_2, ..., B_n\}$
We write $B_1, B_2, ..., B_n \vdash_S E$
instead of $\{B_1, B_2, ..., B_n\} \vdash_S E$

Case 2: $\Gamma = \emptyset$
By the definition of a proof of $E$ from $\Gamma$, $\emptyset \vdash_S E$ means that in the proof of $E$ we use only the logical axioms $\text{LA}$ of $S$
We hence write $\vdash_S E$
to denote that $E$ has a proof from $\Gamma = \emptyset$
Proof from Hypothesis in $H_1$

Show that

$$(A \implies B), \ (B \implies C) \vdash_{H_1} (A \implies C)$$

We construct a formal proof

$$B_1, B_2, \ldots, B_7$$

$B_1 : (B \implies C), \quad B_2 : (A \implies B),$

hypothesis \quad hypothesis

$B_3 : ((A \implies (B \implies C)) \implies ((A \implies B) \implies (A \implies C))),$

axiom $A2$
Proof from Hypothesis in $H_1$

$B_4 : ((B \Rightarrow C) \Rightarrow (A \Rightarrow (B \Rightarrow C)))$,  
axiom A1 for $A = (B \Rightarrow C)$, $B = A$

$B_5 : (A \Rightarrow (B \Rightarrow C))$,  
$B_1$ and $B_4$ and MP

$B_6 : ((A \Rightarrow B) \Rightarrow (A \Rightarrow C))$,  
$B_7 : (A \Rightarrow C)$

MP
Deduction Theorem

In mathematical arguments, one often proves a statement $B$ on the assumption of some other statement $A$ and then concludes that we have proved the implication "if $A$, then $B". This reasoning is justified a theorem, called a Deduction Theorem.

Reminder

We write $\Gamma, A \vdash B$ for $\Gamma \cup \{A\} \vdash B$.

In general, we write $\Gamma, A_1, A_2, ..., A_n \vdash B$ for $\Gamma \cup \{A_1, A_2, ..., A_n\} \vdash B$. 
Deduction Theorem for $H_1$

**Deduction Theorem** for $H_1$

For any $A, B \in \mathcal{F}$ and $\Gamma \subseteq \mathcal{F}$

$$\Gamma, A \vdash_{H_1} B \text{ if and only if } \Gamma \vdash_{H_1} (A \Rightarrow B)$$

In particular

$$A \vdash_{H_1} B \text{ if and only if } \vdash_{H_1} (A \Rightarrow B)$$
The proof of the following Lemma provides a good example of multiple applications of the Deduction Theorem Lemma.

For any $A, B, C \in \mathcal{F}$,

(a) $(A \Rightarrow B), (B \Rightarrow C) \vdash_{H_1} (A \Rightarrow C)$,

(b) $(A \Rightarrow (B \Rightarrow C)) \vdash_{H_1} (B \Rightarrow (A \Rightarrow C))$

Observe that by Deduction Theorem we can re-write (a) as

(a') $(A \Rightarrow B), (B \Rightarrow C), A \vdash_{H_1} C$
Poof of \((a')\)

We construct a formal proof \(B_1, B_2, B_3, B_4, B_5\) of \((A \Rightarrow B), (B \Rightarrow C), A \vdash_{H_1} C\) as follows.

- \(B_1 : (A \Rightarrow B)\)
  hypothesis

- \(B_2 : (B \Rightarrow C)\)
  hypothesis

- \(B_3 : A\)
  hypothesis

- \(B_4 : B\)

- \(B_1, B_3\) and MP

- \(B_5 : C\)

- \(B_2, B_4\) and MP
Thus we proved by **Deduction Theorem** that (a) holds, i.e.

\[(A \Rightarrow B), (B \Rightarrow C) \vdash_{H_1} (A \Rightarrow C)\]

**Proof of Lemma** part (b)

By **Deduction Theorem** we have that

\[(A \Rightarrow (B \Rightarrow C)), B \vdash_{H_1} (A \Rightarrow C)\]
Formal Proofs

We construct a formal proof

$$B_1, B_2, B_3, B_4, B_5, B_6, B_7$$

of $$(A \Rightarrow (B \Rightarrow C)), B \vdash_{H_1} (A \Rightarrow C)$$ as follows.

$$B_1 : (A \Rightarrow (B \Rightarrow C))$$

hypothesis

$$B_2 : B$$

hypothesis

$$B_3 : ((B \Rightarrow (A \Rightarrow B))$$

A1 for $A = B, B = A$

$$B_4 : (A \Rightarrow B)$$

$$B_2, B_3$$ and MP
Formal Proofs

$B_5 : \; ((A \Rightarrow (B \Rightarrow C)) \Rightarrow ((A \Rightarrow B) \Rightarrow (A \Rightarrow C)))$

axiom $A2$

$B_6 : \; ((A \Rightarrow B) \Rightarrow (A \Rightarrow C))$

$B_1, B_5$ and MP

$B_7 : \; (A \Rightarrow C)$

Thus we proved by Deduction Theorem that

$$(A \Rightarrow (B \Rightarrow C)) \vdash_{H_1} (B \Rightarrow (A \Rightarrow C))$$
Simpler Proof

Here is a simpler proof of Lemma part (b).

We apply the Deduction Theorem twice, i.e. we get

\[(A \Rightarrow (B \Rightarrow C)) \vdash_{H_1} (B \Rightarrow (A \Rightarrow C))\]

if and only if

\[(A \Rightarrow (B \Rightarrow C)), B \vdash_{H_1} (A \Rightarrow C)\]

if and only if

\[(A \Rightarrow (B \Rightarrow C)), B, A \vdash_{H_1} C\]
Simpler Proof

We now construct a proof of $\left( A \Rightarrow (B \Rightarrow C) \right), B, A \vdash^H_1 C$ as follows:

- $B_1 \quad (A \Rightarrow (B \Rightarrow C))$
  - hypothesis
- $B_2 \quad B$
  - hypothesis
- $B_3 \quad A$
  - hypothesis
- $B_4 \quad (B \Rightarrow C)$
  - $B_1, B_3$ and MP
- $B_5 \quad C$
  - $B_2, B_4$ and MP
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Slides Set 1

PART 2: Proof of Deduction Theorem for $H_1$
The Deduction Theorem for $H_1$

As we now fix the proof system to be $H_1$, we write $A \vdash B$ instead of $A \vdash_{H_1} B$

Deduction Theorem  (Herbrand, 1930) for $H_1$
For any formulas $A, B \in \mathcal{F}$,

\[
\text{If } A \vdash B, \text{ then } \vdash (A \Rightarrow B)
\]

Deduction Theorem  (General case) for $H_1$
For any formulas $A, B \in \mathcal{F}, \Gamma \subseteq \mathcal{F}$

\[
\Gamma, A \vdash B \quad \text{if and only if} \quad \Gamma \vdash (A \Rightarrow B)
\]
Proof of The Deduction Theorem

Proof:
Part 1  We first prove the ”if” part:

If  \( \Gamma, A \vdash B \)  then  \( \Gamma \vdash (A \Rightarrow B) \)

Assume that
\[ \Gamma, A \vdash B \]

i.e. that we have a formal proof
\[ B_1, B_2, ..., B_n \]

of  \( B \)  from the set of formulas  \( \Gamma \cup \{A\} \)

We have to show that
\[ \Gamma \vdash (A \Rightarrow B) \]
Proof of The Deduction Theorem

In order to prove that
\[ \Gamma \vdash (A \Rightarrow B) \] follows from \( \Gamma, A \vdash B \)
we prove a stronger statement, namely that
\[ \Gamma \vdash (A \Rightarrow B_i) \]
for any \( B_i, 1 \leq i \leq n \) in the formal proof \( B_1, B_2, \ldots, B_n \) of \( B \)
also follows from \( \Gamma, A \vdash B \)

Hence in particular case, when \( i = n \) we will obtain that
\[ \Gamma \vdash (A \Rightarrow B) \] follows from \( \Gamma, A \vdash B \)
and that will end the proof of Part 1
Base Step

The proof of Part 1 is conducted by mathematical induction on \( i \), for \( 1 \leq i \leq n \)

**Step 1** \( i = 1 \) (base step)

Observe that when \( i = 1 \), it means that the formal proof \( B_1, B_2, ..., B_n \) contains only one element \( B_1 \)

By the definition of the formal proof from \( \Gamma \cup \{A\} \), we have that

1. \( B_1 \) is a logical axiom, or \( B_1 \in \Gamma \), or
2. \( B_1 = A \)

This means that \( B_1 \in \{A_1, A_2\} \cup \Gamma \cup \{A\} \)
Now we have **two cases** to consider.

**Case 1:** \( B_1 \in \{A_1, A_2\} \cup \Gamma \)

**Observe** that \( (B_1 \Rightarrow (A \Rightarrow B_1)) \) is the axiom \( A_1 \)

By assumption \( B_1 \in \{A_1, A_2\} \cup \Gamma \)

We get the **required proof** of \( (A \Rightarrow B_1) \) from \( \Gamma \)

by the following application of the **Modus Ponens rule**

\[
(MP) \quad \frac{B_1 ; (B_1 \Rightarrow (A \Rightarrow B_1))}{(A \Rightarrow B_1)}
\]
Base Step

Case 2: $B_1 = A$

When $B_1 = A$ then to prove $\Gamma \vdash (A \Rightarrow B_1)$

This means we have to prove $\Gamma \vdash (A \Rightarrow A)$

This holds by monotonicity of the consequence and the fact that we have shown that $\vdash (A \Rightarrow A)$

The above cases conclude the proof for $i = 1$ of $\Gamma \vdash (A \Rightarrow B_i)$
Inductive Step

Assume that

\[ \Gamma \vdash (A \Rightarrow B_k) \]

for all \( k < i \) (strong induction)

We will show that using this fact we can conclude that also

\[ \Gamma \vdash (A \Rightarrow B_i) \]
Inductive Step

Consider a formula $B_i$ in the formal proof $B_1, B_2, ..., B_n$

By definition of the formal proof we have to show the following tow cases

Case 1: $B_i \in \{A_1, A_2\} \cup \Gamma \cup \{A\}$ and

Case 2: $B_i$ follows by MP from certain $B_j, B_m$ such that $j < m < i$

Consider now the Case 1: $B_i \in \{A_1, A_2\} \cup \Gamma \cup \{A\}$

The proof of $(A \Rightarrow B_i)$ from $\Gamma$ in this case is obtained from the proof of the Step $i = 1$ by replacement $B_1$ by $B_i$

and is omitted here as a straightforward repetition
Inductive Step

Case 2:

$B_i$ is a conclusion of (MP)

If $B_i$ is a conclusion of (MP), then we must have two formulas $B_j, B_m$ in the formal proof $B_1, B_2, ..., B_n$ such that $j < i, m < i, j \neq m$ and

$$\frac{B_j; B_m}{B_i} \quad (MP)$$
Inductive Step

By the **inductive assumption** the formulas $B_j$, $B_m$ are such that $\Gamma \vdash (A \Rightarrow B_j)$ and $\Gamma \vdash (A \Rightarrow B_m)$.

Moreover, by the definition of (MP) rule, the formula $B_m$ has to have a form $(B_j \Rightarrow B_i)$

This means that

$$B_m = (B_j \Rightarrow B_i)$$

**The inductive assumption** can be re-written as follows

$$\Gamma \vdash (A \Rightarrow (B_j \Rightarrow B_i))$$

for $j < i$
Inductive Step

**Observe** now that the formula

\[ (((A \Rightarrow (B_j \Rightarrow B_i)) \Rightarrow ((A \Rightarrow B_j) \Rightarrow (A \Rightarrow B_i))) \]

is a **substitution of the axiom A2** and hence **has a proof**
in our system.

By the **monotonicity** of the consequence, it also has a proof
from the set \( \Gamma \), i.e.

\[ \Gamma \vdash (((A \Rightarrow (B_j \Rightarrow B_i)) \Rightarrow ((A \Rightarrow B_j) \Rightarrow (A \Rightarrow B_i))) \]
Inductive Step

We know that

\[ \Gamma \vdash ((A \Rightarrow (B_j \Rightarrow B_i)) \Rightarrow ((A \Rightarrow B_j) \Rightarrow (A \Rightarrow B_i))) \]

Applying the rule MP i.e. performing the following

\[ (A \Rightarrow (B_j \Rightarrow B_i)) ; ((A \Rightarrow (B_j \Rightarrow B_i)) \Rightarrow ((A \Rightarrow B_j) \Rightarrow (A \Rightarrow B_i))) \]
\[ ((A \Rightarrow B_j) \Rightarrow (A \Rightarrow B_i)) \]

we get that also

\[ \Gamma \vdash ((A \Rightarrow B_j) \Rightarrow (A \Rightarrow B_i)) \]
Inductive Step

Applying again the rule MP i.e. performing the following

\[
\frac{(A \Rightarrow B_j) ; ((A \Rightarrow B_j) \Rightarrow (A \Rightarrow B_i))}{(A \Rightarrow B_i)}
\]

we get that

\[\Gamma \vdash (A \Rightarrow B_i)\]

what ends the proof of the inductive step
Proof of the Deduction Theorem

By the mathematical induction principle, we have proved that

\[ \Gamma \vdash (A \Rightarrow B_i), \quad \text{for all} \quad 1 \leq i \leq n \]

In particular it is true for \( i = n \), i.e. for \( B_n = B \) and we proved that

\[ \Gamma \vdash (A \Rightarrow B) \]

This ends the proof of the first part of the Deduction Theorem:

If \( \Gamma, A \vdash B \), then \( \Gamma \vdash (A \Rightarrow B) \)
Proof of the Deduction Theorem

The proof of the second part, i.e. of the inverse implication:

If \( \Gamma \vdash (A \Rightarrow B) \), then \( \Gamma, A \vdash B \)

is straightforward and goes as follows.

