Chapter 5: Some Three Valued Logics Semantics

First many valued logic (defined semantically only) was formulated by Łukasiewicz in 1920.

We present here five 3-valued logics semantics that are named after their authors: *Łukasiewicz, Kleene, Heyting*, and *Bochvar*.

Three valued logics , when defined semantically, enlist a third logical value $\perp$, or m in Bochvar semantics..

We assume that the third value is intermediate between truth and falsity, i.e. that $F <\perp< T$, or $F < m < T$.

All of presented here semantics take T as designated value, i.e. the value that defines the notion of satisfiability and tautology.

The third value $\perp$ corresponds to some notion of *incomplete information*, or *inconsistent information* or *undefined* or *unknown*.

Historically all these semantics were are called logics, we use the name logic for them, instead saying each time ”logic defined semantically” , or ”semantics for a given logic” .

Łukasiewicz Logic Ł: Motivation

Łukasiewicz developed his semantics (called logic) to deal with future contingent statements.

Contingent statements are not just neither true nor false but are indeterminate in some metaphysical sense.

It is not only that we do not know their truth value but rather that they do not possess one.

The Language :

$$\mathcal{L} = \mathcal{L}_{\{\neg, \Rightarrow, \cup, \cap\}}.$$

Logical Connectives are the following operations in the set $\{F, \bot, T\}$.

For any $a, b \in \{F, \bot, T\}$,

$$\neg \bot = \bot, \quad \neg F = T, \quad \neg T = F,$$

$$a \cup b = max\{a, b\},$$

$$a \cap b = min\{a, b\},$$

$$a \Rightarrow b = \begin{cases} \neg a \cup b & \text{if } a > b \\ T & \text{otherwise} \end{cases}$$

Ł 3-valued truth tables

Ł Negation

¬	F	⊥	T
	T	⊥	F

Ł Conjunction

∩	F	⊥	T
F	F	F	F
⊥	F	⊥	⊥
T	F	⊥	T

Ł Disjunction

∪	F	⊥	T
F	F	⊥	T
⊥	⊥	⊥	T
T	T	T	T

Ł-Implication

⇒	F	⊥	T
F	T	T	T
⊥	⊥	T	T
T	F	⊥	T

A truth assignment is any function

$$v : VAR \longrightarrow \{F, \perp, T\}$$

Extension v^* of v to the set $\mathcal{F}$ of all formulas:

$$v^* : \mathcal{F} \longrightarrow \{F, \perp, T\}.$$

is defined by the induction on the degree of formulas as follows.

$v^*(a) = v(a)$ for $a \in VAR$,

$v^*(\neg A) = \neg v^*(A)$,

$v^*(A \cap B) = (v^*(A) \cap v^*(B))$,

$v^*(A \cup B) = (v^*(A) \cup v^*(B))$,

$v^*(A \Rightarrow B) = (v^*(A) \Rightarrow v^*(B))$.

Ł Model, Counter- Model :

Any truth assignment v, such that $v^*(A) = T$ is called a **Ł model** for the formula $A \in \mathcal{F}$.

Any v such that $v^*(A) \neq T$ is called a **Ł counter-model** for A.

Ł Tautologies : For any $A \in \mathcal{F}$,

A is an **Ł tautology** if and only if $v^*(A) = T$, for all $v : VAR \longrightarrow \{F, \perp, T\}$, i.e.
if all truth assignments v are **Ł models** for A.

Ł tautologies notation:

$$\models_{\text{Ł}} A.$$

Let **ŁT**, **T** denote the sets of all **Ł** and classical tautologies, respectively.

$$\text{Ł}\mathbf{T} = \{A \in \mathcal{F} : \models_{\text{Ł}} A\},$$

$$\mathbf{T} = \{A \in \mathcal{F} : \models A\}.$$

Q1 Is the **Ł** logic really different from the classical logic? It means are theirs sets of tautologies different?

Answer : Consider

$$\models (\neg a \cup a).$$

Take a variable assignment $v$such that

$$v(a) = \perp .$$

Evaluate :

$$v^*(\neg a \cup a) = v^*(\neg a) \cup v^*(a) = \neg v(a) \cup v(a)$$
$$= \neg \perp \cup \perp = \perp \cup \perp = \perp$$

This proves that v is a counter-model for

$$(\neg a \cup a), i.e.$$

$$\not\models_{\text{Ł}} (\neg a \cup a)$$

and we have a property:

$$\text{Ł}\mathbf{T} \neq \mathbf{T}$$

Q2 Do have something in common (besides the same language? Do they share some tautologies?

