CHAPTER 5

Hilbert Proof Systems: Completeness of Classical Propositional Logic

The Hilbert proof systems are systems based on a language with implication and contain a Modus Ponens rule as a rule of inference. They are usually called Hilbert style formalizations. We will call them here Hilbert style proof systems, or Hilbert systems, for short.

Modus Ponens is probably the oldest of all known rules of inference as it was already known to the Stoics (3rd century B.C.). It is also considered as the most natural to our intuitive thinking and the proof systems containing it as the inference rule play a special role in logic. The Hilbert proof systems put major emphasis on logical axioms, keeping the rules of inference to minimum, often in propositional case, admitting only Modus Ponens, as the sole inference rule.

There are many proof systems that describe classical propositional logic, i.e. that are complete proof systems with the respect to the classical semantics.

We present here, after Elliott Mendelson's book *Introduction to Mathematical Logic* (1987), a Hilbert proof system for the classical propositional logic and discuss two ways of proving the Completeness Theorem for it.

Any proof of the Completeness Theorem consists always of two parts. First we have show that *all formulas that have a proof are tautologies.* This implication is also called a Soundness Theorem, or soundness part of the Completeness Theorem. The second implication says: *if a formula is a tautology then it has a proof.* This alone is sometimes called a Completeness Theorem (on assumption that the system is sound). Traditionally it is called a completeness part of the Completeness Theorem.

The proof of the soundness part is standard. We concentrate here on the completeness part of the Completeness Theorem and present two proofs of it.

The first proof is based on the one presented in the Mendelson's book *Introduction to Mathematical Logic* (1987). It is is a straightforward constrictive proof that shows how one can use the assumption that a formula A is a tautology in order to construct its formal proof. It is hence called *a proof - construction method.* It is a beautiful proof

The second proof is non-constrictive. Its strength and importance lies in a fact that the methods it uses can be applied to the proof of completeness for classical predicate logic. We will discuss and apply them in Chapter **??**.

It proves the completeness part of the Completeness Theorem by proving the converse implication to it. It shows how one can deduce that *a formula A is not a tautology from the fact that it does not have a proof.* It is hence called a *counter-model construction proof.*

Both proofs of the Completeness Theorem relay on the Deduction Theorem and so it is the first theorem we are going to prove.

1 Deduction Theorem

We consider first a very simple Hilbert proof system based on a language with implication as the only connective, with two logical axioms (axiom schemas) which characterize the implication, and with Modus Ponens as a sole rule of inference. We call it a Hilbert system H_1 and define it as follows.

$$H_1 = (\ \mathcal{L}_{\{\Rightarrow\}},\ \ \mathcal{F},\ \ A1, A2,\ \ (MP)\ ), \tag{1}$$

where

A1 $(A \Rightarrow (B \Rightarrow A))$,

A2 $((A \Rightarrow (B \Rightarrow C)) \Rightarrow ((A \Rightarrow B) \Rightarrow (A \Rightarrow C)))$,

(MP) is the following rule of inference, called Modus Ponens

$$(MP)\ \frac{A\ ;\ (A \Rightarrow B)}{B},$$

and $A, B, C \in \mathcal{F}$ are any formulas of the propositional language $\mathcal{L}_{\{\Rightarrow\}}$.

Finding formal proofs in this system requires some ingenuity. Let's construct, as an example, the formal proof of such a simple formula as $A \Rightarrow A$.

The formal proof of $(A \Rightarrow A)$ in H_1 is a sequence

$$B_1,\ B_2,\ B_3,\ B_4, B_5 \tag{2}$$

as defined below.

$B_1 = ((A \Rightarrow ((A \Rightarrow A) \Rightarrow A)) \Rightarrow ((A \Rightarrow (A \Rightarrow A)) \Rightarrow (A \Rightarrow A)))$,
axiom A2 for $A = A$, $B = (A \Rightarrow A)$, and $C = A$

$B_2 = (A \Rightarrow ((A \Rightarrow A) \Rightarrow A))$,
axiom A1 for $A = A$, $B = (A \Rightarrow A)$

$B_3 = ((A \Rightarrow (A \Rightarrow A)) \Rightarrow (A \Rightarrow A)))$,
MP application to B_1 and B_2

$B_4 = (A \Rightarrow (A \Rightarrow A))$,
axiom A1 for $A = A$, $B = A$

$B_5 = (A \Rightarrow A)$
MP application to B_3 and B_4

We have hence proved the following.

Fact 1

For any $A \in \mathcal{F}$,

$$\vdash_{H_1} (A \Rightarrow A)$$

and the sequence 2 constitutes its formal proof.

It is easy to see that the above proof wasn't constructed automatically. The main step in its construction was the choice of a proper form (substitution) of logical axioms to start with, and to continue the proof with. This choice is far from obvious for un-experienced prover and impossible for a machine, as the number of possible substitutions is infinite.

Observe that the systems $S_1 - S_4$ from the previous Chapter 4 had inference rules such that it was possible to "reverse" their use; to use them in the reverse manner in order to search for proofs, and we were able to do so in a blind, fully automatic way. We were able to conduct an argument of the type: *if this formula has a proof the only way to construct it is from such and such formulas by the means of one of the inference rules, and that formula can be found automatically.* We called proof systems with such property syntactically decidable and defined them formally as follows.

Definition 1 (Syntactic Decidability)

A proof system $S = (\mathcal{L}, \mathcal{E}, LA, \mathcal{R})$ for which there is an effective mechanical, procedure that finds (generates) a formal proof of any E in S, if it exists, is called **syntactically semi- decidable**. *If additionally there is an effective method of deciding that if a proof of E in S not found, it does not exist, the system S is called* **syntactically decidable**. *Otherwise S is* **syntactically undecidable**.

We will argue now, that one can't apply the above argument to the proof search in Hilbert proof systems as they which contain Modus Ponens as an inference rule.

A *general procedure* for searching for proofs in a proof system S can be stated is as follows. Given an expression B of the system S. If it has a proof, it must be conclusion of the inference rule. Let's say it is a rule r. We find its premisses, with B being the conclusion, i.e. we evaluate $r^{-1}(B)$. If all premisses are axioms, the proof is found. Otherwise we repeat the procedure for any non-axiom premiss.

Search for proof in Hilbert Systems must involve the Modus Ponens. The rule says: given two formulas A and $(A \Rightarrow B)$ we can conclude a formula B. Assume now that we have a formula B and want to find its proof. If it is an axiom, we have the proof: the formula itself. If it is not an axiom, it had to be obtained by the application of the Modus Ponens rule, to certain two formulas A and $(A \Rightarrow B)$. But there is infinitely many of formulas A and $(A \Rightarrow B)$. I.e. for any B, the inverse image of B under the rule MP, $MP^{-1}(B)$ is countably infinit Obviously, we have the following.

Fact 2

Any Hilbert proof system is not syntactically decidable, in particular, the system H_1 is not syntactically decidable.

Semantic Link 1 System H_1 is obviously sound under classical semantics and is sound under **Ł**, **H** semantics and not sound under **K** semantics.

We leave the proof of the following theorem (by induction with respect of the length of the formal proof) as an easy exercise to the reader.

Theorem 1 (Soundness of H_1)

For any $A \in \mathcal{F}$ of H_1, if $\vdash_{H_1} A$, then $\models A$.

Semantic Link 2 System H_1 **is not complete** under classical semantics. It means that not all classical tautologies have a proof in H_1. We have proved in Chapter 3 that one needs negation and one of the other connectives $\cup, \cap, \Rightarrow$ to express all classical connectives, and hence all classical tautologies. Our language contains only implication and one can't express negation in terms of implication and hence we can't provide a proof of any tautology i.e. its logically equivalent form in our language. It means we have proved the following.

Fact 3

The proof system H_1 is sound, but **not complete** *under the classical semantics.*

We have constructed a formal proof (2) of $(A \Rightarrow A)$ in H_1 on a base of logical axioms, as an example of complexity of finding proofs in Hilbert systems.