Assume that \( \Gamma \vdash (A \Rightarrow B) \)

By the monotonicity of the consequence we have also that \( \Gamma, A \vdash (A \Rightarrow B) \)

Obviously \( \Gamma, A \vdash A \)

Applying Modus Ponens to the above, we get the proof of \( B \) from \( \{\Gamma, A\} \)

We have hence proved that \( \Gamma, A \vdash B \)

This ends the proof
Proof of the Deduction Theorem

Deduction Theorem (General case) for $H_1$
For any formulas $A, B \in \mathcal{F}$ and any $\Gamma \subseteq \mathcal{F}$

$$\Gamma, A \vdash B \quad \text{if and only if} \quad \Gamma \vdash (A \Rightarrow B)$$

The particular case we get also the particular case

Deduction Theorem (Herbrand, 1930) for $H_1$
For any formulas $A, B \in \mathcal{F}$,

If $A \vdash B$, then $\vdash (A \Rightarrow B)$

is obtained from the above by assuming that the set $\Gamma$ is empty
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Slides Set 2
PART 3: Proof System $H_2$: Deduction Theorem, Exercises and Examples
Proof System $H_2$

The proof system $H_1$ is **sound** and strong enough to prove the Deduction Theorem, but, as we proved, is **not complete**

We extend now the language and the set of logical axioms of $H_1$ to form a new **Hilbert** system $H_2$ that is **complete** with respect to **classical** semantics

The proof of **Completeness Theorem** for $H_2$ is be presented in the next section (**Slides Set 3**)
Hilbert System $H_2$ Definition

Definition

$$H_2 = ( \mathcal{L}_{\rightarrow,\neg}, \mathcal{F}, \{A1, A2, A3\} (MP) )$$

A1 (Law of simplification)
$$(A \Rightarrow (B \Rightarrow A))$$

A2 (Frege’s Law)
$$(((A \Rightarrow (B \Rightarrow C)) \Rightarrow ((A \Rightarrow B) \Rightarrow (A \Rightarrow C))))$$

A3 $$(((\neg B \Rightarrow \neg A) \Rightarrow (((\neg B \Rightarrow A) \Rightarrow B)))$$

MP (Rule of inference)

$$\text{(MP)} \quad A; (A \Rightarrow B) \quad \frac{}{B}$$

where $A, B, C$ are any formulas of the propositional language $\mathcal{L}_{\rightarrow,\neg}$
Deduction Theorem for System $H_2$

Observation 1
The proof system $H_2$ is obtained by adding axiom $A_3$ to the system $H_1$

Observation 2
The language of $H_2$ is obtained by adding the connective $\neg$ to the language of $H_1$

Observation 3
The use of axioms $A_1, A_2$ in the proof of Deduction Theorem for the system $H_1$ is independent of the connective $\neg$ added to the language of $H_1$

Observation 4
Hence the proof of the Deduction Theorem for the system $H_1$ can be repeated as it is for the system $H_2$
Deduction Theorem for System $H_2$

**Observations 1-4** prove that the **Deduction Theorem** holds for system $H_2$

**Deduction Theorem** for $H_2$

For any $\Gamma \subseteq \mathcal{F}$ and $A, B \in \mathcal{F}$

$$\Gamma, A \vdash_{H_2} B \text{ if and only if } \Gamma \vdash_{H_2} (A \Rightarrow B)$$

In particular

$$A \vdash_{H_2} B \text{ if and only if } \vdash_{H_2} (A \Rightarrow B)$$
Soundness and Completeness Theorems

We get by easy verification that $H_2$ is a **sound** under classical semantics and hence we have the following

**Soundness Theorem** $H_2$

For every formula $A \in \mathcal{F}$

\[
\text{if } \vdash_{H_2} A \text{ then } \models A
\]

We prove in the next section (**Slides Set 3**), that $H_2$ is also **complete** under classical semantics, i.e. we prove

**Completeness Theorem** for $H_2$

For every formula $A \in \mathcal{F}$,

\[
\vdash_{H_2} A \text{ if and only if } \models A
\]
Completeness Theorems

The proof of completeness theorem (for a given semantics) is always a main point in creation of any new logic.

There are many techniques to prove it, depending on the proof system, and on the semantics we define for it.

We present in the next next section (Slides Set 2) two proofs of the Completeness Theorem for the system $H_2$.

These proofs use very different techniques, hence the reason of presenting both of them.
Proof System $H_2$: Exercises and Examples
Examples and Exercises

We present now some examples of formal proofs in $H_2$.

There are two reasons for presenting them.

First reason is that all formulas we provide the formal proofs for play a crucial role in the proof of Completeness Theorem for $H_2$.

The second reason is that they provide a "training ground" for a reader to learn how to develop formal proofs.

For this reason we write some formal proofs in a full detail and we leave some for the reader to complete in a way explained in the following example.
Important Lemma

We write $\vdash$ instead of $\vdash_{H_2}$ for the sake of simplicity.

Reminder
In the construction of the formal proofs we often use the Deduction Theorem and the following Lemma 1 that was proved in the previous section.

Lemma 1

(a) $(A \Rightarrow B), (B \Rightarrow C) \vdash_{H_2} (A \Rightarrow C)$

(b) $(A \Rightarrow (B \Rightarrow C)) \vdash_{H_2} ((B \Rightarrow (A \Rightarrow C)))$
Example 1

Here are consecutive steps

\[ B_1, \ldots, B_5, B_6 \]

of the proof in \( H_2 \) of \((\neg
\neg B \Rightarrow B)\)

\(B_1: \quad (((\neg B \Rightarrow \neg \neg B) \Rightarrow ((\neg B \Rightarrow \neg B) \Rightarrow B)))\)

\(B_2: \quad (((\neg B \Rightarrow \neg B) \Rightarrow ((\neg B \Rightarrow \neg \neg B) \Rightarrow B)))\)

\(B_3: \quad (\neg B \Rightarrow \neg B)\)

\(B_4: \quad (((\neg B \Rightarrow \neg \neg B) \Rightarrow B))\)

\(B_5: \quad (\neg \neg B \Rightarrow (\neg B \Rightarrow \neg \neg B))\)

\(B_6: \quad (\neg \neg B \Rightarrow B)\)
Exercise 1

Complete the proof presented in Example 1 by providing comments how each step of the proof was obtained.

Remark
The solution presented on the next slide shows how to write details of solutions.
Solutions of other problems presented later are less detailed.
Exercise 1 Solution

Solution
The comments that complete the proof are as follows.

\[ B_1 : \quad ((\neg B \Rightarrow \neg \neg B) \Rightarrow ((\neg B \Rightarrow \neg B) \Rightarrow B)) \]

Axiom \textit{A3} for \( A = \neg B, \quad B = B \)

\[ B_2 : \quad ((\neg B \Rightarrow \neg B) \Rightarrow ((\neg B \Rightarrow \neg \neg B) \Rightarrow B)) \]

\( B_1 \) and \textit{Lemma 1 (b)} for
\( A = (\neg B \Rightarrow \neg \neg B), \quad B = (\neg B \Rightarrow \neg B), \quad C = B, \)

i.e. we have

\[ ((\neg B \Rightarrow \neg \neg B) \Rightarrow ((\neg B \Rightarrow \neg B) \Rightarrow B)) \vdash ((\neg B \Rightarrow \neg B) \Rightarrow ((\neg B \Rightarrow \neg \neg B) \Rightarrow B)) \]
Exercise 1 Solution

$B_3 : (\neg B \Rightarrow \neg B)$

We proved for $H_1$ and hence for $H_2$ that $\vdash (A \Rightarrow A)$ and we substitute $A = \neg B$

$B_4 : ((\neg B \Rightarrow \neg \neg B) \Rightarrow B)$

$B_2, B_3$ and MP

$B_5 : (\neg \neg B \Rightarrow (\neg B \Rightarrow \neg \neg B))$

Axiom A1 for $A = \neg \neg B$, $B = \neg B$

$B_6 : (\neg \neg B \Rightarrow B)$

$B_4, B_5$ and Lemma 1 (a) for $A = \neg \neg B$, $B = (\neg B \Rightarrow \neg \neg B)$, $C = B$

i.e. we have

$(\neg \neg B \Rightarrow (\neg B \Rightarrow \neg \neg B)), ((\neg B \Rightarrow \neg \neg B) \Rightarrow B) \vdash (\neg \neg B \Rightarrow B)$
Proofs from Axioms Only

General remark

Observe that in steps

\[ B_2, B_3, B_5, B_6 \]

of the proof we called on previously proved facts and used them as a part of the proof.

We can always obtain a formal proof that uses only axioms of the system by inserting previously constructed formal proofs of these facts into the places occupying by the respective steps \( B_2, B_3, B_5, B_6 \) where these facts were used.
Proofs from Axioms

Example
Consider the step

\[ B_3 : (\neg B \Rightarrow \neg B) \]

The formula \((\neg B \Rightarrow \neg B)\) is a previously proved fact. We replace the formula \((\neg B \Rightarrow \neg B)\) (in step step \(B_3\) by its formal proof that uses only axioms. We obtain this proof from the previously constructed proof of \((A \Rightarrow A)\) by replacing \(A\) by \(\neg B\).

The last step of the inserted proof becomes now "old" step \(B_3\) and we re-numerate all other steps accordingly.
Proofs from Axioms Only

Here are consecutive first THREE steps of the proof of $(\neg\neg B \Rightarrow B)$

$B_1 : \quad ((\neg B \Rightarrow \neg\neg B) \Rightarrow ((\neg B \Rightarrow \neg B) \Rightarrow B))$

$B_2 : \quad ((\neg B \Rightarrow \neg B) \Rightarrow ((\neg B \Rightarrow \neg\neg B) \Rightarrow B))$

$B_3 : \quad (\neg B \Rightarrow \neg B)$

We insert now the proof of $(\neg B \Rightarrow \neg B)$ after step $B_2$ and erase the $B_3$

The last step of the inserted proof becomes the erased $B_3$
Proofs from Axioms Only

A part of new transformed proof is

\[ B_1 : \quad ((\neg B \Rightarrow \neg \neg B) \Rightarrow ((\neg B \Rightarrow \neg B) \Rightarrow B)) \quad (\text{Old } B_1) \]

\[ B_2 : \quad ((\neg B \Rightarrow \neg B) \Rightarrow ((\neg B \Rightarrow \neg \neg B) \Rightarrow B)) \quad (\text{Old } B_2) \]

We insert here the proof from axioms only of Old \( B_3 \)

\[ B_3 : \quad ((\neg B \Rightarrow ((\neg B \Rightarrow \neg B) \Rightarrow \neg B))) \Rightarrow ((\neg B \Rightarrow (\neg B \Rightarrow \neg B))) \Rightarrow (\neg B \Rightarrow \neg B)), \quad (\text{New } B_3) \]

\[ B_4 : \quad (\neg B \Rightarrow ((\neg B \Rightarrow \neg B) \Rightarrow \neg B)) \]

\[ B_5 : \quad ((\neg B \Rightarrow (\neg B \Rightarrow \neg B)) \Rightarrow (\neg B \Rightarrow \neg B))) \]

\[ B_6 : \quad (\neg B \Rightarrow (\neg B \Rightarrow \neg B)) \]

\[ B_7 : \quad (\neg B \Rightarrow \neg B) \quad (\text{Old } B_3) \]
Proofs from Axioms Only

We repeat our procedure by replacing the step $B_2$ by its formal proof as defined in the proof of the Lemma 1 (b).

We continue the process for all other steps which involved application of the Lemma 1 until we get a full formal proof from the axioms of $H_2$ only.

Usually we don’t do it and we don’t need to do it, but it is important to remember that it always can be done.
Example 2

Here are consecutive steps

\[ B_1, \ B_2, \ \ldots, \ B_5 \]

in a proof of \((B \Rightarrow \neg\neg B)\)

\[
B_1 \quad ((\neg\neg\neg B \Rightarrow \neg B) \Rightarrow ((\neg\neg\neg B \Rightarrow B) \Rightarrow \neg\neg B))
\]

\[
B_2 \quad (\neg\neg\neg B \Rightarrow \neg B)
\]

\[
B_3 \quad ((\neg\neg\neg B \Rightarrow B) \Rightarrow \neg\neg B)
\]

\[
B_4 \quad (B \Rightarrow (\neg\neg\neg B \Rightarrow B))
\]

\[
B_5 \quad (B \Rightarrow \neg\neg B)
\]
Exercise 2

Complete the proof presented in Example 2 by providing detailed comments how each step of the proof was obtained.

Solution

The comments that complete the proof are as follows.

\( B_1 \quad ((\neg\neg\neg B \Rightarrow \neg B) \Rightarrow ((\neg\neg\neg B \Rightarrow B) \Rightarrow \neg\neg B)) \)

Axiom A3 for \( A = B, \quad B = \neg\neg B \)

\( B_2 \quad (\neg\neg\neg B \Rightarrow \neg B) \)

Example 1 for \( B = \neg B \)
Exercise 2

\[ B_3 \quad ((\neg\neg B \Rightarrow B) \Rightarrow \neg\neg B) \]
\[ B_1, B_2 \quad \text{and MP} \]
i.e. we have that
\[
\frac{(\neg\neg B \Rightarrow \neg B); ((\neg\neg B \Rightarrow \neg B) \Rightarrow ((\neg\neg B \Rightarrow \neg B) \Rightarrow \neg\neg B))}{((\neg\neg B \Rightarrow \neg B) \Rightarrow \neg\neg B)}
\]
\[ B_4 \quad (B \Rightarrow (\neg\neg B \Rightarrow B)) \]
Axiom A1 for \( A = B, \quad B = \neg\neg B \)
\[ B_5 \quad (B \Rightarrow \neg\neg B) \]
\[ B_3, B_4 \quad \text{and Lemma 1 (a) for} \]
\[ A = B, \quad B = (\neg\neg B \Rightarrow B), \quad C = \neg\neg B, \]
i.e. we have that
\[(B \Rightarrow (\neg\neg B \Rightarrow B)), ((\neg\neg B \Rightarrow B) \Rightarrow \neg\neg B) \vdash (B \Rightarrow \neg\neg B)\]
Example 3

Here are consecutive steps \( B_1, B_2, \ldots, B_{12} \) in a proof of \( (\neg A \Rightarrow (A \Rightarrow B)) \):

- \( B_1 \): \( \neg A \)
- \( B_2 \): \( A \)
- \( B_3 \): \( (A \Rightarrow (\neg B \Rightarrow A)) \)
- \( B_4 \): \( (\neg A \Rightarrow (\neg B \Rightarrow \neg A)) \)
- \( B_5 \): \( (\neg B \Rightarrow A) \)
- \( B_6 \): \( (\neg B \Rightarrow \neg A) \)
- \( B_7 \): \( ((\neg B \Rightarrow \neg A) \Rightarrow ((\neg B \Rightarrow A) \Rightarrow B)) \)
Example 3

$B_8 \quad ((\neg B \Rightarrow A) \Rightarrow B)$
$B_9 \quad B$
$B_{10} \quad \neg A, A \vdash B$
$B_{11} \quad \neg A \vdash (A \Rightarrow B)$
$B_{12} \quad (\neg A \Rightarrow (A \Rightarrow B))$

Exercise 3

1. **Complete** the proof from the Example 3 by providing comments how each step of the proof was obtained.

