LOGICS FOR COMPUTER SCIENCE: Classical and Non-Classical Springer 2019

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Chapter 1
Introduction: Paradoxes and Puzzles

CHAPTER 1 SLIDES

Chapter 1
Introduction: Paradoxes and Puzzles

Slides Set 1

PART 1: Logic for Mathematics

Logical Paradoxes

PART 2: Logic for Mathematics

Semantical Paradoxes

Slides Set 2

PART 3: Logics for Computer Science

Classical, Intuitionistic, Modal, Temporal, Many Valued

PART 4: Computer Science Puzzles

Reasoning in Artificial Intelligence

PART 5: Chapter Short Overview

Chapter 1
Introduction: Paradoxes and Puzzles

Slides Set 1

PART 1: Logic for Mathematics

Logical Paradoxes

Logical Paradoxes

Early **intuitive approach**

Till the end of the 19th century, mathematical theories used to be built in an intuitive, not formal axiomatic way

Historical development of mathematics has shown that it is **not sufficient** to base mathematical theories only on an **intuitive** understanding of their **notions**, as the following **historical** example shows

Example

By a **set**, we mean intuitively, any **collection** of objects

For example, the **set** of all even integers or the **set** of all students in a class

The objects that make up a **set** are called its members (elements)

Sets may themselves be members of **sets**
For example, the **set of all subsets** of integers has **sets** as its members

Example

Most **sets** are not members of **themselves**

The **set** of all students, for example, is not a member of **itself**

The **set** of all students **is not** a student

However, there may be **sets** that do belong to **themselves**

For example, the **set** of all sets

Russell Paradox

Russell Paradox (1902)

Consider the set A of all those sets X such that X **is not** a member of X

Clearly, A **is** a member of A if and only if A **is not** a member of A

So, if A **is** a member of A, the A is also **not** a member of A; and if A **is not** a member of A, then A **is** a member of A

In any case, A **is** a member of A and A **is not** a member of A

Contradiction

Russell Paradox Solution

Russel proposed and developed a **theory of types** as a solution to the Russel Paradox

The **idea** is that every object must have a definite non-negative integer as its **type** assigned to it

An expression: " x **is a member** of the **set** y"

is meaningful if and only if

the **type** of y is one greater than the **type** of x

Russell Paradox Solution

Russell's **theory of types** guarantees that it is meaningless to say that a **set** belongs to **itself**

Hence Russell's **solution** is:

The set A as stated in the Russell Paradox **does not exist**

The **Type Theory** was extensively developed by Whitehead and Russell in years 1910 - 1913

Logical Paradoxes

Logical Paradoxes, also called Logical Antinomies are **paradoxes** concerning the **notion of a set**

A development of **Axiomatic Set Theory** as one of the most important fields of modern Mathematics, or more specifically of Mathematical Logic or Foundations of Mathematics resulted from the **search for solutions** to various **Logical Paradoxes**

First paradoxes free **Axiomatic Set Theory** was developed by Zermello in 1908

Logical Paradoxes

Two of the most known logical paradoxes (antinomies), other then **Russell 's Paradox** are those of **Cantor** and **Burali-Forti**

They were stated at the end of 19th century

Cantor Paradox involves the theory of cardinal numbers

Burali-Forti Paradox is the analogue to Cantor's but in the theory of ordinal numbers

Cardinality of Sets

We say that sets X and Y have the same **cardinality**, $card X = card Y$, or that they are **equinumerous** if and only if there is one-to-one correspondence that maps X onto Y

We say that $card X \leq card Y$ if and only if the set X is **equinumerous** with a **subset** of the set Y

We say that $card X < card Y$ if and only if $card X \leq card Y$ and $card X \neq card Y$

Cantor and Schröder- Berstein Theorems

Cantor Theorem

For any set X,

$card X < card \mathcal{P}(X)$

Schröder- Berstein Theorem

For any sets X and Y,

If $card X \leq card Y$ and $card Y \leq card X$, then

$card X = card Y$

Cantor Paradox

Cantor Paradox (1899)

Let C be the **universal set** - that is, the set of all sets

Now, $\mathcal{P}(C)$ is a subset of C, so it follows easily that

$card\mathcal{P}(C) \leq cardC$

On the other hand, by **Cantor Theorem**,

$cardC < card\mathcal{P}(C) \leq card\mathcal{P}(C)$, so also $cardC \leq card\mathcal{P}(C)$

From **Schröder- Berstein** theorem we have that
$card\mathcal{P}(C) = cardC$, what **contradicts** **Cantor Theorem**

Solution: **Universal set does not exist.**

Burali-Forti Paradox

Ordinal numbers are special measures assigned to ordered sets

Burali-Forti Paradox (1897)
Given any ordinal number, we know that there is a still larger ordinal number