In order to make the construction of formal proofs easier by the use of previously proved formulas we use the notions of a formal proof from some *hypotheses* (and logical axioms) in any proof system $S = (\mathcal{L},\ \mathcal{E},\ LA, \mathcal{R})$ defined as follows in chapter 4.

Definition 2 (Proof from Hypotheses)

Given a proof system $S = (\mathcal{L},\ \mathcal{E}, LA, \mathcal{R})$ *and let* Γ *be any set of expressions of* S*, i.e. let* $\Gamma \subseteq \mathcal{E}$.

A proof of an expression $E \in$ *from the set* Γ *of expressions is a sequence*

$$E_1, E_2,\ \dots\ E_n$$

of expressions, such that

$$E_1 \in LA \cup \Gamma, \qquad E_n = E$$

and for each i, $1 < i \leq n$, *either* $E_i \in LA \cup \Gamma$ *or* E_i *is a direct consequence of some of the preceding expressions in the sequence* $E_1, E_2,\ \dots\ E_n$ *by virtue of one of the rules of inference from* $\mathcal{R}$.

We write

$$\Gamma \vdash_S E$$

to denote that the expression E *has a proof (is provable) from* Γ *in* S *and we write* $\Gamma\ \vdash\ E$, *when the system* S *is fixed.*

When the set of hypothesis Γ is a *finite set* and $\Gamma = \{B_1, B_2, ..., B_n\}$, then we write

$$B_1, B_2, ..., B_n \vdash_S E$$

instead of $\{B_1, B_2, ..., B_n\} \vdash_S E$. The case when Γ is an empty set i.e. when $\Gamma = \emptyset$ is a special one. By the definition of a proof of E from Γ, $\emptyset \vdash_S E$ means that in the proof of E only logical axioms LA of S were used. We hence write as we did before

$$\vdash_S E$$

to denote that E has a proof from the empty set Γ.

Definition 3 (Consequence in S)

Given a proof system $S = (\mathcal{L},\ \mathcal{F}, LA, \mathcal{R})$ *and a set* $\Gamma \subseteq \mathcal{F}$. *Any formula* $A \in \mathcal{F}$ **provable** *from* Γ*, i.e. such that*

$$\Gamma \vdash_S A$$

is called a **consequence** *of* Γ *in* S. *Formulas from* Γ *are called* **hypotheses** *or premisses of a proof of* A *from* Γ *in* S.

The following are simple, but very important properties of the notion of consequence.

Fact 4 (Consequence Properties)

Given a proof system $S = (\mathcal{L},\ \mathcal{F}, LA, \mathcal{R})$*. For any sets* $\Gamma, \Delta \subseteq \mathcal{F}$ *the following holds.*

1. *If* $\Gamma \subseteq \Delta$ *and* $\Gamma \vdash_S A$*, then* $\Delta \vdash_S A$*.* **monotonicity**

2. $\Gamma \vdash_S A$ *if and only if*
there is a **finite** *subset* Γ_0 *of* Γ *such that* $\Gamma_0 \vdash_S A$*.* **finiteness**

3. *If* $\Delta \vdash_S A$*, and, for each* $B \in \Delta$*,* $\Gamma \vdash_S B$*, then* $\Gamma \vdash_S A$*.* **transitivity**

Proof
The properties follow directly from the definition 2 and their proofs are left to the reader as an exercise.

The **monotonicity** property represents the fact that if a formula A is provable from a set Γ of premisses (hypotheses), then if we add still more premisses, A is still provable. It hence is often stated as follows,

$$\text{If } \Gamma \vdash_S A, \text{ then } \Gamma \cup \Delta \vdash_S A, \text{ for any set } \Delta \subseteq \mathcal{F}. \tag{3}$$

The detailed investigation of Tarski general notion of consequence operation, its relationship with proof systems, and hence with the consequence in S introduced here is included in Chapter 4. Here is an application of the proof from hypotheses definition 2 to the system H_1.

Exercise 1

Construct a proof in H_1 *of a formula* $(A \Rightarrow C)$ *from the set of hypotheses* $\Gamma = \{(A \Rightarrow B), (B \Rightarrow C)\}$*. I.e. show that*

$$(A \Rightarrow B), (B \Rightarrow C) \vdash_{H_1} (A \Rightarrow C).$$

Solution
The required formal proof is a sequence

$$B_1, B_2, ...B_7 \tag{4}$$

such that

$B_1 = (B \Rightarrow C)$,
hypothesis

$B_2 = (A \Rightarrow B)$,
hypothesis

$B_3 = ((B \Rightarrow C) \Rightarrow (A \Rightarrow (B \Rightarrow C)))$,
axiom A1 for $A = (B \Rightarrow C)$, $B = A$

$B_4 = (A \Rightarrow (B \Rightarrow C))$
$B_1,\ B_3$ and MP

$B_5 = ((A \Rightarrow (B \Rightarrow C)) \Rightarrow ((A \Rightarrow B) \Rightarrow (A \Rightarrow C)))$,
axiom A2

$B_6 = ((A \Rightarrow B) \Rightarrow (A \Rightarrow C))$,
B_5 and B_4 and MP

$B_7 = (A \Rightarrow C)$.
B_2 and B_6 and MP

Exercise 2

Show, by constructing a formal proof that $A \vdash_{H_1} (A \Rightarrow A)$.

Solution
The required formal proof is a sequence

$$B_1, B_2, B_3 \tag{5}$$

such that

$B_1 = A$,
hypothesis

$B_2 = (A \Rightarrow (A \Rightarrow A))$,
axiom A1 for $B = A$,

$B_3 = (A \Rightarrow A)$
$B_1,\ B_2$ and MP.

We can further simplify the task of constructing formal proofs in H_1 by the use of the following Deduction Theorem.

In mathematical arguments, one often assumes a statement A on the assumption (hypothesis) of some other statement B and then concludes that we have proved the implication "if A, then B". This reasoning is justified by the following theorem, called a Deduction Theorem. It was first formulated and proved for a certain Hilbert proof system S for the classical propositional logic by Herbrand in 1930 in a form stated below.

Theorem 2 (Deduction Theorem for S) *(Herbrand,1930)*

For any formulas A, B of the language of S,

$$\textit{if} \quad A \vdash_S B, \quad \textit{then} \ \vdash_S (A \Rightarrow B).$$

We are going to prove now that for our system H_1 is strong enough to prove the Herbrand Deduction Theorem for it. In fact we formulate and prove a more general version of the Theorem 2.

To formulate it we introduce the following notation. We write $\Gamma, A \vdash_S B$ for $\Gamma \cup \{A\} \vdash_S B$, and in general we write $\Gamma, A_1, A_2, ..., A_n \vdash_S B$ for $\Gamma \cup \{A_1, A_2, ..., A_n\} \vdash_S B$. We are now going to prove the following.

Theorem 3 (Deduction Theorem for H_1)

For any subset Γ of the set of formulas $\mathcal{F}$ of H_1 and for any formulas $A, B \in \mathcal{F}$,

$$\Gamma,\ A \vdash_{H_1} B \ \ if\ and\ only\ if \ \ \Gamma \vdash_{H_1} (A \Rightarrow B).$$

In particular,

$$A \vdash_{H_1} B \ \ if\ and\ only\ if \ \ \vdash_{H_1} (A \Rightarrow B).$$

Proof

We use we use the symbol $\vdash$ instead of $\vdash_{H_1}$. for simplicity.

Part 1
We first prove the "if" part:

$$\text{If} \quad \Gamma,\ A \vdash B \qquad \text{then} \qquad \Gamma \vdash (A \Rightarrow B).$$

Assume that $\Gamma,\ A\ \vdash B$, i.e. that we have a formal proof

$$B_1, B_2, ..., B_n \tag{6}$$

of B from the set of formulas $\Gamma \cup \{A\}$. In order to prove that $\Gamma \vdash (A \Rightarrow B)$ we will prove the following a little bit stronger statement **S**.

$$\mathbf{S}: \quad \Gamma \vdash (A \Rightarrow B_i) \text{ for all } B_i \ (1 \leq i \leq n) \text{ in the proof (6) of } B.$$

Hence, in particular case, when $i = n$, we will obtain that also

$$\Gamma \vdash (A \Rightarrow B).$$

The proof of **S** is conducted by induction on i ($1 \leq i \leq n$).