Answer : Restrict the Truth Tables for **Ł** connectives to the values T and F only.

We get the Truth Tables for classical connectives.

This means that if $v*(A) = T$ for all $v : VAR \longrightarrow \{F, \perp, T\}$, then $v*(A) = T$ for all $v : VAR \longrightarrow \{F, T\}$ and any $A \in \mathcal{F}$.

We have proved a property:

$$\text{Ł}\mathbf{T} \subset \mathbf{T}.$$

Kleene Logic K : Motivation

The third logical value $\perp$, intuitively, represents *undecided*. Its purpose is to signal a state of partial ignorance.

A sentence a is assigned a value $\perp$ just in case it is not *known* to be either true of false.

For example , imagine a detective trying to solve a murder. He may conjecture that Jones killed the victim. He cannot, at present, assign a truth value T or F to his conjecture, so we assign the value $\perp$, but it is certainly either true of false and $\perp$ represents our ignorance rather then total unknown.

The language is the same in case of classical or **Ł** logic.

$$\mathcal{L} = \mathcal{L}_{\{\neg, \Rightarrow, \cup, \cap\}}.$$

Connectives $\neg, \cup, \cap$ of **K** are defined as in **Ł** logic, i.e. for any $a, b \in \{F, \perp, T\}$,

$$\neg \perp = \perp, \quad \neg F = T, \quad \neg T = F,$$

$$a \cup b = max\{a, b\},$$

$$a \cap b = min\{a, b\}.$$

Implication in Kleene's logic is defined as follows.

For any $a, b \in \{F, \bot, T\}$,

$$a \Rightarrow b = \neg a \cup b.$$

The Kleene's 3-valued truth tables differ hence from Łukasiewicz's truth tables only in a case on implication. This table is:

K-Implication

$\Rightarrow$	F	$\bot$	T
F	T	T	T
$\bot$	$\bot$	$\bot$	T
T	F	$\bot$	T

K Tautologies -

$$\mathbf{KT} = \{A \in \mathcal{F} : \quad \models_K A\}$$

Relationship between **Ł**, **K**, and classical logic.

$$\text{Ł}\mathbf{T} \neq \mathbf{KT},$$

$$\mathbf{KT} \subset \mathbf{T}.$$

Proof of $\text{Ł}\mathbf{T} \neq \mathbf{KT}$.

Obviously

$$\models_{\text{Ł}} (a \Rightarrow a).$$

Take v such that

$$v(a) = \perp$$

we have that for **K** semantics

$$v^*(a \Rightarrow a) = v(a) \Rightarrow v(a) = \perp \Rightarrow \perp = \perp .$$

This proves that

$$\not\models_{\mathbf{K}} (a \Rightarrow a)$$

and Ł$\mathbf{T} \neq \mathbf{KT}$

The second property $\mathbf{KT} \subset \mathbf{T}$ follows directly from the the fact that, as in the **Ł** case, if we restrict the K- Truth Tables to the values T and F only, we get the Truth Tables for classical connectives.

Heyting Logic H
Motivation and History

Hayting and Pseudo Bollean Algebras

We call the **H** logic a Heyting logic because its connectives are defined as operations on the set $\{F, \perp, T\}$ in such a way that they form a 3-element Heyting algebra, called also a 3-element pseudo-boolean algebra.

Pseudo-boolean, or Heyting algebras provide algebraic models for the intuitionistic logic. These were the first models ever defined for the intuitionistic logic.

The intuitionistic logic was defined and developed by its inventor Brouwer and his school in 1900s as a proof system only. Heyting provided first axiomatization for the intuitionistic logic.

The semantics for Intuitionistic Logic was discovered some 35 years later by McKinsey and Tarski in 1942 in a form of Pseudo-Boolean (called also later Heyting) algebras.

It took yet another 5-8 years to extend it to predicate logic (Rasiowa, Mostowski, 1957).

The other type of models, called Kripke Models were defined by Kripke in 1964 and were proved later to be equivalent to the pseudo-boolean models.

A formula A is an intutionistic tautology if and only if it is true in *all pseudo-boolean (Heying) algebras*.

Hence, if A is an intuitionistic tatology (true in all algebras) is also true in a 3-element Heyting algebra (a particular algebra). From that we get that all intuitionistic propositional logic tautologies are Heyting 3-valued logic tautologies.