2. **Prove** that

   $\neg A, A \vdash B$
Exercise 4

Example 4

Here are consecutive steps $B_1, ..., B_7$ in a proof of $((\neg B \Rightarrow \neg A) \Rightarrow (A \Rightarrow B))$

$B_1 \ (\neg B \Rightarrow \neg A)$

$B_2 \ ((\neg B \Rightarrow \neg A) \Rightarrow ((\neg B \Rightarrow A) \Rightarrow B))$

$B_3 \ (A \Rightarrow (\neg B \Rightarrow A))$

$B_4 \ ((\neg B \Rightarrow A) \Rightarrow B)$

$B_5 \ (A \Rightarrow B)$

$B_6 \ (\neg B \Rightarrow \neg A) \vdash (A \Rightarrow B)$

$B_7 \ ((\neg B \Rightarrow \neg A) \Rightarrow (A \Rightarrow B))$

Exercise 4

Complete the proof from Example 4 by providing comments how each step of the proof was obtained
Example 5

Example 5
Here are consecutive steps \( B_1, \ldots, B_9 \) in a proof of \( ((A \Rightarrow B) \Rightarrow (\neg B \Rightarrow \neg A)) \)

\[ B_1 \quad (A \Rightarrow B) \]
\[ B_2 \quad (\neg\neg A \Rightarrow A) \]
\[ B_3 \quad (\neg\neg A \Rightarrow B) \]
\[ B_4 \quad (B \Rightarrow \neg\neg B) \]
\[ B_5 \quad (\neg\neg A \Rightarrow \neg\neg B) \]
\[ B_6 \quad ((\neg\neg A \Rightarrow \neg\neg B) \Rightarrow (\neg B \Rightarrow \neg A)) \]
\[ B_7 \quad (\neg B \Rightarrow \neg A) \]
\[ B_8 \quad (A \Rightarrow B) \vdash (\neg B \Rightarrow \neg A) \]
\[ B_9 \quad ((A \Rightarrow B) \Rightarrow (\neg B \Rightarrow \neg A)) \]
Exercise 5

Exercise 5

Complete the proof of Example 5 by providing comments how each step of the proof was obtained.

Solution

\( B_1 \)  \((A \Rightarrow B)\)

Hypothesis

\( B_2 \)  \((\neg \neg A \Rightarrow A)\)

Example 1 for \( B = A \)

\( B_3 \)  \((\neg \neg A \Rightarrow B)\)

Lemma 1 (a) for \( A = \neg \neg A, \ B = A, \ C = B \)

\( B_4 \)  \((B \Rightarrow \neg \neg B)\)

Example 2
Exercise 5

$B_5 \quad (\neg\neg A \Rightarrow \neg\neg B)$

Lemma 1 (a) for $A = \neg\neg A$, $B = B$, $C = \neg\neg B$

$B_6 \quad ((\neg\neg A \Rightarrow \neg\neg B) \Rightarrow (\neg B \Rightarrow \neg A))$

Example 4 for $B = \neg A$, $A = \neg B$

$B_7 \quad (\neg B \Rightarrow \neg A)$

$B_5$, $B_6$ and MP

$B_8 \quad (A \Rightarrow B) \vdash (\neg B \Rightarrow \neg A)$

$B_1 - B_7$

$B_9 \quad ((A \Rightarrow B) \Rightarrow (\neg B \Rightarrow \neg A))$

Deduction Theorem
Example 6

Prove that

\[ \vdash (A \Rightarrow (\neg B \Rightarrow (\neg (A \Rightarrow B)))) \]

Solution

Here are consecutive steps (with comments) of building the formal proof

\[ B_1 \quad A, (A \Rightarrow B) \vdash B \]

This is MP
Example 6

\[ B_2 \quad A \vdash ((A \Rightarrow B) \Rightarrow B) \]

Deduction Theorem

\[ B_3 \quad \vdash (A \Rightarrow ((A \Rightarrow B) \Rightarrow B)) \]

Deduction Theorem

\[ B_4 \quad \vdash (((A \Rightarrow B) \Rightarrow B) \Rightarrow (\neg B \Rightarrow \neg(A \Rightarrow B))) \]

Example 5 for \( A = (A \Rightarrow B), \ B = B \)

\[ B_5 \quad \vdash (A \Rightarrow (\neg B \Rightarrow (\neg(A \Rightarrow B)))) \]

\( B_3, B_4 \) and Lemma 2 (a) for

\( A = A \quad B = ((A \Rightarrow B) \Rightarrow B), \ C = (\neg B \Rightarrow \neg(A \Rightarrow B)) \)

Observe that the proof presented is not the only proof
Example 7

Here are consecutive steps \( B_1, ..., B_{12} \) in a proof of \( ((A \Rightarrow B) \Rightarrow ((\neg A \Rightarrow B) \Rightarrow B)) \)

\begin{align*}
B_1 & : (A \Rightarrow B) \\
B_2 & : (\neg A \Rightarrow B) \\
B_3 & : ((A \Rightarrow B) \Rightarrow (\neg B \Rightarrow \neg A)) \\
B_4 & : (\neg B \Rightarrow \neg A) \\
B_5 & : ((\neg A \Rightarrow B) \Rightarrow (\neg B \Rightarrow \neg \neg A)) \\
B_6 & : (\neg B \Rightarrow \neg \neg A) \\
B_7 & : ((\neg B \Rightarrow \neg \neg A) \Rightarrow ((\neg B \Rightarrow \neg A) \Rightarrow B)))
\end{align*}
Example 7

\[ B_8 \quad ((\neg B \Rightarrow \neg A) \Rightarrow B) \]
\[ B_9 \quad B \]
\[ B_{10} \quad (A \Rightarrow B), (\neg A \Rightarrow B) \vdash B \]
\[ B_{11} \quad (A \Rightarrow B) \vdash ((\neg A \Rightarrow B) \Rightarrow B) \]
\[ B_{12} \quad ((A \Rightarrow B) \Rightarrow ((\neg A \Rightarrow B) \Rightarrow B)) \]

Exercise 7

Complete the proof in Example 7 by providing comments how each step of the proof was obtained
Exercise 7

Solution

\[ B_1 \quad (A \Rightarrow B) \]
Hypothesis

\[ B_2 \quad (\neg A \Rightarrow B) \]
Hypothesis

\[ B_3 \quad ((A \Rightarrow B) \Rightarrow (\neg B \Rightarrow \neg A)) \]

Example 5

\[ B_4 \quad (\neg B \Rightarrow \neg A) \]
\[ B_1, B_3 \quad \text{and MP} \]

\[ B_5 \quad ((\neg A \Rightarrow B) \Rightarrow (\neg B \Rightarrow \neg \neg A)) \]

Example 5 for \( A = \neg A, \ B = B \)

\[ B_6 \quad (\neg B \Rightarrow \neg \neg A) \]
\[ B_2, B_5 \quad \text{and MP} \]
Exercise 7

\[ B_7 \quad ((\neg B \Rightarrow \neg\neg A) \Rightarrow ((\neg B \Rightarrow \neg A) \Rightarrow B)) \]

Axiom A3 for \( B = B, \ A = \neg A \)

\[ B_8 \quad ((\neg B \Rightarrow \neg A) \Rightarrow B) \]

\( B_6, \ B_7 \) and MP

\[ B_9 \quad B \]

\( B_4, \ B_8 \) and MP

\[ B_{10} \quad (A \Rightarrow B), (\neg A \Rightarrow B) \vdash B \]

\( B_{11} \quad B \)

\[ (A \Rightarrow B) \vdash ((\neg A \Rightarrow B) \Rightarrow B) \]

Deduction Theorem

\[ (A \Rightarrow B) \Rightarrow ((\neg A \Rightarrow B) \Rightarrow B) \]

Deduction Theorem
Example 8

Example 8

Here are consecutive steps

\[ B_1, \ldots, B_3 \]

in a proof of

\[ ((\neg A \Rightarrow A) \Rightarrow A) \]

\[ B_1 \quad ((\neg A \Rightarrow \neg A) \Rightarrow ((\neg A \Rightarrow A) \Rightarrow A)) \]

\[ B_2 \quad (\neg A \Rightarrow \neg A) \]

\[ B_3 \quad ((\neg A \Rightarrow A) \Rightarrow A) \]
Exercise 8

Exercise 8

Complete the proof of Example 8 by providing comments how each step of the proof was obtained.

Solution

\[ B_1 \quad ((\neg A \Rightarrow \neg A) \Rightarrow ((\neg A \Rightarrow A) \Rightarrow A)) \]

Axiom A3 for \( B = A \)

\[ B_1 \quad (\neg A \Rightarrow \neg A) \]

Already proved \((A \Rightarrow A)\) for \( A = \neg A \)

\[ B_1 \quad ((\neg A \Rightarrow A) \Rightarrow A) \]

\[ B_1, B_2 \quad \text{and MP} \]
We summarize all the formal proofs in $H_2$ provided in our Examples and Exercises in a form of a following lemma.

**Lemma**

The following formulas are provable in $H_2$

1. $(A \Rightarrow A)$
2. $(\neg\neg B \Rightarrow B)$
3. $(B \Rightarrow \neg\neg B)$
4. $(\neg A \Rightarrow (A \Rightarrow B))$
5. $((\neg B \Rightarrow \neg A) \Rightarrow (A \Rightarrow B))$
6. $((A \Rightarrow B) \Rightarrow (\neg B \Rightarrow \neg A))$
7. $(A \Rightarrow (\neg B \Rightarrow (\neg(A \Rightarrow B))))$
8. $((A \Rightarrow B) \Rightarrow ((\neg A \Rightarrow B) \Rightarrow B))$
9. $((\neg A \Rightarrow A) \Rightarrow A)$
Completeness Theorem for $H_2$

Formulas 1, 3, 4, and 7-9 from the set of provable formulas from the Lemma are all formulas needed together with the logical axioms of $H_2$ to execute the two proofs of the Completeness Theorem for $H_2$.

We present these proofs in the Slides Set 3.

The two proofs represent two different methods of proving the Completeness Theorem.
Chapter 5
Hilbert Proof Systems
Completeness of Classical Propositional Logic

Slides Set 3
PART 4: Completeness Theorem Proof One: Constructive Proof
Completeness Theorem: Proof One

The Proof One of the Completeness Theorem for $H_2$ presented here is similar in its structure to the proof of the Deduction Theorem.

The Proof One is due to Kalmar, 1935 and is a detailed version of the one published in Elliott Mendelson’s book Introduction to Mathematical Logic, 1987.

The Proof One is, as Deduction Theorem was, constructive. It means it defines a method how one can use the assumption that a formula $A$ is a tautology in order to construct its formal proof.
Completeness Theorem: Proof One

The **Proof One** relies heavily on the **Deduction Theorem** and is very elegant and simple but its methods are **applicable only** to the **classical** propositional logic.

The **Proof One** is specific to a propositional language

\[ L\{\neg, \Rightarrow\} \]

and to the proof system \( H_2 \).

Nevertheless, the \( H_2 \) based **Proof One** can be **adopted** and **extended** to other **classical** propositional languages containing **implication** and **negation**.
Completeness Theorem: Proof One

For example we can adopt the Proof One to languages

\[ L\{\neg, \cup, \Rightarrow\}, \quad L\{\neg, \cap, \cup, \Rightarrow\}, \quad L\{\neg, \cap, \cup, \Rightarrow, \Leftrightarrow\} \]

and appropriate proof systems based for them

We do so by adding new special logical axioms to the logical axioms of the proof system \( H_2 \)

Such obtained proof systems are called extensions of the system \( H_2 \)
Completeness Theorem: Proof One

One can think about the system $H_2$ with its axiomatization given by set

$$\{A_1, A_2, A_3\}$$

of logical axioms, and its language

$$\mathcal{L}_{\neg, \Rightarrow}$$

as in a sense, a "minimal" Hilbert System for classical propositional logic.

The Proof One can not be extended to the classical predicate logic, neither to the variety of non-classical logics.
Proof System $H_2$

Reminder: $H_2$ is the following proof system:

$$H_2 = (\mathcal{L}_{\rightarrow, \neg}, \mathcal{F}, \{A_1, A_2, A_3\}, MP)$$

The axioms $A_1 - A_3$ are defined as follows.

A1  $(A \Rightarrow (B \Rightarrow A))$,
A2  $(((A \Rightarrow (B \Rightarrow C)) \Rightarrow ((A \Rightarrow B) \Rightarrow (A \Rightarrow C))))$,
A3  $(((\neg B \Rightarrow \neg A) \Rightarrow ((\neg B \Rightarrow A) \Rightarrow B)))$

$$(MP) \quad \frac{A ; (A \Rightarrow B)}{B}$$
Proof System $H_2$

Obviously, the selected axioms $A_1, A_2, A_3$ are tautologies, and the MP rule leads from tautologies to tautologies.