Base Step $i = 1$.
When $i = 1$, it means that the formal proof (6) contains only one element B_1. By the definition of the formal proof from $\Gamma \cup \{A\}$, we have that $B_1 \in LA$, or $B_1 \in \Gamma$, or $B_1 = A$, i.e.

$$B_1 \in \{A1, A2\} \cup \Gamma \cup \{A\}.$$

Here we have two cases.

Case 1. $B_1 \in \{A1, A2\} \cup \Gamma$.
Observe that $(B_1 \Rightarrow (A \Rightarrow B_1))$ is the axiom $A1$ and by assumption $B_1 \in$

$\{A1, A2\} \cup \Gamma$, hence we get the required proof of $(A \Rightarrow B_1)$ from Γ by the following application of the Modus Ponens rule

$$(MP)\ \frac{B_1\ ;\ (B_1 \Rightarrow (A \Rightarrow B_1))}{(A \Rightarrow B_1)}.$$

Case 2. $B_1 = A$.
When $B_1 = A$, then to prove $\Gamma \vdash (A \Rightarrow B)$ means to prove $\Gamma \vdash (A \Rightarrow A)$. This holds by the monotonicity of the consequence in H_1 (Fact 4), and the fact that we have proved (Fact 1) that $\vdash(A \Rightarrow A)$. The above cases conclude the proof of the Base case $i = 1$.

Inductive step
Assume that $\Gamma \vdash(A \Rightarrow B_k)$ for all $k < i$, we will show that using this fact we can conclude that also $\Gamma \vdash(A \Rightarrow B_i)$.

Consider a formula B_i in the sequence 6. By the definition, $B_i \in \{A1, A2\} \cup \Gamma \cup \{A\}$ or B_i follows by MP from certain B_j, B_m such that $j < m < i$. We have to consider again two cases.

Case 1. $B_i \in \{A1, A2\} \cup \Gamma \cup \{A\}$.
The proof of $(A \Rightarrow B_i)$ from Γ in this case is obtained from the proof of the Base Step for $i = 1$ by replacement B_1 by B_i and will be omitted here as a straightforward repetition.

Case 2. B_i is a conclusion of MP.
If B_i is a conclusion of MP, then we must have two formulas B_j, B_m in the sequence 6 such that $j < i, m < i,\ j \neq m$ and

$$(MP)\ \frac{B_j\ ;\ B_m}{B_i}.$$

By the inductive assumption, the formulas B_j, B_m are such that

$$\Gamma \vdash (A \Rightarrow B_j) \tag{7}$$

and

$$\Gamma \vdash (A \Rightarrow B_m). \tag{8}$$

Moreover, by the definition of the Modus Ponens rule, the formula B_m has to have a form $(B_j \Rightarrow B_i)$, i.e. $B_m = (B_j \Rightarrow B_i)$, and the the inductive assumption (8) can be re-written as follows.

$$\Gamma \vdash (A \Rightarrow (B_j \Rightarrow B_i)), \quad for\ j < i. \tag{9}$$

Observe now that the formula

$$((A \Rightarrow (B_j \Rightarrow B_i)) \Rightarrow ((A \Rightarrow B_j) \Rightarrow (A \Rightarrow B_i)))$$

is a substitution of the axiom schema A2 and hence has a proof in our system. By the monotonicity of the consequence (3), it also has a proof from the set Γ, i.e.

$$\Gamma \vdash ((A \Rightarrow (B_j \Rightarrow B_i)) \Rightarrow ((A \Rightarrow B_j) \Rightarrow (A \Rightarrow B_i))). \tag{10}$$

Applying the rule MP to formulas (10) and (9,) i.e. performing the following

$$(MP)\ \frac{(A \Rightarrow (B_j \Rightarrow B_i))\ ;\ ((A \Rightarrow (B_j \Rightarrow B_i)) \Rightarrow ((A \Rightarrow B_j) \Rightarrow (A \Rightarrow B_i)))}{((A \Rightarrow B_j) \Rightarrow (A \Rightarrow B_i))}$$

we get that also

$$\Gamma \vdash ((A \Rightarrow B_j) \Rightarrow (A \Rightarrow B_i)). \tag{11}$$

Applying again the rule MP to formulas 7 and 11, i.e. performing the following

$$(MP)\ \frac{(A \Rightarrow B_j)\ ;\ ((A \Rightarrow B_j) \Rightarrow (A \Rightarrow B_i))}{(A \Rightarrow B_i)}$$

we get that

$$\Gamma \vdash (A \Rightarrow B_i)$$

what ends the proof of the Inductive Step. By the mathematical induction principle, we hence have proved that $\Gamma \vdash (A \Rightarrow B_j)$ for all i such that $1 \leq i \leq n$. In particular it is true for $i = n$, what means for $B_n = B$ and we have proved that

$$\Gamma \vdash (A \Rightarrow B).$$

This ends the proof of the **Part 1**.

Part 2

The proof of the inverse implication

$$\text{if}\ \ \Gamma \vdash (A \Rightarrow B)\ \ \text{then}\ \ \Gamma, A \vdash B$$

is straightforward. Assume that $\Gamma \vdash (A \Rightarrow B)$, hence by the monotonicity of the consequence (3) we have also that $\Gamma, A \vdash (A \Rightarrow B)$. Obviously, $\Gamma, A \vdash A$. Applying Modus Ponens to the above, we get the proof of B from $\{\Gamma, A\}$ i.e. we have proved that $\Gamma,\ A \vdash B$. That ends the proof of the deduction theorem for any set $\Gamma \subseteq \mathcal{F}$ and any formulas $A, B \in \mathcal{F}$. The particular case is obtained from the above by assuming that the set Γ is empty. This ends the proof of the Deduction Theorem for H_1.

The proof of the following useful lemma provides a good example of multiple applications of the Deduction Theorem 3.

Lemma 1

For any $A, B, C \in \mathcal{F}$,

(a) $(A \Rightarrow B), (B \Rightarrow C) \vdash_{H_1} (A \Rightarrow C)$,

(b) $(A \Rightarrow (B \Rightarrow C)) \vdash_{H_1} (B \Rightarrow (A \Rightarrow C))$.

Proof of **(a)**.
Deduction theorem says:
$(A \Rightarrow B), (B \Rightarrow C) \vdash_{H_1} (A \Rightarrow C)$ if and only if $(A \Rightarrow B), (B \Rightarrow C), A \vdash_{H_1} C$.

We construct a formal proof

$$B_1,\ B_2,\ B_3,\ B_4,\ B_5$$

of $(A \Rightarrow B), (B \Rightarrow C), A \vdash_{H_1} C$ as follows.

$B_1 = (A \Rightarrow B)$,
hypothesis

$B_2 = (B \Rightarrow C)$,
hypothesis

$B_3 = A$,
hypothesis

$B_4 = B$,
$B_1,\ B_3$ and MP

$B_5 = C$.
$B_2,\ B_4$ and MP

Thus $(A \Rightarrow B), (B \Rightarrow C) \vdash_{H_1} (A \Rightarrow C)$ by Deduction Theorem.

Proof of **(b)**.

By Deduction Theorem,

$(A \Rightarrow (B \Rightarrow C)) \vdash_{H_1} (B \Rightarrow (A \Rightarrow C))$ if and only if $(A \Rightarrow (B \Rightarrow C)), B \vdash_{H_1} (A \Rightarrow C)$.

We construct a formal proof

$$B_1,\ B_2,\ B_3,\ B_4,\ B_5,\ B_6,\ B_7$$

of $(A \Rightarrow (B \Rightarrow C)), B \vdash_{H_1} (A \Rightarrow C)$. as follows.