But the ordinal number determined by the set of **all ordinal numbers** is the largest ordinal number

Solution: the set of all ordinal numbers do not exist

Logical Paradoxes

Another **solution** to Logical Paradoxes is to **reject** the assumption that for **every** property $P(x)$, there exists a corresponding set of all objects x that **satisfy** $P(x)$

The **Russell's Paradox** then proves that there **is no** set A defined by a property $P(X)$: X is a set of all sets that **do not** belong to themselves

Logical Paradoxes

Cantor Paradox shows that

there **is no** set A defined by a property

$P(X)$: there is an universal set X

Burali-Forti Paradox shows that

there **is no** set A defined by a property

$P(X)$: there is a set X that contains **all** ordinal numbers

Intuitionism

A more radical interpretation of the paradoxes has been advocated by Brouwer and his **intuitionist school**

Intuitionists refuse to accept the universality of certain basic logical laws, such as the law of **excluded middle:**
A or not A
For **Intuitionists** the **excluded middle** law is true for finite sets, but it is invalid to extend it to all other sets

The **Intuitionists'** concept of **infinite** set differs from that of classical mathematicians

Intuitionists' Mathematics

The basic difference between **classical** and **intuitionists'** mathematics lies also in the interpretation of the word **exists**

In classical mathematics proving **existence** of an object x such that $P(x)$ holds **does not mean** that one is able to indicate a method of **construction** of it

In the **intuitionists' universe** we are justified in asserting the **existence** of an object having a certain property **only if** we prove existence of an **effective method** for constructing, or finding such an object

Intuitionists' Mathematics

In **intuitionistic** mathematics the logical paradoxes are **not derivable**, or even meaningful

The **Intuitionism**, because of its constructive flavor, has found a lot of applications in **computer science**, for example in the theory of **programs correctness**

Intuitionistic Logic (to be studied in the book) reflects intuitionists ideas in a form a formalized deductive system

Chapter 1
Introduction: Paradoxes and Puzzles

Slides Set 1

PART 2: Logic for Mathematics

Semantic Paradoxes

Semantic Paradoxes

The development of **axiomatic theories** solved some, but not all problems brought up by the **Logical Paradoxes.**

Even the consistent sets of axioms, as the following examples show, do not prevent the occurrence of another kind of paradoxes, called **Semantic Paradoxes**

The **Semantic Paradoxes** deal with the notion of **truth**

Semantic Paradoxes

Berry Paradox, 1906:

Let A denote the set of all positive integers which can be defined in the English language by means of a sentence containing at most 1000 letters

The set A is finite since the set of all sentences containing at most 1000 letters is finite. Hence, there exist positive integer which do not belong to A

Consider a sentence: n is the least positive integer which cannot be defined by means of a sentence of the English language containing at most 1000 letters

This sentence contains **less than 1000 letters** and defines a positive integer n

Therefore $n \in A$ - but $n \notin A$ by the definition of n

CONTRADICTION!

Berry Paradox Analysis

The paradox resulted entirely from the fact that we **did not** say precisely what notions and sentences belong TO the arithmetic and what notions and sentences concern the arithmetic

Of course we didn't talk about and examine arithmetic as a fix and closed deductive system

We also **incorrectly** mixed the natural language with mathematical language of arithmetic

Berry Paradox Solution

We have to always clearly distinguish between the **language** of the theory (arithmetic) and the **language** in which we talk **about** the theory, which is called a metalanguage

In general we must clearly distinguish a formal **theory** from the **meta-theory**

In well and correctly defined **theory** such paradoxes **can not** appear

The Liar Paradox

Liar Paradox

A man says: I am lying

If he is lying, then what he says is **true**, and so he is not lying

If he is not lying, then what he says is **not true**, and so he is lying

Contradiction

Liar Paradoxes

These **paradoxes arise** because the **concepts** of the type

" I am true", **" this sentence is true"**, " I am lying"

should not occur in the **language** of the theory

They belong to a **metalanguage** of the theory

It it means they belong to a language that talks **about** the theory

Cretan Paradox

The **Liar Paradox** is a corrected version of a following paradox stated in antiquity by a philosopher Epimenides

Cretan Paradox

The Cretan philosopher Epimenides said:

All Cretans are liars

If what he said is **true**, then, since he is a Cretan,

it must be **false** and what he said is **false**

Thus, **there is** a Cretan who **is not** a liar

Contradiction

Chapter 1
Introduction: Paradoxes and Puzzles

Slides Set 2

PART 3: Logics for Computer Science:

Classical, Intuitionistic, Modal, Temporal, Many Valued

Classical and Intuitionistic Logics

The use of Classical Logic in computer science is known, indisputable, and well established.