Hence our proof system $H_2$ is sound and the following theorem holds

**Soundness Theorem**

For every formula $A \in \mathcal{F}$,

If $\vdash_{H_2} A$, then $\models A$
System $H_2$ Lemma

We have proved and presented in *Slides Set 2* the following Lemma:

The following formulas are provable in $H_2$:

1. $(A \Rightarrow A)$
2. $(\neg\neg B \Rightarrow B)$
3. $(B \Rightarrow \neg\neg B)$
4. $(\neg A \Rightarrow (A \Rightarrow B))$
5. $((\neg B \Rightarrow \neg A) \Rightarrow (A \Rightarrow B))$
6. $((A \Rightarrow B) \Rightarrow (\neg B \Rightarrow \neg A))$
7. $(A \Rightarrow (\neg B \Rightarrow (\neg(A \Rightarrow B))))$
8. $((A \Rightarrow B) \Rightarrow ((\neg A \Rightarrow B) \Rightarrow B))$
9. $((\neg A \Rightarrow A) \Rightarrow A)$
Proof One

The Proof One of Completeness Theorem presented here is very elegant and simple, but is applicable only to the classical propositional logic.

This proof is, as was the proof of Deduction Theorem, a fully constructive.

The technique it uses, because of its specifics can’t be used even in a case of classical predicate logic, not to mention variety of non-classical logics.
Completeness Theorem

The Proof One is similar in its structure to the proof of the Deduction Theorem and is due to Kalmar, 1935.

It is a constructive proof and relies heavily on the Deduction Theorem.

It is possible to prove the Completeness Theorem independently of the Deduction Theorem and we will discuss such proofs in later chapters.
Main Lemma

Some Notations
We write $\vdash A$ instead of $\vdash_S A$ as the system $S$ is fixed.
Let $A$ be a formula and $b_1, b_2, ..., b_n$ be all propositional variables that occur in $A$, we write it as $A = A(b_1, b_2, ..., b_n)$

Lemma Definition
Let $v$ be a truth assignment $v : \text{VAR} \rightarrow \{T, F\}$
We define, for $A, b_1, b_2, ..., b_n$ and truth assignment $v$ corresponding formulas $A', B_1, B_2, ..., B_n$ as follows:

$$A' = \begin{cases} A & \text{if } v^*(A) = T \\ \neg A & \text{if } v^*(A) = F \end{cases}$$

$$B_i = \begin{cases} b_i & \text{if } v(b_i) = T \\ \neg b_i & \text{if } v(b_i) = F \end{cases}$$

for $i = 1, 2, ..., n$
Examples

Example
Let $A$ be a formula $(a \Rightarrow \neg b)$
Let $v$ be such that $v(a) = T$, $v(b) = F$
In this case we have that $b_1 = a$, $b_2 = b$, and
$v^*(A) = v^*(a \Rightarrow \neg b) = v(a) \Rightarrow \neg v(b) = T \Rightarrow \neg F = T$
The corresponding $A', B_1, B_2$ are:
$A' = A$ as $v^*(A) = T$
$B_1 = a$ as $v(a) = T$
$B_2 = \neg b$ as $v(b) = F$
Examples

Example 2
Let $A$ be a formula $((\neg a \Rightarrow \neg b) \Rightarrow c)$
and let $v$ be such that $v(a) = T$, $v(b) = F$, $v(c) = F$
Evaluate $A'$, $B_1$, $B_2$, $B_3$ as defined by the definition 1
In this case $n = 3$ and $b_1 = a$, $b_2 = b$, $b_3 = c$
and we evaluate
$v^*(A) = v^*((\neg a \Rightarrow \neg b) \Rightarrow c) = ((\neg v(a) \Rightarrow \neg v(b)) \Rightarrow v(c)) = ((\neg T \Rightarrow \neg F) \Rightarrow F) = (T \Rightarrow F) = F$
The corresponding $A'$, $B_1$, $B_2$, $B_3$ are:
$A' = \neg((\neg a \Rightarrow \neg b) \Rightarrow c)$ as $v^*(A) = F$
$B_1 = a$ as $v(a) = T$, $B_2 = \neg b$ as $v(b) = F$, and
$B_3 = \neg c$ as $v(c) = F$
Main Lemma

The **Main Lemma** stated below describes a method of transforming a **semantic** notion of a **tautology** into a **syntactic** notion of provability.

It defines, for any formula $A$ and a truth assignment $v$ a corresponding **deducibility relation**.

**Main Lemma**

For any formula $A = A(b_1, b_2, ..., b_n)$ and any truth assignment $v$.

If $A', B_1, B_2, ..., B_n$ are corresponding formulas defined by **Lemma Definition**, then

$$B_1, B_2, ..., B_n \vdash A'$$
Examples

Example
Let $A$ be a formula $(a \Rightarrow \neg b)$
Let $v$ be such that $v(a) = T$, $v(b) = F$
We have that $A' = A$, $B_1 = a$, $B_2 = \neg b$
Main Lemma asserts that

$$a, \neg b \vdash (a \Rightarrow \neg b)$$

Example
Let $A$ be a formula $((\neg a \Rightarrow \neg b) \Rightarrow c)$ and let $v$ be such that $v(a) = T$, $v(b) = F$, $v(c) = F$
Main Lemma asserts that

$$a, \neg b, \neg c \vdash \neg((\neg a \Rightarrow \neg b) \Rightarrow c)$$
Proof of the Main Lemma

The proof is by induction on the degree of the formula $A$

**Base Case**  $n = 0$

In this case $A$ is atomic and so consists of a single propositional variable, say $a$

If $v^*(A) = T$ then we have by Lemma Definition $A' = A = a$, $B_1 = a$

We obtain, by definition of provability from a set $\Gamma$ of hypothesis for $\Gamma = \{a\}$ that

$$a \vdash a$$
Proof of the Main Lemma

If $v^*(A) = F$ we have by Lemma Definition that

$$A' = \neg A = \neg a \quad \text{and} \quad B_1 = \neg a$$

We obtain, by definition of provability from a set $\Gamma$ of hypothesis for $\Gamma = \{\neg a\}$ that

$$\neg a \vdash \neg a$$

This proves that Main Lemma holds for $n=0$
Proof of the Main Lemma

Inductive Step
Assume that the Main Lemma holds for any formula with \( j < n \) connectives

Need to prove: the Main Lemma holds for \( A \) with \( n \) connectives

There are several sub-cases to deal with

Case: \( A \) is \( \neg A_1 \)
By the inductive assumption we have the formulas

\[ A'_1, B_1, B_2, ..., B_n \]

corresponding to the \( A_1 \) and the propositional variables \( b_1, b_2, ..., b_n \) in \( A_1 \), such that

\[ B_1, B_2, ..., B_n \vdash A'_1 \]
Proof of the Main Lemma

Observe that the formulas \( A \) and \( \neg A_1 \) have the same propositional variables.

So the corresponding formulas

\[
B_1, \ B_2, \ldots, \ B_n
\]

are the same for both of them.

We are going to show that the inductive assumption allows us to prove that

\[
B_1, B_2, \ldots, B_n \vdash A'
\]

There are two cases to consider.
Proof of the Main Lemma

Case: \( v^*(A_1) = T \)

If \( v^*(A_1) = T \) then by Lemma Definition \( A'_1 = A_1 \) and by the inductive assumption

\[
B_1, B_2, ..., B_n \vdash A_1
\]

In this case: \( v^*(A) = v^*(\neg A_1) = \neg v^*(T) = F \)

So we have that

\[
A' = \neg A = \neg \neg A_1
\]
Proof of the Main Lemma

By Lemma formula 3. we have that

\[ \vdash (A_1 \Rightarrow \neg\neg A_1) \]

we obtain by the monotonicity that also

\[ B_1, B_2, \ldots, B_n \vdash (A_1 \Rightarrow \neg\neg A_1) \]

By inductive assumption

\[ B_1, B_2, \ldots, B_n \vdash A_1 \]

and by MP we have

\[ B_1, B_2, \ldots, B_n \vdash \neg\neg A_1 \]

and as \( A' = \neg A = \neg\neg A_1 \) we get \( B_1, B_2, \ldots, B_n \vdash \neg A \) and so we proved that

\[ B_1, B_2, \ldots, B_n \vdash A' \]
Proof of the Main Lemma

Case: \( v^*(A_1) = F \)

If \( v^*(A_1) = F \) then \( A'_1 = \neg A_1 \) and \( v^*(A) = T \) so
\[
A' = A
\]

Therefore by the **inductive assumption** we have that
\[
B_1, B_2, ..., B_n \vdash \neg A_1
\]
as \( A' = \neg A_1 \) we get
\[
B_1, B_2, ..., B_n \vdash A'
\]
Proof of the Main Lemma

Case:  $A$ is $(A_1 \Rightarrow A_2)$

If $A$ is $(A_1 \Rightarrow A_2)$ then $A_1$ and $A_2$ have less than $n$ connectives.

$A = A(b_1, \ldots, b_n)$ so there are some subsequences $c_1, \ldots, c_k$ and $d_1, \ldots, d_m$ for $k, m \leq n$ of the sequence $b_1, \ldots, b_n$ such that

$$A_1 = A_1(c_1, \ldots, c_k) \quad \text{and} \quad A_2 = A(d_1, \ldots, d_m)$$
Proof of the Main Lemma

$A_1$ and $A_2$ have less than $n$ connectives and so by the **inductive assumption** we have appropriate formulas $C_1, \ldots, C_k$ and $D_1, \ldots, D_m$ such that

$$C_1, C_2, \ldots, C_k \vdash A_1' \quad \text{and} \quad D_1, D_2, \ldots, D_m \vdash A_2'$$

and $C_1, C_2, \ldots, C_k, D_1, D_2, \ldots, D_m$ are **subsequences** of formulas $B_1, B_2, \ldots, B_n$ corresponding to the propositional variables in $A$

By **monotonicity** we have the also

$$B_1, B_2, \ldots, B_n \vdash A_1' \quad \text{and} \quad B_1, B_2, \ldots, B_n \vdash A_2'$$

Now we have the following **sub-case** to consider
Proof of the Main Lemma

Case: \( v^*(A_1) = v^*(A_2) = T \)

If \( v^*(A_1) = T \) then \( A_1' = A_1 \) and if \( v^*(A_2) = T \) then \( A_2' = A_2 \)

We also have \( v^*(A_1 \Rightarrow A_2) = T \) and so \( A' = (A_1 \Rightarrow A_2) \)

By the above and the \textbf{inductive assumption}

\[
B_1, B_2, ..., B_n \vdash A_2
\]

and By \textbf{Axiom 1} and by \textbf{monotonicity} we have

\[
B_1, B_2, ..., B_n \vdash (A_2 \Rightarrow (A_1 \Rightarrow A_2))
\]

By above and \textbf{MP} we have \( B_1, B_2, ..., B_n \vdash (A_1 \Rightarrow A_2) \)

that is

\[
B_1, B_2, ..., B_n \vdash A'
\]
Proof of the Main Lemma

Case: \( v^*(A_1) = T, v^*(A_2) = F \)

If \( v^*(A_1) = T \) then \( A_1' = A_1 \) and

if \( v^*(A_2) = F \) then \( A_2' = \neg A_2 \)

Also we have in this case \( v^*(A_1 \Rightarrow A_2) = F \) and so \( A' = \neg(A_1 \Rightarrow A_2) \)

By the above, the inductive assumption and monotonicity

\( B_1, B_2, ..., B_n \vdash \neg A_2 \)

By Lemma 7. and by monotonicity we have

\[ B_1, B_2, ..., B_n \vdash (A_1 \Rightarrow (\neg A_2 \Rightarrow \neg(A_1 \Rightarrow A_2))) \]

By above and MP twice we have

\( B_1, B_2, ..., B_n \vdash \neg(A_1 \Rightarrow A_2) \) that is

\[ B_1, B_2, ..., B_n \vdash A' \]
Proof of the Main Lemma

Case: \( v^*(A_1) = F \)

Observe that if \( v^*(A_1) = F \) then \( A_1' \) is \( \neg A_1 \) and, whatever value \( v \) gives \( A_2 \), we have

\[
v^*(A_1 \Rightarrow A_2) = T
\]

So \( A' \) is \( (A_1 \Rightarrow A_2) \)

Therefore

\[
B_1, B_2, \ldots, B_n \vdash \neg A_1
\]

From Lemma formula 4. and by monotonicity we have

\[
B_1, B_2, \ldots, B_n \vdash (\neg A_1 \Rightarrow (A_1 \Rightarrow A_2))
\]
Proof of the Main Lemma

By Modus Ponens we get that

\[ B_1, B_2, ..., B_n \vdash (A_1 \Rightarrow A_2) \]

that is

\[ B_1, B_2, ..., B_n \vdash A' \]

We have covered all cases and, by mathematical induction on the degree of the formula \( A \) we got

\[ B_1, B_2, ..., B_n \vdash A' \]

This ends the proof of the Main Lemma
Proof One of Completeness Theorem
Proof of Completeness Theorem

Now we use the **Main Lemma** to prove the following

**Completeness Theorem** (Completeness Part)
For any formula \( A \in \mathcal{F} \)

\[
\text{if } \models A \text{ then } \vdash A
\]

**Proof**

Assume that \( \models A \)

Let \( b_1, b_2, \ldots, b_n \) be all propositional variables that occur in the formula \( A \), i.e.

\[
A = A(b_1, b_2, \ldots, b_n)
\]

By the **Main Lemma** we know that, for any truth assignment \( \nu \), the corresponding formulas \( A', B_1, B_2, \ldots, B_n \) can be found such that

\[
B_1, B_2, \ldots, B_n \vdash A'
\]
Proof Completeness Theorem

Note that in this case \( A' = A \) for any \( v \) since \( \models A \)

We have two cases.

1. If \( v \) is such that \( v(b_n) = T \), then \( B_n = b_n \) and

\[
B_1, B_2, ..., b_n \vdash A
\]

2. If \( v \) is such that \( v(b_n) = F \), then \( B_n = \neg b_n \) and by the Main Lemma

\[
B_1, B_2, ..., \neg b_n \vdash A
\]

So, by the Deduction Theorem we have

\[
B_1, B_2, ..., B_{n-1} \vdash (b_n \Rightarrow A)
\]

and

\[
B_1, B_2, ..., B_{n-1} \vdash (\neg b_n \Rightarrow A)
\]
Proof of Completeness Theorem

By Lemma formula 8.

\[ \vdash ((A \implies B) \implies ((\neg A \implies B) \implies B)) \]

for \( A = b_n, \ B = A \)

By monotonicity we have that

\[ B_1, B_2, ..., B_{n-1} \vdash ((b_n \implies A) \implies ((\neg b_n \implies A) \implies A)) \]

Applying Modus Ponens twice we get that

\[ B_1, B_2, ..., B_{n-1} \vdash A \]

Similarly, \( v^*(B_{n-1}) \) may be T or F

Applying the Main Lemma, the Deduction Theorem, monotonicity, Lemma formula 8. and Modus Ponens twice we can eliminate \( B_{n-1} \) just as we have eliminated \( B_n \)

After \( n \) steps, we finally obtain proof of \( A \) in \( H_2 \), i.e. we proved that

\[ \vdash A \]
Constructiveness of the Proof

Observe that the proof of the Completeness Theorem is constructive.