$B_1 = (A \Rightarrow (B \Rightarrow C))$,
hypothesis

$B_2 = B$,
hypothesis

$B_3 = ((B \Rightarrow (A \Rightarrow B))$,
$A1$ for $A = B,\ B = A$

$B_4 = (A \Rightarrow B)$,
$B_2,\ B_3$ and MP

$B_5 = ((A \Rightarrow (B \Rightarrow C)) \Rightarrow ((A \Rightarrow B) \Rightarrow (A \Rightarrow C)))$,
$axiom A2$

$B_6 = ((A \Rightarrow B) \Rightarrow (A \Rightarrow C))$,
$B_1,\ B_5$ and MP

$B_7 = (A \Rightarrow C)$.

Thus $(A \Rightarrow (B \Rightarrow C)) \vdash_{H_1} (B \Rightarrow (A \Rightarrow C))$ by Deduction Theorem.

Hilbert System H_2

The proof system H_1 is sound and strong enough to admit the Deduction Theorem, but is *not complete*as proved in Fact 3. We define now a proof system H_2 that is complete with respect to classical semantics. The proof of Completeness Theorem for H_2 is to be presented in the next section.

H_2 is defined as follows.

$$H_2 = (\ \mathcal{L}_{\{\neg,\ \Rightarrow\}},\ \mathcal{F},\ A1, A2, A3,\ \ MP\ \frac{A\ ;\ (A \Rightarrow B)}{B}), \tag{12}$$

where for any formulas $A, B, C \in \mathcal{F}$ of $\mathcal{L}_{\{\neg,\ \Rightarrow\}}$ we define

A1 $(A \Rightarrow (B \Rightarrow A))$,

A2 $((A \Rightarrow (B \Rightarrow C)) \Rightarrow ((A \Rightarrow B) \Rightarrow (A \Rightarrow C)))$,

A3 $((\neg B \Rightarrow \neg A) \Rightarrow ((\neg B \Rightarrow A) \Rightarrow B)))$,

Observation 1 *Here are some simple, straightforward facts about the proof system H_2.*

1. The language of H_2 is obtained from the language of H_1 by adding the connective $\neg$ to it.

2. H_2 is obtained from H_1 by adding axiom to it the axiom A_3 that characterizes negation.

3. The use of axioms $A1, A2$ in the proof of Deduction Theorem 3 for H_1 is independent of the negation connective $\neg$ added to the language of H_1.

4. The proof of Deduction Theorem 3 for the system H_1 can be repeated as it is for the system H_2.

Directly from the above Observation 1 we get the following.

Theorem 4 (Deduction Theorem for H_2)

For any subset Γ of the set of formulas $\mathcal{F}$ of H_2 and for any formulas $A, B \in \mathcal{F}$,

$$\Gamma,\ A \vdash_{H_2} B \ \ if\ and\ only\ if\ \ \Gamma \vdash_{H_2} (A \Rightarrow B).$$

In particular,

$$A \vdash_{H_2} B \ \ if\ and\ only\ if\ \ \vdash_{H_2} (A \Rightarrow B).$$

Observe that for the same reason the Lemma 1 holds also for H_2. It is a very i useful lemma for creating proofs in H_2 so we re-state it for it here.

Lemma 2

For any $A, B, C \in \mathcal{F}$,

(a) $(A \Rightarrow B), (B \Rightarrow C) \vdash_{H_2} (A \Rightarrow C)$,

(b) $(A \Rightarrow (B \Rightarrow C)) \vdash_{H_2} (B \Rightarrow (A \Rightarrow C))$.

We know that the axioms $A1, A2$ are tautologies and the Modus Ponens rule is sound. We get by simple verification that $\models A3$, hence the proof system H_2 is *sound*, and the following holds.

Theorem 5 (Soundness Theorem for H_2)

For every formula $A \in \mathcal{F}$, if $\vdash_{H_2} A$, then $\models A$.

The soundness theorem proves that the system "produces" only tautologies. We show, in the next chapter, that our proof system H_2 "produces" not only tautologies, but that all tautologies are provable in it. This is called a *completeness theorem for classical logic*.

Theorem 6 (Completeness Theorem for H_2)

For every $A \in \mathcal{F}$,

$$\vdash_{H_2} A \ \ if\ and\ only\ if\ \ \models A.$$

The proof of completeness theorem (for a given semantics) is always a main point in any logic creation. There are many ways (techniques) to prove it, depending on the proof system, and on the semantics we define for it.

We present in the next sections two proofs of the completeness theorem for our system H_2. The proofs use very different techniques, hence the reason of presenting both of them. Both proofs relay heavily on some of the formulas proved in the next section 1.1 and stated in Lemma 3.

1.1 Formal Proofs

We present here some examples of formal proofs in H_2. There are two reasons for presenting them. First reason is that all formulas we prove here to be provable play a crucial role in the proof of Completeness Theorem for H_2, or are needed to find formal proofs of those needed. The second reason is that they provide a "training" ground for a reader to learn how to develop formal proofs. For this second reason we write some proofs in a full detail and we leave some others for the reader to complete in a way explained in the following example.

We write, were needed $\vdash$ instead of $\vdash_{H_2}$.

Example 1

We prove that

$$\vdash_{H_2} (\neg\neg B \Rightarrow B) \tag{13}$$

by constructing its formal proof $B_1, \ldots, B_5, B_6$ *as follows.*

$B_1 = ((\neg B \Rightarrow \neg\neg B) \Rightarrow ((\neg B \Rightarrow \neg B) \Rightarrow B))$,

$B_2 = ((\neg B \Rightarrow \neg B) \Rightarrow ((\neg B \Rightarrow \neg\neg B) \Rightarrow B))$,

$B_3 = \neg B \Rightarrow \neg B)$,

$B_4 = ((\neg B \Rightarrow \neg\neg B) \Rightarrow B)$,

$B_5 = \neg\neg B \Rightarrow (\neg B \Rightarrow \neg\neg B))$,

$B_6 = (\neg\neg B \Rightarrow B)$.

Exercise 3

Complete the proof $B_1, \ldots, B_5, B_6$ *of (83) by providing comments how each step of the proof was obtained.*

Solution
The proof of (83) with comments complementing it is as follows.

$B_1 = ((\neg B \Rightarrow \neg\neg B) \Rightarrow ((\neg B \Rightarrow \neg B) \Rightarrow B))$,
axiom $A3$ for $A = \neg B,\ B = B$

$B_2 = ((\neg B \Rightarrow \neg B) \Rightarrow ((\neg B \Rightarrow \neg\neg B) \Rightarrow B))$,
B_1 and Lemma 2 **b** for $A = (\neg B \Rightarrow \neg\neg B),\ B = (\neg B \Rightarrow \neg B),\ C = B$.
Lemma 2 application is: $((\neg B \Rightarrow \neg\neg B) \Rightarrow ((\neg B \Rightarrow \neg B) \Rightarrow B)) \vdash ((\neg B \Rightarrow \neg B) \Rightarrow ((\neg B \Rightarrow \neg\neg B) \Rightarrow B))$

$B_3 = (\neg B \Rightarrow \neg B)$,
Fact 1 for $A = \neg B$

$B_4 = ((\neg B \Rightarrow \neg\neg B) \Rightarrow B)$,
$B_2,\ B_3$ and MP

$B_5 = (\neg\neg B \Rightarrow (\neg B \Rightarrow \neg\neg B))$,
axiom $A1$ for $A = \neg\neg B,\ B = \neg B$

$B_6 = (\neg\neg B \Rightarrow B)$
$B_4,\ B_5$ and Lemma 2 **a** for $A = \neg\neg B, B = (\neg B \Rightarrow \neg\neg B), C = B$.
Lemma 2 application is:
$(\neg\neg B \Rightarrow (\neg B \Rightarrow \neg\neg B)),\ ((\neg B \Rightarrow \neg\neg B) \Rightarrow B) \vdash (\neg\neg B \Rightarrow B)$

Remark 1

Observe that in In step $B_2,\ B_3,\ B_5,\ B_6$ of the proof $B_1,\ \ldots,\ B_5,\ B_6$ we call previously proved results and use their results as a part of our proof. We can insert previously constructed formal proofs of the results we call upon into our formal proof.