The existence of PROLOG and Logic Programming as a separate field of computer science is the best example of it

Intuitionistic Logic in the form of Martin-Löf's theory of types (1982), provides a complete theory of the process of program specification, construction, and verification

A similar theme has been developed by Constable (1971) and Beeson (1983)

Modal Logics

Modal Logic was created by C.I. Lewis in 1918

In an attempt to avoid, what some felt, the paradoxes of classical implication (a false sentence implies any sentence) he proposed a new interpretation of the logical implication

The idea was to distinguish two sorts of truth: necessary truth and mere possible truth

As a consequence a new, modal logic was created

Modal Logics for Computer Science

Modal Logics in Computer Science are used as as a tool for analyzing such notions as knowledge, belief, tense

Modal logics have been also employed in a form of Dynamic logic (Harel, 1979) to facilitate the statement and proof of properties of programs

Temporal Logics

Temporal Logics were created for the specification and verification of concurrent programs by Harel (1979) and Parikh (1983)

For a specification of hardware circuits by Halpern, Manna, Maszkowski (1983)

Temporal Logics were also used to specify and clarify the concept of causation and its role in commonsense reasoning by Shoham (1988)

Other Non-classical Logics

The development of new logics and the applications of logics to different areas of **Computer Science** and in particular to **Artificial Intelligence** is a subject of a book in itself but is beyond the scope of this book

The **book** examines in detail the classical logic and some aspects of the intuitionistic logic and its **relationship** with the classical logic

It introduces some of the most standard many valued logics, and examines modal S4, S5 logics
It also shows the relationship between the modal S4 and the intuitionistic logics

Chapter 1
Introduction: Paradoxes and Puzzles

Slides Set 2

PART 4: Computer Science Puzzles

Reasoning in Artificial Intelligence

Reasoning in Distributive Systems

Problem by Grey (1978), Halpern, Moses (1984)

Two divisions of an army are camped on **two** hilltops overlooking a common valley

In the valley awaits the enemy

If **both** divisions attack the enemy **simultaneously** they will win the battle

If only **one** division attacks it will be defeated

Coordinated Attack

The divisions **do not** initially have plans for launching an attack on the **enemy**

The commanding general of the **first** division wishes to coordinate a **simultaneous** attack
Neither general will decide to attack unless he is **sure** that the other will attack with **him**

The generals can only **communicate** by means of a messenger.

Coordinated Attack

It takes a messenger one hour to get from one
encampment
to the other
However, it is possible that the messenger will **get lost**
in the dark or, worst yet, be **captured** by the enemy

Fortunately on this particular night, everything **goes** smoothly

Question

How long will it take them to coordinate an **attack?**

Coordinated Attack

Suppose the **messenger** sent by General A makes it to General B with a **message** saying Attack at dawn

Will General B **attack?**

No, since General A does not know General B **got** the message, and thus may **not attack**

Coordinated Attack

General B sends the **messenger** back with an acknowledgment

Suppose the **messenger** makes it

Will General A **attack?**

No , because now A is worried that General B does not know A got the message, that General B thinks A may think that B did not get the original message, and thus General A does **not attack**

Coordinated Attack

General A sends the **messenger** back with an acknowledgment. This is not enough

No amount of acknowledgments sent back and forth will ever **guarantee** agreement
Even in a case that the **messenger** succeeds in delivering the message every time

All that is **required** in this (informal) reasoning is the **possibility** that the messenger **does not** succeed

Coordinated Attack Solution

To **solve** this problem Halpern and Moses (1985) created a propositional modal logic with m agents

They **proved** this **logic** to be essentially a multi-agent version of the standard modal logic S5

They also **proved** that formally defined **common knowledge** is not attainable in systems where communication is **not guaranteed**

Communication in Distributed Systems

The **common knowledge** is also not attainable in systems where communication is **guaranteed**, as long as there is some uncertainty in massage delivery time

In distributed systems where communication is **not guaranteed** common knowledge is **not attainable**

But we often **do reach** agreement!