Moreover, we have used in it only Main Lemma and Deduction Theorem which both have constructive proofs.

We can hence reconstruct proofs in each case when we apply these theorems back to the original axioms of $H_2$. 

Constructiveness of the Proof

The same applies to the proofs in $H_2$ of all formulas 1. - 9. of the Lemma

It means that for any $A$, such that

$$\models A$$

the set $V_A$ of all $v$ restricted to $A$ provides a method of a construction of the formal proof of $A$ in $H_2$
Example

The proof of **Completeness Theorem** defines a **method** of efficiently combining truth assignments \( v \in V_A \) restricted to \( A \) while **constructing** the proof of \( A \).

Let’s consider a **tautology** \( A \), where the formula \( A \) is

\[
A(a, b, c) = (((\neg a \Rightarrow b) \Rightarrow (\neg (\neg a \Rightarrow b)) \Rightarrow c)
\]

We **present** on the next slides all steps of the **Proof One** as applied to \( A \).
Example

Given

\[ A(a, b, c) = ((\neg a \Rightarrow b) \Rightarrow (\neg(\neg a \Rightarrow b) \Rightarrow c) \]

By the Main Lemma and the assumption that \[ \models A(a, b, c) \]

any \( v \in V_A \) defines formulas \( B_a, B_b, B_c \) such that

\[ B_a, B_b, B_c \vdash A \]

The proof is based on a method of using all \( v \in V_A \) (there are 8 of them) to define a process of elimination of all hypothesis \( B_a, B_b, B_c \) to construct the proof of \( A \), i.e. to prove that

\[ \vdash A \]
Example

Step 1: elimination of $B_c$

Observe that by definition, $B_c$ is $c$ or $\neg c$ depending on the choice of $v \in V_A$

We choose two truth assignments $v_1 \neq v_2 \in V_A$ such that

$$v_1 \mid \{a, b\} = v_2 \mid \{a, b\} \text{ and } v_1(c) = T, \ v_2(c) = F$$

Case 1: $v_1(c) = T$

By definition $B_c = c$

By our choice, the assumption that $\models A$ and the Main Lemma applied to $v_1$

$$B_{a, b, c} \vdash A$$

By Deduction Theorem we have that

$$B_{a, b} \vdash (c \Rightarrow A)$$
Example

Case 2: \( v_2(c) = F \)

By definition \( B_c = \neg c \)

By our **choice**, assumption that \( \models A \), and the **Main Lemma** applied to \( v_2 \)

\[
B_a, B_b, \neg c \vdash A
\]

By the **Deduction Theorem** we have that

\[
B_a, B_b \vdash (\neg c \Rightarrow A)
\]
Example

By Lemma formula 8. for $A = c$, $B = A$ we have that

$$\vdash ((c \Rightarrow A) \Rightarrow ((\neg c \Rightarrow A) \Rightarrow A))$$

By monotonicity we have that

$$B_a, B_b \vdash ((c \Rightarrow A) \Rightarrow ((\neg c \Rightarrow A) \Rightarrow A))$$

Applying Modus Ponens twice to the above property and properties on the previous slide we get that

$$B_a, B_b \vdash A$$

We have eliminated $B_c$
Example

Step 2: elimination of $B_b$ from $B_a, B_b \vdash A$

We repeat the Step 1

As before we have 2 cases to consider: $B_b = b$ or $B_b = \neg b$

We choose two truth assignments $w_1 \neq w_2 \in V_A$ such that

$$w_1| \{a\} = w_2| \{a\} = v_1| \{a\} = v_2| \{a\} \quad \text{and} \quad w_1(b) = T, \ w_2(b) = F$$

Case 1: $w_1(b) = T$ and by definition $B_b = b$

By our choice, assumption that $\models A$ and the Main Lemma applied to $w_1$

$$B_a, b \vdash A$$

By Deduction Theorem we have that

$$B_a \vdash (b \Rightarrow A)$$
Example

Case 2: \( w_2(b) = F \) and by definition \( B_b = \neg b \)

By choice, assumption that \( \models A \) and the **Main Lemma** applied to \( w_2 \)

\[
B_a, \neg b \vdash A
\]

By the **Deduction Theorem** we have that

\[
B_a \vdash (\neg b \Rightarrow A)
\]
Example

By Lemma formula 8. for $A = b$, $B = A$ we have that

$$\vdash ((b \Rightarrow A) \Rightarrow ((\neg b \Rightarrow A) \Rightarrow A))$$

By monotonicity

$$B_a \vdash ((b \Rightarrow A) \Rightarrow ((\neg b \Rightarrow A) \Rightarrow A))$$

Applying Modus Ponens twice to the above property and properties from the previous slide we get that

$$B_a \vdash A$$

We have eliminated $B_b$
Example

**Step 3:** elimination of $B_a$ from $B_a \vdash A$

We repeat the **Step 2**

As before we have **2 cases** to consider: $B_a = a$ or $B_a = \neg a$

We choose two truth assignments $g_1 \neq g_2 \in VA$ such that

$$g_1(a) = T \quad \text{and} \quad g_2(a) = F$$

**Case 1:** $g_1(a) = T$, and by definition $B_a = a$

By the choice, assumption that $\models A$, and the **Main Lemma** applied to $g_1$

$$a \vdash A$$

By **Deduction Theorem** we have that

$$\vdash (a \Rightarrow A)$$
Example

Case 2: \( g_2(a) = F \) and by definition \( B_a = \neg a \)

By the choice, assumption that \( \models A \), and the Main Lemma applied to \( g_2 \)

\[ \neg a \vdash A \]

By the Deduction Theorem we have that

\[ \vdash (\neg a \Rightarrow A) \]
Example

By Lemma formula 8. for $A = a, B = A$ we have that

$$\vdash ((a \Rightarrow A) \Rightarrow ((\neg a \Rightarrow A) \Rightarrow A))$$

Applying Modus Ponens twice to the above property and properties from previous slides we get that

$$\vdash A$$

We have eliminated $B_a, B_b, B_c$ and constructed the proof of $A$ in $S$
Exercises

Exercise 1

The Lemma listed formulas 1. - 9. that we said they were needed for both proofs of the Completeness Theorem. List all the formulas from Lemma that are needed for the Proof One alone.
Exercises

Exercise 2
The system $H_2$ was defined and the Proof One was carried out for the language $L_{\{\Rightarrow, \neg\}}$

Extend the system $H_2$ and the Proof One to the language $L_{\{\Rightarrow, \cup, \neg\}}$ by adding all new cases concerning the new connective $\cup$

List all new formulas needed to be added as new Axioms to $H_2$ to be able to follow the methods of the original Proof One

Exercise 3
Repeat the Exercise 2 for the language $L_{\{\Rightarrow, \cup, \cap, \neg\}}$
Chapter 5
Hilbert Proof Systems
Completeness of Classical Propositional Logic

Slides Set 4

PART 6: Completeness Theorem Proof Two:
A Counter-Model Existence Method
Completeness Theorem Proof Two

Our goal now is to prove the following

**Completeness Theorem** (Completeness Part)

For any formula \( A \in \mathcal{F} \) of \( H_2 \)

\[
\text{if } \models A \quad \text{then} \quad \vdash A
\]

We do so by **proving** its **logically equivalent** opposite implication:

\[
\text{If } \not\models A, \quad \text{then} \quad \not\models A
\]

Hence the **Proof Two** consists of using the information that a formula \( A \) is **not provable** to show the **existence** of a **counter-model** for \( A \).
Completeness Theorem Proof Two

The **Proof Two** is much more complicated than the **Proof One**.

The **main point** of the proof is a general, non-constructive method for proving **existence** of a **counter-model** for any non-provable formula $A$.

The **generality** of the method makes it possible to **adopt** it for other cases of **predicate** and some **non-classical** logics.

This is why we call the **Proof Two** a **counter-model existence** method.
Proof Two Steps

The construction of a counter-model for any non-provable formula $A$ presented in this proof is abstract, not constructive, as it was in the Proof One.

It can be generalized to the case of predicate logic, and many of non-classical logics; propositional and predicate.

This is the reason we present it here.
Proof Two Steps

We remind that \( \not\models A \) means that there is a truth assignment \( \nu : \text{VAR} \to \{T, F\} \), such that (as we are in classical semantics) \( \nu^*(A) = F \)

We assume that \( A \) does not have a proof i.e. \( \not \vdash A \) we use this information in order to define a general method of constructing \( \nu \), such that \( \nu^*(A) = F \)

This is done in the following steps.
Proof Two Steps

Step 1

Definition of a special set of formulas $\Delta^*$

We use the information $\not\models A$ to define a set of formulas $\Delta^*$ such that $\neg A \in \Delta^*$

Step 2

Definition of the counter-model

We define the variable truth assignment $v : VAR \rightarrow \{T, F\}$ as follows:

$$v(a) = \begin{cases} 
T & \text{if } \Delta^* \vdash a \\
F & \text{if } \Delta^* \vdash \neg a 
\end{cases}$$
Proof 2 Steps

Step 3
We prove that \( \nu \) is a counter-model for \( A \)
We first prove a following more general property of \( \nu \)

Property
The set \( \Delta^* \) and \( \nu \) defined in the Steps 1 and 2 are such that for every formula \( B \in \mathcal{F} \)

\[
\nu^*(B) = \begin{cases} 
T & \text{if } \Delta^* \vdash B \\
F & \text{if } \Delta^* \vdash \neg B 
\end{cases}
\]

We then use the Step 3 to prove that \( \nu^*(A) = F \)
Main Notions

The definition, construction and the properties of the set $\Delta^*$ and hence the Step 1, are the most essential for the Proof Two.

The other steps have mainly technical character.

The main notions involved in the proof are: consistent set, complete set and a consistent complete extension of a set of formulas.

We are going prove some essential facts about them.
Consistent and Inconsistent Sets

There exist two definitions of consistency; semantical and syntactical.

Semantical definition uses the notion of a model and says:

A set is **consistent** if it has a **model**

Syntactical definition uses the notion of provability and says:

A set is **consistent** if one can’t prove a **contradiction** from it.
Consistent and Inconsistent Sets

In our proof of the **Completeness Theorem** we use the following formal **syntactical definition** of consistency of a set of formulas

**Definition** of a **consistent** set

We say that a set $\Delta \subseteq \mathcal{F}$ of formulas is **consistent** if and only if there is no a formula $A \in \mathcal{F}$ such that

$$\Delta \vdash A \quad \text{and} \quad \Delta \vdash \neg A$$
Consistent and Inconsistent Sets

**Definition of an inconsistent set**

A set $\Delta \subseteq \mathcal{F}$ is **inconsistent** if and only if there is a formula $A \in \mathcal{F}$ such that

$$\Delta \vdash A \quad \text{and} \quad \Delta \vdash \neg A$$

The notion of consistency, as defined above, is characterized by the following **Consistency Lemma**
Consistency Condition Lemma

**Lemma** Consistency Condition

For every set \( \Delta \subseteq \mathcal{F} \) of formulas, the following conditions are equivalent

(i) \( \Delta \) is consistent

(ii) there is a formula \( A \in \mathcal{F} \) such that \( \Delta \nvdash A \)
Proof of Consistency Lemma

Proof
To establish the equivalence of (i) and (ii) we prove the corresponding opposite implications

We prove the following two cases

Case 1  not (ii)  implies  not (i)

Case 2  not (i)  implies  not (ii)
Proof of Consistency Lemma

Case 1

Assume that not (ii)
It means that for all formulas $A \in \mathcal{F}$ we have that

$$\Delta \vdash A$$

In particular it is true for a certain $A = B$ and for a certain $A = \neg B$ i.e.

$$\Delta \vdash B \quad \text{and} \quad \Delta \vdash \neg B$$

and hence it proves that $\Delta$ is inconsistent i.e. not (i) holds
Proof of Consistency Lemma

Case 2
Assume that not (i), i.e. that $\Delta$ is inconsistent
Then there is a formula $A$ such that $\Delta \vdash A$ and $\Delta \vdash \neg A$
Let $B$ be any formula
We proved (Lemma formula 6.) that $\vdash (\neg A \Rightarrow (A \Rightarrow B))$
By monotonicity

$$\Delta \vdash (\neg A \Rightarrow (A \Rightarrow B))$$

Applying Modus Ponens twice to $\neg A$ first, and to $A$ next we get that $\Delta \vdash B$ for any formula $B$
Thus not (ii) and it ends the proof of the Consistency Condition Lemma
Inconsistency Condition Lemma

Inconsistent sets are hence characterized by the following fact.

**Lemma** Inconsistency Condition

For every set $\Delta \subseteq F$ of formulas, the following conditions are equivalent:

(i) $\Delta$ is inconsistent,

(ii) for any formula $A \in F$ $\Delta \vdash A$
Finite Consequence Lemma

We remind here property of the finiteness of the consequence operation.