For example we adopt previously constructed proof (2) of $(A \Rightarrow A)$ in H_1 to the proof of $(\neg B \Rightarrow \neg B)$ in H_2 by replacing A by $\neg B$ and we insert the proof of $(\neg B \Rightarrow \neg B)$ after B_2.

The "old" step B_3 becomes now B_7, the "old" step B_4 becomes now B_8, etc..... Such "completed" original proof $B_1,\ \ldots,\ B_5,\ B_6$ is now $B_1,\ \ldots,\ B_9,\ B_{10}$ looks now as follows.

$B_1 =\ ((\neg B \Rightarrow \neg\neg B) \Rightarrow ((\neg B \Rightarrow \neg B) \Rightarrow B))$, (original B_1),
axiom $A3$ for $A = \neg B, B = B$

$B_2 =\ ((\neg B \Rightarrow \neg B) \Rightarrow ((\neg B \Rightarrow \neg\neg B) \Rightarrow B))$, (original B_2)
B_1 and Lemma 2 **b** for $A = (\neg B \Rightarrow \neg\neg B), B = (\neg B \Rightarrow \neg B), C = B$,

$B_3 = ((\neg B \Rightarrow ((\neg B \Rightarrow \neg B) \Rightarrow \neg B)) \Rightarrow ((\neg B \Rightarrow (\neg B \Rightarrow \neg B)) \Rightarrow (\neg B \Rightarrow \neg B)))$, (new proof of B_3 inserted)
axiom A2 for $A = \neg B$, $B = (\neg B \Rightarrow \neg B)$, and $C = \neg B$

$B_4 = (\neg B \Rightarrow ((\neg B \Rightarrow \neg B) \Rightarrow \neg B))$,
axiom A1 for $A = \neg B$, $B = (\neg B \Rightarrow \neg B)$

$B_5 = ((\neg B \Rightarrow (\neg B \Rightarrow \neg B)) \Rightarrow (\neg B \Rightarrow \neg B)))$,
MP application to B_4 and B_3

$B_6 = (\neg B \Rightarrow (\neg B \Rightarrow \neg B))$, (end of proof inserted)
axiom A1 for $A = \neg B$, $B = \neg B$

$B_7 = (\neg B \Rightarrow \neg B)$ ("old" B_3),
MP application to B_5 and B_4

$B_8 = ((\neg B \Rightarrow \neg\neg B) \Rightarrow B)$, ("old" B_4) ("old" B_4)
B_2, B_3 and MP

$B_9 =$ ("old B_5) $(\neg\neg B \Rightarrow (\neg B \Rightarrow \neg\neg B))$, ("old" B_5) Axiom $A1$ for $A = \neg\neg B, B = \neg B$

$B_{10} = (\neg\neg B \Rightarrow B)$. ("old B_6)
B_8, B_9 and Lemma 2 **a** for $A = \neg\neg B, B = (\neg B \Rightarrow \neg\neg B), C = B$

We repeat our procedure by replacing the step B_2 by its formal proof as defined in the proof of the Lemma 1 **b**, and continue the process for all other steps which involved application of Lemma 2 until we get a full **formal proof from the axioms** of H_2 only.

Usually we don't need to do it, but it is important to remember that it always can be done, if we wished to take time and space to do so.

Example 2

We prove that

$$\vdash_{H_2} (B \Rightarrow \neg\neg B) \tag{14}$$

by constructing its formal proof $B_1, \ldots, B_5$ *as follows.*

$B_1 = ((\neg\neg\neg B \Rightarrow \neg B) \Rightarrow ((\neg\neg\neg B \Rightarrow B) \Rightarrow \neg\neg B))$,

$B_2 = (\neg\neg\neg B \Rightarrow \neg B)$,

$B_3 = ((\neg\neg\neg B \Rightarrow B) \Rightarrow \neg\neg B)$,

$B_4 = (B \Rightarrow (\neg\neg\neg B \Rightarrow B))$,

$B_5 = (B \Rightarrow \neg\neg B)$.

Exercise 4

Complete the proof $B_1, \ \ldots, \ B_5$ *of (85) by providing comments how each step of the proof was obtained.*

Solution
The proof of (85) with comments complementing it is as follows.

$B_1 = ((\neg\neg\neg B \Rightarrow \neg B) \Rightarrow ((\neg\neg\neg B \Rightarrow B) \Rightarrow \neg\neg B))$,
axiom $A3$ for $A = B, \ B = \neg\neg B$

$B_2 = (\neg\neg\neg B \Rightarrow \neg B)$,
Example 10 for $B = \neg B$

$B_3 = ((\neg\neg\neg B \Rightarrow B) \Rightarrow \neg\neg B)$,
B_1, B_2 and MP, i.e.

$$\frac{(\neg\neg\neg B \Rightarrow \neg B); ((\neg\neg\neg B \Rightarrow \neg B) \Rightarrow ((\neg\neg\neg B \Rightarrow B) \Rightarrow \neg\neg B))}{((\neg\neg\neg B \Rightarrow B) \Rightarrow \neg\neg B)}$$

$B_4 = (B \Rightarrow (\neg\neg\neg B \Rightarrow B))$,
axiom $A1$ for $A = B, B = \neg\neg\neg B$

$B_5 = (B \Rightarrow \neg\neg B)$,
B_3, B_4 and Lemma 2**a** for $A = B, B = (\neg\neg\neg B \Rightarrow B), C = \neg\neg B$, i.e.

$$(B \Rightarrow (\neg\neg\neg B \Rightarrow B)), ((\neg\neg\neg B \Rightarrow B) \Rightarrow \neg\neg B) \vdash_{H_2} (B \Rightarrow \neg\neg B)$$

Example 3

We prove that

$$\vdash_{H_2} (\neg A \Rightarrow (A \Rightarrow B)) \tag{15}$$

by constructing its formal proof $B_1, \ \ldots, \ B_{12}$ *as follows.*

$B_1 = \neg A$,

$B_2 = A$,

$B_3 = (A \Rightarrow (\neg B \Rightarrow A))$,

$B_4 = (\neg A \Rightarrow (\neg B \Rightarrow \neg A))$,

$B_5 = (\neg B \Rightarrow A)$,

$B_6 = (\neg B \Rightarrow \neg A)$,

$B_7 = ((\neg B \Rightarrow \neg A) \Rightarrow ((\neg B \Rightarrow A) \Rightarrow B))$,

$B_8 = ((\neg B \Rightarrow A) \Rightarrow B)$,

$B_9 = \ B,$

$B_{10} = \ \neg A, A \vdash B,$

$B_{11} = \ \neg A \vdash (A \Rightarrow B),$

$B_{12} = \ (\neg A \Rightarrow (A \Rightarrow B)).$

Example 4

We prove that

$$\vdash_{H_2} ((\neg B \Rightarrow \neg A) \Rightarrow (A \Rightarrow B)) \tag{16}$$

by constructing its formal proof $B_1, \ \ldots, \ B_7$ *as follows. Here are consecutive steps*

$B_1 = \ (\neg B \Rightarrow \neg A),$

$B_2 = \ ((\neg B \Rightarrow \neg A) \Rightarrow ((\neg B \Rightarrow A) \Rightarrow B)),$

$B_3 = \ (A \Rightarrow (\neg B \Rightarrow A)),$

$B_4 = \ ((\neg B \Rightarrow A) \Rightarrow B),$

$B_5 = \ (A \Rightarrow B),$

$B_6 = \ (\neg B \Rightarrow \neg A) \vdash (A \Rightarrow B),$

$B_7 = \ ((\neg B \Rightarrow \neg A) \Rightarrow (A \Rightarrow B)).$

Example 5

We prove that

$$\vdash_{H_2} ((A \Rightarrow B) \Rightarrow (\neg B \Rightarrow \neg A)) \tag{17}$$

by constructing its formal proof $B_1, \ \ldots, \ B_9$ *as follows. Here are consecutive steps*

$B_1 = \ (A \Rightarrow B),$

$B_2 = \ (\neg\neg A \Rightarrow A),$

$B_3 = \ (\neg\neg A \Rightarrow B),$

$B_4 = \ (B \Rightarrow \neg\neg B),$

$B_5 = \ (\neg\neg A \Rightarrow \neg\neg B),$

$B_6 = \ ((\neg\neg A \Rightarrow \neg\neg B) \Rightarrow (\neg B \Rightarrow \neg A)),$

$B_7 = \ (\neg B \Rightarrow \neg A),$

$B_8 = \ (A \Rightarrow B) \vdash (\neg B \Rightarrow \neg A),$

$B_9 = \ ((A \Rightarrow B) \Rightarrow (\neg B \Rightarrow \neg A)).$

Exercise 5

Complete the proof $B_1, \ \ldots, \ B_9$ *of (17) by providing comments how each step of the proof was obtained.*

Solution
The proof of (17) with comments complementing it is as follows.