Communication in Distributed Systems

They proved that formally defined common knowledge is attainable in such models of reality where we assume, for example, events can be guaranteed to happen simultaneously

Moreover, there are some variants of the definition of common knowledge that are **attainable** under more reasonable assumptions

So, we can formally prove that in fact we often **do reach agreement!**

Reasoning in Artificial Intelligence

Assumption 1:

Flexibility of reasoning is one of the key property of intelligence

Assumption 2:

Commonsense inference is **defeasible** in its nature; we are all capable of drawing conclusions, acting on them, and then **retracting** them if necessary in the face of new evidence

Reasoning in Artificial Intelligence

If computer programs are to act **intelligently**, they will need to be similarly flexible

Goal: development of **formal systems** (logics) that describe commonsense flexibility

Flexible Reasoning

Example: Reiter, 1987

Consider a statement Birds fly

Tweety, we are told, is a bird. From this, and the fact that birds fly, we **conclude** that Tweety can fly

This conclusion is **defeasible:** Tweety may be an ostrich, a penguin, a bird with a broken wing, or a bird whose feet have been set in concrete

This is a **non-monotonic** reasoning: on learning a **new fact** (that Tweety has a broken wing), we are forced to **retract** our conclusion (that he could fly)

Non-Monotonic and Default Reasoning

Definition

A non-monotonic reasoning is a reasoning in which the introduction of a new information can **invalidate** old facts

Definition

A default reasoning (logic) is a reasoning that let us draw plausible inferences from less-than- conclusive evidence in the absence of information to the contrary

Observe that non-monotonic reasoning is an example of default reasoning

Believe Reasoning

Example Moore, 1983

Consider my reason for believing that I do not have an older brother

It is surely not that one of my parents once casually remarked,

you know, you don't have any older brothers,

nor have I pieced it together by carefully sifting other evidence

I simply believe that if I did have an older brother

I would know about it;

therefore since I don't know of any older brothers of mine,

I must not have any

Auto-epistemic Reasoning

The brother **example** reasoning is not **default** reasoning

nor **non-monotonic** reasoning

It is a reasoning about one's own knowledge or belief

Definition

Any reasoning about one's own **knowledge** or **belief** is called an **auto-epistemic** reasoning

Auto-epistemic reasoning **models** the reasoning of an ideally rational agent reflecting upon his **beliefs** or **knowledge**

Logics which describe it are called **auto-epistemic logics**

Computer Science Puzzles
Missionaries and Cannibals

Example McCarthy, 1985

Here is the old **Cannibals Problem**

Three missionaries and three cannibals come to the river.

A rowboat that seats two is available

If the cannibals ever outnumber the missionaries on either bank of the river, the missionaries will be **eaten**

How shall they cross the river?

Traditional Solution

Traditionally the puzzler is expected to devise a strategy of rowing the boat back and forth that gets them all across and avoids the disaster

A **state** is a triple comprising the number of missionaries, cannibals and boats on the starting bank of the river

The **initial** state is 331,
the **desired** state is 000
A **solution** is given by the sequence:

331, 220, 321, 300, 311, 110, 222, 020, 031, 010, 021, **000**

Missionaries and Cannibals Revisited

Imagine now giving someone the problem, and after he puzzles for a while, he suggests going upstream half a mile and crossing on a bridge

What a bridge? you say
No bridge is mentioned in the statement of the problem
He replies: **Well, they don't say the isn't a bridge**

So you modify the problem to **exclude** the bridges and pose it again
He proposes a helicopter, and after you **exclude** that, he proposes a winged horse....

Missionaries and Cannibals Revisited

So you tell him the **solution**

He **attacks** your solution on the grounds that the boat might have a leak

After you **rectify** that omission from the statement of the problem, he suggests that a sea monster may **swim** up the river and may **swallow** the boat

Finally, you must look for a **mode** of reasoning that will **settle** his hash once and for all

McCarthy Solution

McCarthy proposes circumscription as a technique for solving his puzzle

He argues that it is a part of **common knowledge** that a boat **can** be used to cross the river **unless** there is something **wrong** with it or something else **prevents** using it

If our facts **do not** require that there be something that **prevents** crossing the river, the circumscription will **generate** the conjecture that there isn't

Chapter 1
Introduction: Paradoxes and Puzzles

Slides Set 2

PART 5: A Short Chapter Overview

Definitions and Facts

Definition

Logical Paradoxes, also called Logical Antinomies are paradoxes concerning the **notion of a set**

Definition

Semantic Paradoxes are paradoxes that deal with the notion of **truth**

Definition

A non-monotonic inference is a reasoning in which **introduction** of a new information can **invalidate** old facts

Definitions and Facts

Fact

Non-monotonic reasoning is an example of the default reasoning

Definition

An auto-epistemic reasoning is any reasoning about one's own **knowledge** or **belief**

Auto-epistemic reasoning **models** the reasoning of an ideally rational agent **reflecting** upon his beliefs or knowledge

Definitions and Facts

Facts

The main difference between **classical** and **intuitionists'** mathematics lies in the **interpretation** of the word **exists**

In classical mathematics proving **existence** of an object x such that a property $P(x)$ holds **does not** always mean that one is able to indicate a method of its construction

In the **intuitionists' universe** we are justified in asserting the **existence** of an object having a certain property only if we know an **effective method** for constructing, or finding such an object