**Lemma  Finite Consequence**

For every set $\Delta$ of formulas and for every formula $A \in F$
$\Delta \vdash A$ if and only if there is a finite set $\Delta_0 \subseteq \Delta$ such that $\Delta_0 \vdash A$

**Proof**

If $\Delta_0 \vdash A$ for a certain $\Delta_0 \subseteq \Delta$, hence by the monotonicity of the consequence, also $\Delta \vdash A$
Finite Consequence Lemma

Assume now that $\Delta \vdash A$ and let

$$A_1, A_2, ..., A_n$$

be a formal proof of $A$ from $\Delta$

Let

$$\Delta_0 = \{A_1, A_2, ..., A_n\} \cap \Delta$$

Obviously, $\Delta_0$ is finite and $A_1, A_2, ..., A_n$ is a formal proof of $A$ from $\Delta_0$
Finite Inconsistency Theorem

The following theorem is a simple corollary of just proved Finite Consequence Lemma

Theorem  Finite Inconsistency

(1.) If a set \( \Delta \) is inconsistent, then it has a finite inconsistent subset \( \Delta_0 \)

(2.) If every finite subset of a set \( \Delta \) is consistent then the set \( \Delta \) is also consistent
Finite Inconsistency Theorem

Proof
If \( \Delta \) is inconsistent, then for some formula \( A \),

\[
\Delta \vdash A \quad \text{and} \quad \Delta \vdash \neg A
\]

By the Finite Consequence Lemma, there are finite subsets \( \Delta_1 \) and \( \Delta_2 \) of \( \Delta \) such that

\[
\Delta_1 \vdash A \quad \text{and} \quad \Delta_2 \vdash \neg A
\]

The union \( \Delta_1 \cup \Delta_2 \) is a finite subset of \( \Delta \) and by monotonicity

\[
\Delta_1 \cup \Delta_2 \vdash A \quad \text{and} \quad \Delta_1 \cup \Delta_2 \vdash \neg A
\]

Hence we proved that \( \Delta_1 \cup \Delta_2 \) is a finite inconsistent subset of \( \Delta \)

The second implication (2.) is the opposite to the one just proved and hence also holds
Consistency Lemma

The following Lemma links the notion of non-provability and consistency.

It will be used as an important step in our Proof Two of the Completeness Theorem.

Lemma

For any formula $A \in \mathcal{F}$, if $\not\vdash A$ then the set $\{\neg A\}$ is consistent.
Consistency Lemma

Proof We prove the opposite implication
If \( \{\neg A \} \) is inconsistent, then \( \vdash A \)
Assume that \( \{\neg A \} \) is inconsistent
By the Inconsistency Condition Lemma we have that
\( \{\neg A \} \vdash B \) for any formula \( B \), and hence in particular
\( \{\neg A \} \vdash A \)

By Deduction Theorem we get
\( \vdash (\neg A \Rightarrow A) \)

We proved (Lemma formula 9.) that
\( \vdash ((\neg A \Rightarrow A) \Rightarrow A) \)

By Modus Ponens we get
\( \vdash A \)

This ends the proof
Complete and Incomplete Sets

Another important notion, is that of a **complete set** of formulas.

**Complete sets**, as defined here are sometimes called **maximal**, but we use the first name for them.

They are defined as follows.

**Definition**  **Complete set**

A set $\Delta$ of formulas is called **complete** if for every formula $A \in \mathcal{F}$

$$
\Delta \vdash A \quad \text{or} \quad \Delta \vdash \neg A
$$

**Godel** used this notion of complete sets in his **Incompleteness of Arithmetic Theorem**

The **complete sets** are characterized by the following fact.
Complete and Incomplete Sets

Complete Set Condition Lemma

For every set $\Delta \subseteq \mathcal{F}$ of formulas, the following conditions are equivalent

(i) The set $\Delta$ is complete

(ii) For every formula $A \in \mathcal{F}$,
    
    if $\Delta \not\vdash A$ then the set $\Delta \cup \{A\}$ is inconsistent

Proof

We consider two cases

Case 1 We show that (i) implies (ii) and

Case 2 we show that (ii) implies (i)
Complete Set Condition Lemma

Proof of **Case 1**
Assume (i) and not(ii) i.e.
assume that $\Delta$ is complete and there is a formula $A \in \mathcal{F}$ such that $\Delta \not\vdash A$ and the set $\Delta \cup \{A\}$ is consistent
We have to show that we get a contradiction
But if $\Delta \not\vdash A$, then from the assumption that $\Delta$ is complete we get that
$$\Delta \vdash \neg A$$
By the monotonicity of the consequence we have that
$$\Delta \cup \{A\} \vdash \neg A$$
We proved (Lemma formula 4.) \( \vdash (A \Rightarrow A) \)

By monotonicity \( \Delta \vdash (A \Rightarrow A) \) and by Deduction Theorem

\[ \Delta \cup \{ A \} \vdash A \]

We hence proved that there is a formula \( A \in \mathcal{F} \) such that

\[ \Delta \cup \{ A \} \quad \text{and} \quad \Delta \cup \{ A \} \vdash \neg A \]

i.e. that the set \( \Delta \cup \{ A \} \) is inconsistent

Contradiction
Complete Set Condition Lemma

Proof of **Case 2**
Assume (ii), i.e. that for every formula $A \in F$ if $\Delta \not\models A$ then the set $\Delta \cup \{A\}$ is inconsistent
Let $A$ be any formula.
We want to show (i), i.e. to show that the following condition

$$ C : \quad \Delta \vdash A \quad \text{or} \quad \Delta \vdash \neg A $$

is satisfied.

Observe that if

$$ \Delta \vdash \neg A $$

then the condition $C$ is obviously satisfied
Complete Set Condition Lemma

If, on the other hand,

\[ \Delta \not\vdash \neg A \]

then we are going to show now that it must be, under the assumption of (ii), that \( \Delta \vdash A \) i.e. that (i) holds

Assume that

\[ \Delta \not\vdash \neg A \]

then by (ii) the set \( \Delta \cup \{\neg A\} \) is inconsistent
Complete Set Condition Lemma

The Inconsistency Condition Lemma says
For every set $\Delta \subseteq \mathcal{F}$ of formulas, the following conditions are equivalent:

(i) $\Delta$ is inconsistent,

(i) for any formula $A \in \mathcal{F}$, $\Delta \vdash A$

We just proved that the set $\Delta \cup \{\neg A\}$ is inconsistent
So by the the above Lemma we get

$$\Delta \cup \{\neg A\} \vdash A$$
Complete Set Condition Lemma

By the **Deduction Theorem** \( \Delta \cup \{\neg A\} \vdash A \) implies that

\[ \Delta \vdash (\neg A \Rightarrow A) \]

**Observe** that by **Lemma** formula 4.

\[ \vdash ((\neg A \Rightarrow A) \Rightarrow A) \]

By monotonicity

\[ \Delta \vdash ((\neg A \Rightarrow A) \Rightarrow A) \]

Detaching, by **MP** the formula \((\neg A \Rightarrow A)\) we obtain that

\[ \Delta \vdash A \]

This **ends** the proof that (i) holds.
Incomplete Sets

**Definition**  Incomplete Set

A set $\Delta$ of formulas is called **incomplete** if it is **not complete** i.e. when the following condition holds

There exists a formula $A \in \mathcal{F}$ such that

$$\Delta \not\vDash A \quad \text{and} \quad \Delta \not\vDash \neg A$$
Incomplete Set Condition Lemma

We get as a direct consequence of the Complete Set Condition Lemma the following characterization of incomplete sets

**Lemma**  Incomplete Set Condition

For every set $\Delta \subseteq \mathcal{F}$ of formulas, the following conditions are equivalent:

(i) $\Delta$ is **incomplete**,

(ii) there is formula $A \in \mathcal{F}$ such that $\Delta \not\vdash A$ and the set $\Delta \cup \{A\}$ is **consistent**.


Main Lemma: Complete Consistent Extension

Now we are going to prove a Main Lemma that is essential to the construction of the special set $\Delta^*$ mentioned in the Step 1 of the proof of the Completeness Theorem and hence to the proof of the theorem itself.

Let’s first introduce one more notion.
Complete Consistent Extension

**Definition**  
Extension $\Delta^*$ of the set $\Delta$

A set $\Delta^*$ of formulas is called an extension of a set $\Delta$ of formulas if the following condition holds

$$\{A \in \mathcal{F} : \Delta \vdash A\} \subseteq \{A \in \mathcal{F} : \Delta^* \vdash A\}$$

i.e.

$$Cn(\Delta) \subseteq Cn(\Delta^*)$$

In this case we say also that $\Delta$ extends to the set of formulas $\Delta^*$
Main Lemma
Main Lemma

Complete Consistent Extension

Every **consistent** set \( \Delta \) of formulas can be **extended** to a **complete consistent** set \( \Delta^* \) of formulas i.e.

For every **consistent** set \( \Delta \) there is a set \( \Delta^* \) that is **complete** and **consistent** and is an **extension** of \( \Delta \) i.e.

\[
Cn(\Delta) \subseteq Cn(\Delta^*)
\]
Proof of the Main Lemma

Proof
Assume that the lemma does not hold, i.e. that there is a consistent set $\Delta$, such that all its consistent extensions are not complete.

In particular, as $\Delta$ is an consistent extension of itself, we have that $\Delta$ is not complete.

The proof consists of a construction of a particular set $\Delta^*$ and proving that it forms a complete consistent extension of $\Delta$.

This is contrary to the assumption that all its consistent extensions are not complete.
Construction of $\Delta^*$

As we know, the set $\mathcal{F}$ of all formulas is enumerable; they can hence be put in an infinite sequence

$$\mathcal{F} \quad A_1, A_2, \ldots, A_n, \ldots$$

such that every formula of $\mathcal{F}$ occurs in that sequence exactly once.

We define, by mathematical induction, an infinite sequence

$$D \quad \{\Delta_n\}_{n \in \mathbb{N}}$$

of consistent subsets of formulas together with a sequence

$$B \quad \{B_n\}_{n \in \mathbb{N}}$$

of formulas as follows.
Construction of $\Delta^*$

Initial Step
In this step we define the sets $\Delta_1, \Delta_2$ and the formula $B_1$ and prove that $\Delta_1$ and $\Delta_2$ are consistent, incomplete extensions of $\Delta$.

We take as the first set in $\mathcal{D}$ the set $\Delta$, i.e. we define $\Delta_1 = \Delta$. 
Construction of $\Delta^*$

By assumption the set $\Delta$, and hence also $\Delta_1$ is not complete.

From the Incomplete Set Condition Lemma we get that there is a formula $B \in \mathcal{F}$ such that

$$\Delta_1 \nvdash B \quad \text{and} \quad \Delta_1 \cup \{B\} \quad \text{is consistent}$$

Let $B_1$ be the first formula with this property in the sequence $\mathcal{F}$ of all formulas.

We define

$$\Delta_2 = \Delta_1 \cup \{B_1\}$$
Construction of $\Delta^*$

Observe that the set $\Delta_2$ is consistent and

$$\Delta_1 = \Delta \subseteq \Delta_2$$

By monotonicity $\Delta_2$ is a \textit{consistent extension} of $\Delta$

Hence, as we assumed that all consistent extensions of $\Delta$ are \textit{not complete}, we get that $\Delta_2$ cannot be complete, i.e.

$\Delta_2$ is incomplete
Construction of $\Delta^*$

Inductive Step

Suppose that we have defined a sequence

$\Delta_1, \Delta_2, \ldots, \Delta_n$

of incomplete, consistent extensions of $\Delta$ and a sequence

$B_1, B_2, \ldots, B_{n-1}$

of formulas, for $n \geq 2$
Construction of $\Delta^*$

Since $\Delta_n$ is incomplete, it follows from the Incomplete Set Condition Lemma that there is a formula $B \in \mathcal{F}$ such that

$$\Delta_n \not\subset B \text{ and } \Delta_n \cup \{B\} \text{ is consistent}$$
Construction of $\Delta^*$

Let $B_n$ be the first formula with this property in the sequence $F$ of all formulas.

We define

$$\Delta_{n+1} = \Delta_n \cup \{B_n\}$$

By the definition

$$\Delta \subseteq \Delta_n \subseteq \Delta_{n+1}$$

and the set $\Delta_{n+1}$ is a consistent extension of $\Delta$.

Hence by our assumption that all consistent extensions of $\Delta$ are incomplete we get that

$$\Delta_{n+1}$$

is an incomplete consistent extension of $\Delta$. 
Construction of \( \Delta^* \)

By the principle of mathematical induction we have defined an infinite sequence

\[ \Delta = \Delta_1 \subseteq \Delta_2 \subseteq \ldots, \subseteq \Delta_n \subseteq \Delta_{n+1} \subseteq \ldots \]

such that for all \( n \in N \), \( \Delta_n \) is consistent, and each \( \Delta_n \) an incomplete consistent extension of \( \Delta \)

Moreover, we have also defined a sequence

\[ B_1, B_2, \ldots, B_n, \ldots \]

of formulas, such that for all \( n \in N \),

\[ \Delta_n \nvdash B_n \quad \text{and} \quad \Delta_n \cup \{ B_n \} \quad \text{is consistent} \]

Observe that \( B_n \in \Delta_{n+1} \) for all \( n \geq 1 \)
Definition of $\Delta^*$

Now we are ready to define $\Delta^*$

**Definition** of $\Delta^*$

$$\Delta^* = \bigcup_{n \in N} \Delta_n$$

To complete the proof our theorem we have now to prove that $\Delta^*$ is a complete consistent extension of $\Delta$. 

\[ \Delta^* \text{ Consistent} \]

**Obviously** directly from the definition \[ \Delta \subseteq \Delta^* \] and hence we have the following

**Fact 1** \( \Delta^* \) is an **extension** of \( \Delta \).

By Monotonicity of Consequence \( Cn(\Delta) \subseteq Cn(\Delta^*) \), hence extension.