$B_1 = \ (A \Rightarrow B),$
hypothesis

$B_2 = \ (\neg\neg A \Rightarrow A),$
Example 10 for $B = A$

$B_3 = \ (\neg\neg A \Rightarrow B),$
Lemma 2 **a** for $A = \neg\neg A, \ B = A, \ C = B$

$B_4 = \ (B \Rightarrow \neg\neg B),$
Example 11

$B_5 = \ (\neg\neg A \Rightarrow \neg\neg B),$
Lemma 2 **a** for $A = \neg\neg A, \ B = B, \ C = \neg\neg B$

$B_6 = \ ((\neg\neg A \Rightarrow \neg\neg B) \Rightarrow (\neg B \Rightarrow \neg A)),$
Example 4 for $B = \neg A, \ A = \neg B$

$B_7 = \ (\neg B \Rightarrow \neg A),$
$B_5, \ B_6$ and MP

$B_8 = \ (A \Rightarrow B) \vdash (\neg B \Rightarrow \neg A),$
$B_1 - B_7$

$B_9 = \ ((A \Rightarrow B) \Rightarrow (\neg B \Rightarrow \neg A)).$
Deduction Theorem 31

Example 6

We prove that

$$\vdash_{H_2} ((A \Rightarrow B) \Rightarrow ((\neg A \Rightarrow B) \Rightarrow B)) \tag{18}$$

by constructing its formal proof $B_1, \ \ldots, \ B_{12}$ *as follows. Here are consecutive steps.*

$B_1 = \ (A \Rightarrow B),$

$B_2 = \ (\neg A \Rightarrow B),$

$B_3 = ((A \Rightarrow B) \Rightarrow (\neg B \Rightarrow \neg A))$,

$B_4 = (\neg B \Rightarrow \neg A)$,

$B_5 = ((\neg A \Rightarrow B) \Rightarrow (\neg B \Rightarrow \neg\neg A))$,

$B_6 = (\neg B \Rightarrow \neg\neg A)$,,

$B_7 = ((\neg B \Rightarrow \neg\neg A) \Rightarrow ((\neg B \Rightarrow \neg A) \Rightarrow B)))$,

$B_8 = ((\neg B \Rightarrow \neg A) \Rightarrow B)$,

$B_9 = B$,

$B_{10} = (A \Rightarrow B), (\neg A \Rightarrow B) \vdash B$,

$B_{11} = (A \Rightarrow B) \vdash ((\neg A \Rightarrow B) \Rightarrow B)$,

$B_{12} = ((A \Rightarrow B) \Rightarrow ((\neg A \Rightarrow B) \Rightarrow B))$.

Exercise 6

Complete the proof $B_1, \ldots, B_{12}$ *of (18) by providing comments how each step of the proof was obtained.*

Solution
The proof of (18) with comments complementing it is as follows.

$B_1 = (A \Rightarrow B)$,
hypothesis

$B_2 = (\neg A \Rightarrow B)$,
hypothesis

$B_3 = ((A \Rightarrow B) \Rightarrow (\neg B \Rightarrow \neg A))$,
Example 5

$B_4 = (\neg B \Rightarrow \neg A)$,
$B_1,\ B_3$ and MP

$B_5 = ((\neg A \Rightarrow B) \Rightarrow (\neg B \Rightarrow \neg\neg A))$
Example 5 for $A = \neg A,\ B = B$

$B_6 = (\neg B \Rightarrow \neg\neg A)$,
$B_2,\ B_5$ and MP

$B_7 = ((\neg B \Rightarrow \neg\neg A) \Rightarrow ((\neg B \Rightarrow \neg A) \Rightarrow B)))$,
axiom A3 for $B = B,\ A = \neg A$

$B_8 = ((\neg B \Rightarrow \neg A) \Rightarrow B)$,
$B_6,\ B_7$ and MP

$B_9 = \ B,$
$B_4, \ B_8$ and MP

$B_{10} = \ (A \Rightarrow B), (\neg A \Rightarrow B) \vdash_{H_2} B,$
$B_1 - B_9$

$B_{11} = \ (A \Rightarrow B) \vdash ((\neg A \Rightarrow B) \Rightarrow B),$
Deduction Theorem 31

$B_{12} = \ ((A \Rightarrow B) \Rightarrow ((\neg A \Rightarrow B) \Rightarrow B)).$ Deduction Theorem 31

Example 7

We prove that

$$\vdash_{H_2} ((\neg A \Rightarrow A) \Rightarrow A) \tag{19}$$

by constructing its formal proof $B_1, \ B_2, \ B_3$ *as follows. Here are consecutive steps.*

$B_1 = \ ((\neg A \Rightarrow \neg A) \Rightarrow ((\neg A \Rightarrow A) \Rightarrow A))),$

$B_2 = \ (\neg A \Rightarrow \neg A),$

$B_3 = \ ((\neg A \Rightarrow A) \Rightarrow A)).$

Exercise 7

Complete the proof $B_1, \ B_2, \ B_3$ *of (19) by providing comments how each step of the proof was obtained.*

Solution
The proof of (19) with comments complementing it is as follows.

$B_1 = \ ((\neg A \Rightarrow \neg A) \Rightarrow ((\neg A \Rightarrow A) \Rightarrow A))),$
axiom A3 for $B = A$

$B_2 = \ (\neg A \Rightarrow \neg A),$
Lemma 2 for $A = \neg A$

$B_3 = \ ((\neg A \Rightarrow A) \Rightarrow A)).$
$B_1, \ B_2$ and MP

The above Examples 10 - 7 and the Fact 1 provide a proof of the following lemma.

Lemma 3

For any formulas $A, B, Cin\mathcal{F}$ *of the system* H_2 *the following holds.*

1. $\vdash_{H_2} (A \Rightarrow A)$;
2. $\vdash_{H_2} (\neg\neg B \Rightarrow B)$;
3. $\vdash_{H_2} (B \Rightarrow \neg\neg B)$;
4. $\vdash_{H_2} (\neg A \Rightarrow (A \Rightarrow B))$;
5. $\vdash_{H_2} ((\neg B \Rightarrow \neg A) \Rightarrow (A \Rightarrow B))$;
6. $\vdash_{H_2} ((A \Rightarrow B) \Rightarrow (\neg B \Rightarrow \neg A))$;
7. $\vdash_{H_2} (A \Rightarrow (\neg B \Rightarrow (\neg(A \Rightarrow B)))$;
8. $\vdash_{H_2} ((A \Rightarrow B) \Rightarrow ((\neg A \Rightarrow B) \Rightarrow B))$;
9. $\vdash_{H_2} ((\neg A \Rightarrow A) \Rightarrow A$.

The set of provable formulas from the above Lemma 3 includes a set of provable formulas needed, with H_2 axioms to execute two proofs of the Completeness Theorem 6 for H_2. These two proofs represent two very different methods of proving Completeness Theorem.

2 Completeness Theorem: Proof One

The Proof One of the Completeness Theorem 6 for H_2. presented here is similar in its structure to the proof of the Deduction Theorem 3 and is due to Kalmar, 1935. It is, as Deduction Theorem was, a *constructive proof.* It means it defines a method how one can use the assumption that a formula A is a tautology in order to *construct* its formal proof. We hence call it a *proof construction method.* It relies heavily on the Deduction Theorem.