Denote by **IT**, H**T** the sets of all tautologies of the intuitionistic semantics and Heyting 3-valued semantics, respectively we can write it symbolically as:

$$\mathbf{IT} \subset \mathbf{HT}.$$

Conclude that for any formula A,

$$If \quad \not\models_H A \;\; then \not\models_I A.$$

If we can show that a formula A has a Heying 3-valued counter-model, then we have proved that it is not an intuitionistic tautology.

Hayting 3 valued logic semantics is formally defined as follows.

The Language :

$\mathcal{L} = \mathcal{L}_{\{\neg, \Rightarrow, \cup, \cap\}}$.

Logical connectives : $\cup$ and $\cap$ are the same as in the case of **Ł** and **K** logics, i.e.

For any $a, b \in \{F, \perp, T\}$ we define

$$a \cup b = max\{a, b\},$$

$$a \cap b = min\{a, b\}.$$

Implication :

$$a \Rightarrow b = \begin{cases} T & \text{if } a \leq b \\ b & \text{otherwise} \end{cases}$$

Negation :

$$\neg a = a \Rightarrow F.$$

H-Implication

$\Rightarrow$	F	$\bot$	T
F	T	T	T
$\bot$	F	T	T
T	F	$\bot$	T

H Negation

$\neg$	F	$\bot$	T
	T	F	F

Notation : **HT**, **T**, **ŁT**, **KT** denote the set of all tautologies of the **H**, classical, **Ł**, and **K** logic, respectively.

Relationship : The

$$\mathbf{HT} \neq \mathbf{T} \neq \mathbf{ŁT} \neq \mathbf{KT},$$

$$\mathbf{HT} \subset \mathbf{T}. \tag{1}$$

Proof For the formula $(\neg a \cup a)$ we have:

$$\models (\neg a \cup a)$$

and

$$\not\models_H (\neg a \cup a)$$

Take the variable assignment v such that

$$v(a) = \perp .$$

A formula $(A \Rightarrow A)$ is a **H** logic tautology

$$\models_H A(\Rightarrow A$$

but is not a **K** logic tautology.

Take the variable assignment v such that $v(a) = v(b) = \perp$.It proves that

$$\not\models_K (\neg(a \cap b) \Rightarrow (\neg a \cup \neg b))$$

but

$$\models_{\mathbf{H}} (\neg(a \cap b) \Rightarrow (\neg a \cup \neg b)).$$

Observe now that if we restrict the truth tables for **H** connectives to the values T and F only, we get the truth tables for classical connectives.

Bochvar 3-valued logic B: Motivation

Consider a semantic paradox given by a sentence: *this sentence is false*.

If it is true it must be false, if it is false it must be true.

Bohvar's proposal adopts a strategy of a change of logic.

According to Bochvar, such sentences are neither true of false but rather *paradoxical* or *meaningless*.

The semantics follows the principle that the third logical value, denoted now by m is in some sense " infectious" ; if one one component of the formula is assigned the value m then the formula is also assigned the value m.

Bohvar also adds an one argument *assertion operator S* that asserts the logical value of T and F, i.e. $SF = F$, $ST = T$ and it asserts that meaningfulness is false, i.e $Sm = F$.

Language : $\mathcal{L}_{\{\neg, S, \Rightarrow, \cup, \cap\}}$.

Logical connectives :

B Negation

$\neg$	F	m	T
	F	m	T

B Conjunction

$\cap$	F	m	T
F	F	m	F
m	m	m	m
T	F	m	T

B Assertion :

S	F	m	T
	F	F	T

B Disjunction

$\cup$	F	m	T
F	F	m	T
m	m	m	m
T	T	m	T

B Implication

$\Rightarrow$	F	m	T
F	T	m	T
m	m	m	m
T	F	m	T

Observe that none of the formulas of $\mathcal{L}_{\{\neg,\Rightarrow,\cup,\cap\}}$ is a **B** tautology.

Any v such that $v(a) = m$ for at least one variable in a formula is a counter-model for that formula. I. e we have that

$$\mathbf{T} \cap \mathbf{BT} = \emptyset.$$

For a formula to be a **B** tautology, it must contain the connective S.

Examples :

$$\not\models_{\mathbf{B}} (a \cup \neg a)$$

as $v(a) = m$ gives: $m \cup \neg m = m$.

For the same $v(a) = m$ we have that

$$\not\models_{\mathbf{B}} (a \cup \neg Sa),$$

$$\not\models_{\mathbf{B}} (Sa \cup \neg a),$$

$$\not\models_{\mathbf{B}} (Sa \cup S\neg a),$$

but it is easy to verify that

$$\models_{\mathbf{B}} (Sa \cup \neg Sa).$$