As the next step we prove

**Fact 2** The set \( \Delta^* \) is **consistent**.
\[ \Delta^* \text{ Consistent} \]

**Proof** that \( \Delta^* \) is **consistent**

Assume that \( \Delta^* \) is **inconsistent**

By the **Finite Inconsistency Theorem** there is a **finite** subset \( \Delta_0 \) of \( \Delta^* \) that is **inconsistent**, i.e.

\[ \Delta_0 \subseteq \bigcup_{n \in \mathbb{N}} \Delta_n, \quad \Delta_0 = \{C_1, \ldots, C_n\}, \quad \Delta_0 \text{ is inconsistent} \]
Proof of $\Delta^*$ Consistent

We have $\Delta_0 = \{C_1, \ldots, C_n\}$

By the definition of $\Delta^*$ for each formula $C_i \in \Delta_0$

$$C_i \in \Delta_{k_i}$$

for certain $\Delta_{k_i}$ in the sequence

$$D \quad \Delta = \Delta_1 \subseteq \Delta_2 \subseteq \ldots, \subseteq \Delta_n \subseteq \Delta_{n+1} \subseteq \ldots$$

Hence $\Delta_0 \subseteq \Delta_m$ for $m = \max\{k_1, k_2, \ldots k_n\}$
Proof of $\Delta^*$ Consistent

But we proved that all sets of the sequence $D$ are consistent

This contradicts the fact that $\Delta_m$ is consistent as it contains an inconsistent subset $\Delta_0$

This contradiction ends the proof that $\Delta^*$ is consistent
Proof of $\Delta^*$ Complete

**Fact 3**  The set $\Delta^*$ is complete

**Proof**  Assume that $\Delta^*$ is not complete.
By the Incomplete Set Condition, there is a formula $B \in \mathcal{F}$ such that
$\Delta^* \not\vdash B$, and the set $\Delta^* \cup \{B\}$ is consistent
By definition of the sequence $D$ and the sequence $B$ of formulas we have that for every $n \in \mathbb{N}$

$$\Delta_n \not\vdash B_n \quad \text{and the set} \quad \Delta_n \cup \{B_n\} \quad \text{is consistent}$$

Moreover $B_n \in \Delta_{n+1}$ for all $n \geq 1$
Proof of $\Delta^*$ Complete

Since the formula $B$ is one of the formulas of the sequence $\mathcal{B}$ so we get that $B = B_j$ for certain $j$.

By definition, $B_j \in \Delta_{j+1}$ and it proves that

$$B \in \Delta^* = \bigcup_{n \in \mathbb{N}} \Delta_n$$

But this means that $\Delta^* \vdash B$.

This is a contradiction with the assumption $\Delta^* \nvdash B$ and it ends the proof of the Fact 3.
Main Lemma

Facts 1-3 prove that \( \Delta^* \) is a complete consistent extension of \( \Delta \).

We hence completed the proof of the Main Lemma.

Main Lemma
Every consistent set \( \Delta \) of formulas can be extended to a complete consistent set \( \Delta^* \) of formulas.
Proof Two of Completeness Theorem
Proof Two of Completeness Theorem

We proved already that $H_2$ is sound, so we have to prove only the Completeness part of the Completeness Theorem:

For any formula $A \in \mathcal{F}$,

\[
\text{If } \models A, \text{ then } \vdash A
\]

We prove it by proving its logically equivalent opposite implication form, i.e. we prove now the following

Completeness Theorem

For any formula $A \in \mathcal{F}$,

\[
\text{If } \not\models A, \text{ then } \not\vdash A
\]
Proof Two of Completeness Theorem

Proof
Assume that $A$ does not have a proof, we want to define a counter-model for $A$

But if $\not\models A$, then by the Inconsistency Lemma the set $\{\neg A\}$ is consistent

By the Main Lemma there is a complete, consistent extension of the set $\{\neg A\}$

This means that there is a set $\Delta^*$ such that $\{\neg A\} \subseteq \Delta^*$, i.e.

$E \quad \neg A \in \Delta^*$ and $\Delta^*$ is complete and consistent
Proof Two of Completeness Theorem

Since $\Delta^*$ is a consistent, complete set, it satisfies the following form of

Consistency Condition

For any $A \in \mathcal{F}$,

$$\Delta^* \nvdash A \quad \text{or} \quad \Delta^* \nvdash \neg A$$

$\Delta^*$ is also complete i.e. satisfies

Completeness Condition

For any $A \in \mathcal{F}$,

$$\Delta^* \vdash A \quad \text{or} \quad \Delta^* \vdash \neg A$$
Proof Two of Completeness Theorem

Directly from the **Completeness** and **Consistency** Conditions we get the following

**Separation Condition**

For any \( A \in \mathcal{F} \), **exactly one** of the following conditions is satisfied:

\[
(1) \quad \Delta^* \vdash A, \quad \text{or} \quad (2) \quad \Delta^* \vdash \neg A
\]

In **particular case** we have that for every propositional variable \( a \in \text{VAR} \) **exactly one** of the following conditions is satisfied:

\[
(1) \quad \Delta^* \vdash a, \quad \text{or} \quad (2) \quad \Delta^* \vdash \neg a
\]

This **justifies** the **correctness** of the following definition
Proof Two of Completeness Theorem

**Definition**

We define the variable truth assignment $v : \text{VAR} \rightarrow \{T, F\}$ as follows:

$$v(a) = \begin{cases} T & \text{if } \Delta^* \vdash a \\ F & \text{if } \Delta^* \vdash \neg a. \end{cases}$$

We show, as a separate Lemma below, that such defined variable assignment $v$ has the following property
Property of $v$ Lemma

**Lemma**  Property of $v$

Let $v$ be the variable assignment defined above and $v^*$ its extension to the set $\mathcal{F}$ of all formulas $B \in \mathcal{F}$, the following is true:

$$v^*(B) = \begin{cases} T & \text{if } \Delta^* \vdash B \\ F & \text{if } \Delta^* \vdash \neg B \end{cases}$$
Proof 2 of Completeness Theorem

Given the Property of $v$ Lemma (still to be proved) we now prove that the $v$ is in fact, a counter model for any formula $A$, such that $\not\models A$

Let $A$ be such that $\not\models A$

By the Property $E$ we have that $\neg A \in \Delta^*$

So obviously

$$\Delta^* \vdash \neg A$$

Hence by the Property of $v$ Lemma

$$v^*(A) = F$$

what proves that $v$ is a counter-model for $A$ and it ends the proof of the Completeness Theorem
Proof of Property of $v$ Lemma

Proof of the Property of $v$ Lemma
The proof is conducted by the induction on the degree of the formula $A$

Initial step $A$ is a propositional variable so the Lemma holds by definition of $v$

Inductive Step
If $A$ is not a propositional variable, then $A$ is of the form $
eg C$ or $(C \Rightarrow D)$, for certain formulas $C, D$

By the inductive assumption the Lemma holds for the formulas $C$ and $D$
Case \( A = \neg C \)

By the Separation Condition for \( \Delta^* \) we consider two possibilities

1. \( \Delta^* \vdash A \)
2. \( \Delta^* \vdash \neg A \)

Consider case 1. i.e. we assume that \( \Delta^* \vdash A \)

It means that

\[
\Delta^* \vdash \neg C
\]

Then from the fact that \( \Delta^* \) is consistent it must be that

\[
\Delta^* \not\vdash C
\]
Proof of Property of $v$ Lemma

By the inductive assumption we have that $v^*(C) = F$ and accordingly $v^*(A) = v^*(\neg C) = \neg v^*(C) = \neg F = T$

Consider case 2. i.e. we assume that $\Delta^* \vdash \neg A$

Then from the fact that $\Delta^*$ is consistent it must be that $\Delta^* \not\vdash A$ and

$$\Delta^* \not\vdash \neg C$$

If so, then $\Delta^* \vdash C$, as the set $\Delta^*$ is complete

By the inductive assumption, $v^*(C) = T$, and accordingly

$$v^*(A) = v^*(\neg C) = \neg v^*(C) = \neg T = F$$

Thus $A$ satisfies the Property of $v$ Lemma
Proof of Property of $\nu$ Lemma

**Case** $A = (C \Rightarrow D)$

As in the previous case, we assume that the **Lemma** holds for the formulas $C, D$ and we consider by the **Separation Condition** for $\Delta^*$ two possibilities:

1. $\Delta^* \vdash A$ and 2. $\Delta^* \vdash \neg A$

**Case 1.** Assume $\Delta^* \vdash A$

It means that $\Delta^* \vdash (C \Rightarrow D)$

If at the same time $\Delta^* \not\vdash C$, then $\nu^*(C) = F$, and accordingly

$$\nu^*(A) = \nu^*(C \Rightarrow D) = \nu^*(C) \Rightarrow \nu^*(D) = F \Rightarrow \nu^*(D) = T$$
Proof of Property of $v$ Lemma

If at the same time $\Delta^* \vdash C$, then since $\Delta^* \vdash (C \Rightarrow D)$, we infer, by Modus Ponens, that

$$\Delta^* \vdash D$$

If so, then $v^*(C) = v^*(D) = T$

and accordingly

$$v^*(A) = v^*(C \Rightarrow D) =$$

$$v^*(C) \Rightarrow v^*(D) = T \Rightarrow T = T$$

Thus if $\Delta^* \vdash A$, then $v^*(A) = T$
Proof of Property of $\lor$ Lemma

Case 2. Assume now, as before, that $\Delta^* \vdash \neg A$.
Then from the fact that $\Delta^*$ is consistent it must be that $\Delta^* \not\vdash A$, i.e.,

$$\Delta^* \not\vdash (C \Rightarrow D)$$

It follows from this that $\Delta^* \not\vdash D$
For if $\Delta^* \vdash D$, then, as $(D \Rightarrow (C \Rightarrow D))$ is provable formula 1. in $S$, by monotonicity also

$$\Delta^* \vdash (D \Rightarrow (C \Rightarrow D))$$

Applying Modus Ponens we obtain

$$\Delta^* \vdash (C \Rightarrow D)$$

which is contrary to the assumption, so it must be $\Delta^* \not\vdash D$
Proof of Property of \( \nu \) Lemma

Also we must have

\[ \Delta^* \vdash C \]

for otherwise, as \( \Delta^* \) is **complete** we would have \( \Delta^* \vdash \neg C \)

This this is **impossible** since by **Lemma** formula 9.

\[ \vdash (\neg C \Rightarrow (C \Rightarrow D)) \]

By **monotonicity**

\[ \Delta^* \vdash (\neg C \Rightarrow (C \Rightarrow D)) \]

Applying **Modus Ponens** we would get

\[ \Delta^* \vdash (C \Rightarrow D) \]

which is **contrary** to the assumption \( \Delta^* \not\vdash (C \Rightarrow D) \)
Proof Two of Completeness Theorem

This **ends** the proof of the **Property of v Lemma** and the **Proof Two** of the **Completeness Theorem** is also **completed**
Chapter 5
Hilbert Proof Systems
Completeness of Classical Propositional Logic

Slides Set 5
PART 6: Some Other Axiomatizations and Examples and Exercises
Some Other Axiomatizations

We present here some of the most known, and historically important axiomatizations of classical propositional logic.

It means the Hilbert proof systems that are proven to be complete under classical semantics.
Lukasiewicz

Lukasiewicz (1929)

The Lukasiewicz proof system (axiomatization) is

\[ L = ( \mathcal{L}_{\neg, \Rightarrow}, \mathcal{F}, A_1, A_2, A_3, MP ) \]

where

A1 \quad ((\neg A \Rightarrow A) \Rightarrow A)

A2 \quad (A \Rightarrow (\neg A \Rightarrow B))

A3 \quad ((A \Rightarrow B) \Rightarrow ((B \Rightarrow C) \Rightarrow (A \Rightarrow C))))

for any formulas \( A, B, C \in \mathcal{F} \)
Hilbert and Ackermann (1928)

\[ HA = ( \mathcal{L}_{\neg, \cup}, \mathcal{F}, A1 - A4, MP ) \]

where for any \( A, B, C \in \mathcal{F} \)

- A1 \( (\neg(A \cup A) \cup A) \)
- A2 \( (\neg A \cup (A \cup B)) \)
- A3 \( (\neg(A \cup B) \cup (B \cup A)) \)
- A4 \( (\neg(\neg B \cup C) \cup (\neg(A \cup B) \cup (A \cup C))) \)

The Modus Ponens rule in the language \( \mathcal{L}_{\neg, \cup} \) has a form

\[
MP \quad A ; (\neg A \cup B) \quad B
\]
Observe that also the **Deduction Theorem** is now formulated as follow.

**Deduction Theorem for HA**

For any subset $\Gamma$ of the set of formulas $\mathcal{F}$ of $HA$ and for any formulas $A, B \in \mathcal{F}$,

$$\Gamma, A \vdash_{HA} B \text{ if and only if } \Gamma \vdash_{HA} (\neg A \cup B)$$

In particular,

$$A \vdash_{HA} B \text{ if and only if } \vdash_{HA} (\neg A \cup B)$$
Hilbert (1928)

\[ H = ( \mathcal{L}_{\neg, \cup, \cap, \Rightarrow}, \ F, \ A_1 - A_{15}, \ MP ) \]

where for any \( A, B, C \in \mathcal{F} \)

A1 \( (A \Rightarrow A) \)

A2 \( (A \Rightarrow (B \Rightarrow A)) \)

A3 \( ((A \Rightarrow B) \Rightarrow ((B \Rightarrow C) \Rightarrow (A \Rightarrow C))) \)

A4 \( ((A \Rightarrow (A \Rightarrow B)) \Rightarrow (A \Rightarrow B)) \)

A5 \( ((A \Rightarrow (B \Rightarrow C)) \Rightarrow (B \Rightarrow (A \Rightarrow C))) \)

A6 \( ((A \Rightarrow B) \Rightarrow ((B \Rightarrow C) \Rightarrow (A \Rightarrow C))) \)

A7 \( ((A \cap B) \Rightarrow A) \)

A8 \( ((A \cap B) \Rightarrow B) \)
Hilbert

A9 \(((A \Rightarrow B) \Rightarrow ((A \Rightarrow C) \Rightarrow (A \Rightarrow (B \cap C))))\)

A10 \((A \Rightarrow (A \cup B))\)

A11 \((B \Rightarrow (A \cup B))\)

A12 \(((A \Rightarrow C) \Rightarrow ((B \Rightarrow C) \Rightarrow ((A \cup B) \Rightarrow C)))\)

A13 \(((A \Rightarrow B) \Rightarrow ((A \Rightarrow \neg B) \Rightarrow \neg A))\)

A14 \((\neg A \Rightarrow (A \Rightarrow B))\)

A1 - A14 are the axioms Hilbert proposed and were accepted as axioms defining Intuitionistic logic.

They were later proved to be complete when the intuitionistic semantics was discovered.

Hilbert obtained his classical axiomatization by adding as the last axiom the excluded middle law rejected by intuitionists.