Proof One, the first proof of the Completeness Theorem 6 presented here is very elegant and simple, but is applicable only to the classical propositional logic. Methods it uses are specific to a propositional language $\mathcal{L}_{\{\neg,\, \Rightarrow\}}$ and the proof system H_2. Nevertheless, it can be adopted and extended to other classical propositional languages $\mathcal{L}_{\{\neg,\, \cup,\, \Rightarrow\}}$, $\mathcal{L}_{\{\neg,\, \cap,\, \cup, \Rightarrow\}}$, $\mathcal{L}_{\{\neg,\, \cap,\, \cup, \Rightarrow, \Leftrightarrow\}}$, and proof systems based on them. We do so by adding appropriate new logical axioms to the logical axioms of H_2 (section 2.1). Such obtained proof systems are called *extentions* of the system H_2. It means that one can think about the system H_2, i.e. an axiomatization given by set $\{A1, A2, A3\}$ of logical axioms of H_2, and its language $\mathcal{L}_{\{\neg,\, \Rightarrow\}}$ as in a sense, a "minimal one" for classical propositional logic and its languages that contain implication.

Proof One, i.e. the methods of carrying it, can't be extended to the classical predicate logic, not to mention variety of non-classical logics. Hence we present, in the next section3 another, more general proof, called Proof Two, that can.

We have already proved the Soundness Theorem 5 for H_2, so in order to prove the Completeness Theorem 6 we need to prove only the completeness part of the completeness theorem, i.e. the following implication.

For any formula A of H_2,

$$\text{if} \quad \models A, \quad \text{then} \quad \vdash_S A. \tag{20}$$

In order to prove (20), i.e. to prove that any tautology has a formal proof in H_2, we need first to present one definition and prove one lemma stated below. We write $\vdash\ A$ instead of $\vdash_{H_2} A$, as the system H_2 is fixed.

Definition 4

Let A be a formula and $b_1, b_2, ..., b_n$ be all propositional variables that occur in A. Let v be variable assignment $v : VAR \longrightarrow \{T, F\}$. We define, for $A, b_1, b_2, ..., b_n$ and v a corresponding formulas A', $B_1, B_2, ..., B_n$ as follows:

$$A' = \begin{cases} A & \text{if } v^*(A) = T \\ \neg A & \text{if } v^*(A) = F \end{cases}$$

$$B_i = \begin{cases} b_i & \text{if } v(b_i) = T \\ \neg b_i & \text{if } v(b_i) = F \end{cases}$$

for $i = 1, 2, ..., n$.

Example 8

Let A be a formula

$$(a \Rightarrow \neg b) \tag{21}$$

and let v be such that

$$v(a) = T, \quad v(b) = F. \tag{22}$$

In this case $b_1 = a$, $b_2 = b$, and $v^*(A) = v^*(a \Rightarrow \neg b) = v(a) \Rightarrow \neg v(b) = T \Rightarrow \neg F = T$. The corresponding A', B_1, B_2 are: $A' = A$ (as $v^*(A) = T$), $B_1 = a$ (as $v(a) = T$), $B_2 = \neg b$ (as $v(b) = F$).

Here is a simple exercise.

Exercise 8

Let A be a formula $((\neg a \Rightarrow \neg b) \Rightarrow c)$ and let v be such that $v(a) = T$, $v(b) = F, v(c) = F$.

Evaluate A', $B_1, ...B_n$ as defined by the definition 4.

Solution
In this case $n = 3$ and $b_1 = a,\ b_2 = b, b_3 = c$, and $v^*(A) = v^*((\neg a \Rightarrow \neg b) \Rightarrow c)$ $=((\neg v(a) \Rightarrow \neg v(b)) \Rightarrow v(c)) = ((\neg T \Rightarrow \neg F) \Rightarrow F) = (T \Rightarrow F) = F$. The corresponding A', B_1, B_2, B_2 are: $A' = \neg A = \neg((\neg a \Rightarrow \neg b) \Rightarrow c)$ (as $v^*(A) = F$), $B_1 = a$ (as $v(a) = T$), $B_2 = \neg b$ (as $v(b) = F$). $B_3 = \neg c$ (as $v(c) = F$).

The lemma stated below describes a method of transforming a semantic notion of a tautology into a syntactic notion of provability. It defines, for any formula A and a variable assignment v a corresponding deducibility relation $\vdash$.

Lemma 4 (Main Lemma)

For any formula A and a variable assignment v, if $A^{'}$, B_1 , B_2, ..., B_n are corresponding formulas defined by 4, then

$$B_1, B_2, ..., B_n \ \vdash \ A'. \tag{23}$$

Example 9 *Let A, v be as defined by (21) and (22), respectively.*

1. The Lemma 4 asserts that $a, \neg b \ \vdash \ (a \Rightarrow \neg b)$.

Let A, v be as defined in Exercise 8.

2. The Lemma 4 asserts that $a, \neg b, \neg c \ \vdash \ \neg((\neg a \Rightarrow \neg b) \Rightarrow c)$.

Proof of the Main Lemma

The Main Lemma 4 states: for any formula A and a variable assignment v, if $A^{'}$, B_1 , B_2, ..., B_n are corresponding formulas defined by Definition 4, then

$$B_1, B_2, ..., B_n \ \vdash \ A'.$$

Proof We carry the proof by mathematical induction on the degree of A i.e. a number n of logical connectives in A.

Case: $n = 0$

In the case that $n = 0$ A is atomic and so consists of a single propositional variable, say a. We have to cases to consider, $v^*(A) = T$ or $v^*(A) = F$. Clearly, if $v^*(A) = T$ then we $A' = A = a$, $B_1 = a$, and $a \vdash a$ holds by the Deduction Theorem and **??**. I.e. $\vdash (a \Rightarrow a)$ holds by **??**). Applying the the Deduction Theorem we get $a \vdash a$.

If $v^*(A) = F$ then we $A' = \neg A = \neg a$, $B_1 = \neg a$, and $\vdash (\neg a \Rightarrow \neg a)$ holds by Lemma 3. Applying the the Deduction Theorem we get $\neg a \vdash \neg a$. So the lemma holds for the case $n = 0$.

Now assume that the lemma holds for any A with $j < n$ logical connectives (any A of the degree $j < n$). The goal is to prove that it holds for A with the

degree n.
There are several sub-cases to deal with.

Case: A **is** $\neg A_1$

If A is of the form $\neg A_1$ then A_1 has less then n connectives and by the inductive assumption we have the formulas $A_1^{'}$, B_1 , B_2, ..., B_n corresponding to the A_1 and the propositional variables $b_1, b_2, ..., b_n$ in A_1, as defined by the definition 4, such that

$$B_1, B_2, ..., B_n \vdash A_1^{'}. \tag{24}$$

Observe, that the formulas A and $\neg A_1$ have the same propositional variables, so the corresponding formulas B_1 , B_2, ..., B_n are the same for both of them. We are going to show that the inductive assumption (24) allows us to prove that the lemma holds for A, ie. that

$$B_1, B_2, ..., B_n \vdash A^{'}.$$

There two cases to consider.

Case: $v^*(A_1) = T$

If $v^*(A_1) = T$ then by definition 4 $A_1^{'} = A_1$ and by the inductive assumption (24)

$$B_1, B_2, ..., B_n \vdash A_1. \tag{25}$$

In this case $v^*(A) = v^*(\neg A_1) = \neg v^*(T) = F$ and so $A^{'} = \neg A = \neg\neg A_1$. We have by Lemma 3, $\vdash (A_1 \Rightarrow \neg\neg A_1)$, By the monotonicity, $B_1, B_2, ..., B_n \vdash (A_1 \Rightarrow \neg\neg A_1)$. By inductive assumption (25) and Modus Ponens we have that also $B_1, B_2, ..., B_n \vdash \neg\neg A_1$, that is $B_1, B_2, ..., B_n \vdash \neg A$, that is $B_1, B_2, ..., B_n \vdash A^{'}$.