A15 \((A \cup \neg A)\)
Kleene

Kleene (1952)

\[ K = ( \mathcal{L}_{\neg, \cup, \cap, \Rightarrow}, \mathcal{F}, A1 - A10, MP ) \]

where for any \( A, B, C \in \mathcal{F} \)

A1 \( (A \Rightarrow (B \Rightarrow A)) \)

A2 \( ((A \Rightarrow (B \Rightarrow C)) \Rightarrow (B \Rightarrow (A \Rightarrow C))) \)

A3 \( ((A \cap B) \Rightarrow A) \)

A4 \( ((A \cap B) \Rightarrow B) \)

A5 \( (A \Rightarrow (B \Rightarrow (A \cap B))) \)
Kleene

A6 \((A \Rightarrow (A \cup B))\)
A7 \((B \Rightarrow (A \cup B))\)
A8 \(((A \Rightarrow C) \Rightarrow ((B \Rightarrow C) \Rightarrow ((A \cup B) \Rightarrow C)))\)
A9 \(((A \Rightarrow B) \Rightarrow ((A \Rightarrow \neg B) \Rightarrow \neg A))\)
A10 \((\neg \neg A \Rightarrow A)\)

Kleene proved that when A10 is replaced by
A10’ \((\neg A \Rightarrow (A \Rightarrow B))\)
the resulting system is a complete axiomatization of
Intuitionistic Logic
Rasiowa-Sikorski

Rasiowa-Sikorski (1950)

\[ RS = ( \mathcal{L}_{\neg,\cup,\cap,\Rightarrow}, \mathcal{F}, A1 - A12, \text{MP} ) \]

where for any \( A, B, C \in \mathcal{F} \)

A1 \( ((A \Rightarrow B) \Rightarrow ((B \Rightarrow C) \Rightarrow (A \Rightarrow C))) \)

A2 \( (A \Rightarrow (A \cup B)) \)

A3 \( (B \Rightarrow (A \cup B)) \)

A4 \( ((A \Rightarrow C) \Rightarrow ((B \Rightarrow C) \Rightarrow ((A \cup B) \Rightarrow C))) \)
Rasiowa-Sikorski

A5 \(((A \cap B) \Rightarrow A)\)
A6 \(((A \cap B) \Rightarrow B)\)
A7 \(((C \Rightarrow A) \Rightarrow ((C \Rightarrow B) \Rightarrow (C \Rightarrow (A \cap B))))\)
A8 \(((A \Rightarrow (B \Rightarrow C))) \Rightarrow ((A \cap B) \Rightarrow C))\)
A9 \(((A \cap B) \Rightarrow C) \Rightarrow (A \Rightarrow (B \Rightarrow C))\)
A10 \((A \cap \neg A) \Rightarrow B)\)
A11 \(((A \Rightarrow (A \cap \neg A)) \Rightarrow \neg A)\)
A12 \((A \cup \neg A)\)
Rasiowa-Sikorski

Rasiowa - Sikorski proved A1 - A11 to be a complete axiomatization for the Intuitionistic Logic.

They obtained the classical axiomatization by adding A12, the excluded middle law rejected by intuitionists, as Hilbert did.

Both classical and intuitionistic completeness proofs were carried under respective Boolean and Pseudo-Boolean algebras semantics what is reflected in the choice of axioms A1 - A12.
Here is the shortest axiomatization for the language

\[ \mathcal{L}_{\neg, \Rightarrow} \]

It contains just one axiom

**Meredith** (1953)

\[ M = ( \mathcal{L}_{\neg, \Rightarrow}, \mathcal{F}, A1 \text{ MP} ) \]

where

\[ A1 \quad ((((((A \Rightarrow B) \Rightarrow (\neg C \Rightarrow \neg D)) \Rightarrow C) \Rightarrow E)) \Rightarrow ((E \Rightarrow A) \Rightarrow (D \Rightarrow A))) \]
Shortest Axiomatizations

Here is another axiomatization that uses only one axiom

**Nicod** (1917)

\[ N = ( \mathcal{L}_{\{\uparrow}\}, \mathcal{F}, \ A1, \ (r) ) \]

where
\[ A1 \ ((((A \uparrow (B \uparrow C)) \uparrow ((D \uparrow (D \uparrow D)) \uparrow ((E \uparrow B) \uparrow ((A \uparrow E) \uparrow (A \uparrow E)))))))) \]

and
\[ (r) \ \frac{A \uparrow (B \uparrow C)}{A} \]

Reminder

We have proved in chapter 3 that

\[ \mathcal{L}_{\{\neg,\cup,\cap,\Rightarrow\}} \equiv \mathcal{L}_{\{\uparrow\}} \]
Exercises

Here are few exercises designed to help with understanding the notions of **completeness**, **monotonicity** of the **consequence operation**, the **role** of the **deduction theorem** and the **importance** of some basic **tautologies**
Complete Hilbert System \( S \)

Let \( S \) be any Hilbert proof system

\[
S = (\mathcal{L}_{\{\cap, \cup, \Rightarrow, \neg\}}, \mathcal{F}, \text{LA}, \text{MP} \quad \frac{A, (A \Rightarrow B)}{B})
\]

with the set \( \text{LA} \) of logical axioms such that \( S \) is complete under classical semantics.

Let \( X \subseteq \mathcal{F} \) be any subset of the set \( \mathcal{F} \) of formulas of the language

\[
\mathcal{L}_{\{\cap, \cup, \Rightarrow, \neg\}}
\]

We define, as we did in chapter 4, a set \( \text{Cn}(X) \) of all consequences of the set \( X \) as

\[
\text{Cn}(X) = \{ A \in \mathcal{F} : X \vdash_S A \} 
\]
Exercises

Reminder

The proof system

\[ S = (\mathcal{L}_{\{\cap, \cup, \Rightarrow, \neg\}}, \mathcal{F}, \text{LA, MP} \quad \frac{A, (A \Rightarrow B)}{B} ) \]

in all exercises is complete
Exercises

Exercise 1

1. Prove that for any subsets $X, Y$ of the set $\mathcal{F}$ of formulas of $S$ the following monotonicity property holds

$$\text{If } X \subseteq Y, \text{ then } Cn(X) \subseteq Cn(Y)$$

Solution

1. Let $A \in \mathcal{F}$ be any formula such that $A \in Cn(X)$

By the consequence definition, we have that $X \vdash_S A$ and $A$ has a formal proof from the set $X \cup LA$

But $X \subseteq Y$, hence this proof is also a proof from the set $Y \cup LA$, i.e. $Y \vdash_S A$ and $A \in Cn(Y)$

This proves that $Cn(X) \subseteq Cn(Y)$
Exercises

Exercise 1

2. Do we need the completeness of $S$ to prove that the monotonicity property holds for $S$?

Solution

2. No, we do not need the completeness of $S$ for the monotonicity property to hold.

We have used only the definition of a formal proof from the hypothesis $X$ and the definition of the consequence operation.
Exercises

Exercise 2

1. Prove that for any set \( X \subseteq F \), the set \( T \subseteq F \) of all classical tautologies of the language \( L_{\{\cap, \cup, \Rightarrow, \neg\}} \) of the system \( S \) is a subset of \( Cn(X) \); i.e. prove that

\[
T \subseteq Cn(X)
\]

2. Do we need the completeness of \( S \) to prove that the property \( T \subseteq Cn(X) \) holds for \( S \)?
Exercises

Solution
1. The proof system $S$ is complete, so by the completeness theorem we have that

$$T = \{ \in \mathcal{F} : \vdash_S A \}$$

By definition of the consequence,

$$\{ A \in \mathcal{F} : \vdash_S A \} = Cn(\emptyset)$$

and hence $Cn(\emptyset) = T$

But $\emptyset \subseteq X$ for any set $X$, so by monotonicity property

$$T \subseteq Cn(X)$$

2. Yes, the completeness of $S$ in the main property used in the proof of 1.

The other property is the monotonicity
Exercises

Exercise 3
Prove that for any formulas $A, B \in \mathcal{F}$, and for any set $X \subseteq \mathcal{F}$,

$$(A \cap B) \in Cn(X) \text{ if and only if } A \in Cn(X) \text{ and } B \in Cn(X)$$

List all properties essential to the proof
Exercises

Solution

(1) Proof of the implication:

if \((A \cap B) \in Cn(X)\), then \(A \in Cn(X)\) and \(B \in Cn(X)\)

Assume \((A \cap B) \in Cn(X)\), i.e. \(X \vdash_S (A \cap B)\)

From **monotonicity** property proved in **Exercise 1**, **completeness** of \(S\), and the fact that

\[\vdash ((A \cap B) \Rightarrow A)\] and \[\vdash ((A \cap B) \Rightarrow B)\]

we get that

\(X \vdash_S ((A \cap B) \Rightarrow A)\) and \(X \vdash_S ((A \cap B) \Rightarrow B)\)

From the **assumption** \(X \vdash_S (A \cap B)\) and the above

\(X \vdash_S ((A \cap B) \Rightarrow A)\)

we get by **Modus Ponens**

\(X \vdash_S A\)
Exercises

Similarly, from the assumption \( X \vdash_S (A \cap B) \) and the above property

\( X \vdash_S ((A \cap B) \Rightarrow B) \)

we get by Modus Ponens

\( X \vdash_S B \)

This proves that \( A \in Cn(X) \) and \( B \in Cn(X) \) and ends the proof of the implication (1)
Exercises

(2) Proof of the implication:

if \( A \in Cn(X) \) and \( B \in Cn(X) \), then \( (A \cap B) \in Cn(X) \)

Assume now \( A \in Cn(X) \) and \( B \in Cn(X) \), i.e.

\[ X \vdash_S A \quad \text{and} \quad X \vdash_S B \]

By the monotonicity property, completeness of \( S \), and tautology

\( (A \Rightarrow (B \Rightarrow (A \cap B))) \)

we get that

\[ X \vdash_S (A \Rightarrow (B \Rightarrow (A \cap B))) \]
Exercises

By the **assumption** we have that

\[ X \vdash_S A, \quad X \vdash_S B \]

and the above

\[ X \vdash_S (A \Rightarrow (B \Rightarrow (A \cap B))) \]

we get by **Modus Ponens**

\[ X \vdash_S (B \Rightarrow (A \cap B)) \]

Applying **Modus Ponens** again we obtain

\[ X \vdash_S (A \cap B) \]

This proves

\[(A \cap B) \in Cn(X)\]

and **ends** the **proof** and the implication (2) and the **proof** of Exercise 3
Exercises

Exercise 4

Prove that classical completeness of a Hilbert proof system implies the Deduction Theorem, i.e prove that the following theorem holds for the system $S$

Deduction Theorem

For any subset $\Gamma$ of the set of formulas $\mathcal{F}$ of $S$ and for any formulas $A, B \in \mathcal{F}$,

$$\Gamma, A \vdash_S B \text{ if and only if } \Gamma \vdash_S (A \Rightarrow B)$$
Exercises

Solution

The formulas

\[ A_1 = (A \Rightarrow (B \Rightarrow A)) \] and
\[ A_2 = (((A \Rightarrow (B \Rightarrow C)) \Rightarrow ((A \Rightarrow B) \Rightarrow (A \Rightarrow C))) \]

are basic classical autologies

By the completeness of S we have that

\[ \vdash_S (A \Rightarrow (B \Rightarrow A)) \] and
\[ \vdash_S (((A \Rightarrow (B \Rightarrow C)) \Rightarrow ((A \Rightarrow B) \Rightarrow (A \Rightarrow C))) \]

The formulas \( A_1, A_2 \) are the axioms of the Hilbert system \( H_1 \)

By the completeness of S, we have that both axioms of \( H_1 \) are provable in S

These axioms were sufficient for the proof of the Deduction Theorem for \( H_1 \) and so the \( H_1 \) proof can be repeated for the system S
Exercises

Exercise 5
Prove that for any \( A, B \in \mathcal{F} \)

\[ Cn(\{A, B\}) = Cn(\{(A \cap B)\}) \]

Solution
(1) Proof of the inclusion

\[ Cn(\{A, B\}) \subseteq Cn(\{(A \cap B)\}) \]

Assume \( C \in Cn(\{A, B\}) \), i.e. we assume \( A, B \vdash \neg S \neg C \) (\( S \))

By Exercise 4 the Deduction Theorem holds for \( S \) and we apply it twice to get an equivalent form

\[ \vdash_S (A \Rightarrow (B \Rightarrow C)) \]

of the assumption
Exercises

We use completeness of $S$, the fact that the formula

$(((A \Rightarrow (B \Rightarrow C)) \Rightarrow ((A \cap B) \Rightarrow C)))$

is a tautology and get that

$\vdash_S (((A \Rightarrow (B \Rightarrow C)) \Rightarrow ((A \cap B) \Rightarrow C)))$

Applying Modus Ponens to the above and the assumption

$\vdash_S (A \Rightarrow (B \Rightarrow C))$

we get

$\vdash_S ((A \cap B) \Rightarrow C)$

This is equivalent by Deduction Theorem to

$(A \cap B) \vdash_S C$

We have proved that

$C \in Cn(\{(A \cap B)\})$

and this ends the proof of the inclusion (1)
Exercises

(2) Proof of the inclusion

\[ Cn(\{(A \cap B)\}) \subseteq Cn(\{A, B\}) \]

Assume that \( C \in Cn(\{(A \cap B)\}) \), i.e.

\[ (A \cap B) \vdash S C \]

By Deduction Theorem

\[ \vdash_S ((A \cap B) \Rightarrow C) \]

We want to prove that \( C \in Cn(\{A, B\}) \)

This is equivalent, by Deduction Theorem applied twice to proving that

\[ \vdash_S (A \Rightarrow (B \Rightarrow C)) \]
Exercises

The proof is similar to the previous case
We use completeness of $S$, the fact that the formula

$$(((A \cap B) \Rightarrow C) \Rightarrow (A \Rightarrow (B \Rightarrow C)))$$

is a tautology to get

$$\vdash_S (((A \cap B) \Rightarrow C) \Rightarrow (A \Rightarrow (B \Rightarrow C)))$$

Applying Modus Ponens to above and the assumption

$$\vdash_S ((A \cap B) \Rightarrow C)$$

we get

$$\vdash_S (A \Rightarrow (B \Rightarrow C))$$

what ends the proof