Case: $v^*(A_1) = F$

If $v^*(A_1) = F$ then $A_1^{'} = \neg A_1$ and $v^*(A) = T$ so $A^{'} = A$. Therefore the inductive assumption (24) $B_1, B_2, ..., B_n \vdash \neg A_1$, that is $B_1, B_2, ..., B_n \vdash A^{'}$.

Case: A **is** $(A_1 \Rightarrow A_2)$

If A is of the form $(A_1 \Rightarrow A_2)$ then A_1 and A_2 have less than n connectives.

$A = A(b_1, ... b_n)$ so there are some subsequences $c_1, ..., c_k$ and $d_1, ...d_m$, for $k, m \leq n$, of the sequence $b_1, ..., b_n$ such that $A_1 = A_1(c_1, ..., c_k)$ and $A_2 = A(d_1, ...d_m)$. A_1 and A_2 have less than n connectives and so by the inductive assumption we have appropriate formulas $C_1, ..., C_k$ and $D_1, ...D_m$ such that $C_1, C_2, \ldots, C_k \vdash {A_1}^{'}$ and $D_1, D_2, \ldots, D_m \vdash {A_2}^{'}$. The formulas $C_1, C_2, ..., C_k$ and $D_1, D_2, ..., D_m$ are *subsequences* of formulas $B_1, B_2, ..., B_n$ corresponding to the propositional variables in A. Hence by monotonicity we have also that have $B_1, B_2, ..., B_n \vdash {A_1}^{'}$ and

$B_1, B_2, ..., B_n \vdash A_2{}^{'}$, where $B_1, B_2, ..., B_n$ are formulas corresponding to the propositional variables in A.

Now we have the following sub-cases to consider.

Case: $v^*(A_1) = v^*(A_2) = T$

If $v^*(A_1) = T$ then $A_1{}^{'}$ is A_1 and if $v^*(A_2) = T$ then $A_2{}^{'}$ is A_2. We also have $v^*(A_1 \Rightarrow A_2) = T$ and so $A^{'}$ is $(A_1 \Rightarrow A_2)$. By the above and the inductive assumption, therefore, $B_1, B_2, ..., B_n \vdash A_2$ and by Lemma 3, i.e. $\vdash (A_2 \Rightarrow (A_1 \Rightarrow A_2))$. By monotonicity and Modus Ponens, that $B_1, B_2, ..., B_n \vdash (A_1 \Rightarrow A_2)$, that is $B_1, B_2, ..., B_n \vdash A^{'}$.

Case: $v^*(A_1) = T, v^*(A_2) = F$

If $v^*(A_1) = T$ then $A_1{}^{'}$ is A_1 and if $v^*(A_2) = F$ then $A_2{}^{'}$ is $\neg A_2$. Also we have in this case $v^*(A_1 \Rightarrow A_2) = F$ and so $A^{'}$ is $\neg(A_1 \Rightarrow A_2)$. By the above and the inductive assumption, therefore, $B_1, B_2, ..., B_n \vdash \neg A_2$. By Lemma 3, $\vdash (A_1 \Rightarrow (\neg A_2 \Rightarrow \neg(A_1 \Rightarrow A_2)))$. By monotonicity and Modus Ponens twice, we have that $B_1, B_2, ..., B_n \vdash \neg(A_1 \Rightarrow A_2)$, that is $B_1, B_2, ..., B_n \vdash A^{'}$.

Case: $v^*(A_1) = F$

If $v^*(A_1) = F$ then $A_1{}^{'}$ is $\neg A_1$ and, whatever value v gives A_2, we have $v^*(A_1 \Rightarrow A_2) = T$ and so $A^{'}$ is $(A_1 \Rightarrow A_2)$. Therefore, $B_1, B_2, ..., B_n \vdash \neg A_1$ and by Lemma 3, $\vdash (\neg A_1 \Rightarrow (A_1 \Rightarrow A_2))$. By monotonicity and Modus Ponens we get that $B_1, B_2, ..., B_n \vdash (A_1 \Rightarrow A_2)$, that is $B_1, B_2, ..., B_n \vdash A^{'}$.

With that we have covered all cases and, by induction on n, the proof of the lemma is complete.

Proof of the Completeness Theorem

Now we use the Main Lemma 4 to prove the completeness part of the Completeness Theorem 6, i.e. to prove the implication (20):

$$\text{For any formula } A \in \mathcal{F}, \text{ if } \models A, \text{then } \vdash A.$$

Proof Assume that $\models A$. Let $b_1, b_2, ..., b_n$ be all propositional variables that occur in A, i.e. $A = A(b_1, b_2, ..., b_n)$.

Let $v : VAR \to \{T, F\}$ be any variable assignment, and

$$v_A : \quad \{b_1, b_2, ..., b_n\} \quad \to \{T, F\} \tag{26}$$

its restriction to the formula A, i.e. $v_A = v|\{b_1, b_2, ..., b_n\}$. Let

$$V_A = \{v_A : \quad v_A : \{b_1, b_2, ..., b_n\} \quad \to \{T, F\}\} \tag{27}$$

By the Main Lemma 4 and the assumption that $\models A$ any $v \in V_A$ defines formulas B_1 , B_2, ..., B_n such that

$$B_1, B_2, ..., B_n \vdash A. \tag{28}$$

The proof is based on a method of using all $v \in V_A$ to define a process of elimination of all hypothesis $B_1, B_2, ..., B_n$ in (28) to finally construct the proof of A in H_2 i.e. to prove that $\vdash A$.

Step 1: elimination of B_n.

Observe that by definition 4, each B_i is b_i or $\neg b_i$ depending on the choice of $v \in V_A$. In particular $B_n = b_n$ or $B_n = \neg b_n$. We choose two truth assignments $v_1 \neq v_2 \in V_A$ such that

$$v_1|\{b_1, ..., b_{n-1}\} = v_2|\{b_1, ..., b_{n-1}\} \tag{29}$$

and $v_1(b_n) = T$ and $v_2(b_n) = F$.

Case 1: $v_1(b_n) = T$, by definition 4 $B_n = b_n$. By the property (29), assumption that $\models A$, and the Main Lemma 4 applied to v_1

$$B_1, B_2, ..., B_{n-1}, b_n \vdash A.$$

By Deduction Theorem 3 we have that

$$B_1, B_2, ..., B_{n-1} \vdash (b_n \Rightarrow A). \tag{30}$$

Case 2: $v_2(b_n) = F$ hence by definition 4 $B_n = \neg b_n$. By the property (29), assumption that $\models A$, and the Main Lemma 4 applied to v_2

$$B_1, B_2, ...B_{n-1}, \neg b_n \vdash A.$$

By the Deduction Theorem 3 we have that

$$B_1, B_2, ..., B_{n-1} \vdash (\neg b_n \Rightarrow A). \tag{31}$$

By Lemma 3 of the formula $\vdash ((A \Rightarrow B) \Rightarrow ((\neg A \Rightarrow B) \Rightarrow B))$. Hence for for $A = b_n, B = A$ we have that

$$\vdash ((b_n \Rightarrow A) \Rightarrow ((\neg b_n \Rightarrow A) \Rightarrow A)).$$

By monotonicity we have that

$$B_1, B_2, ..., B_{n-1} \vdash ((b_n \Rightarrow A) \Rightarrow ((\neg b_n \Rightarrow A) \Rightarrow A)). \tag{32}$$

Applying Modus Ponens twice to the above property (32) and properties (30), (31) we get that

$$B_1, B_2, ..., B_{n-1} \vdash A. \tag{33}$$

We have eliminated B_n.

Step 2: elimination of B_{n-1} from (33). We repeat the Step 1.

As before we have 2 cases to consider: $B_{n-1} = b_{n-1}$ or $B_{n-1} = \neg b_{n-1}$. We choose two truth assignments $w_1 \neq w_2 \in V_A$ such that

$$w_1|\{b_1, ..., b_{n-2}\} = w_2|\{b_1, ..., b_{n-2}\} = v_1|\{b_1, ..., b_{n-2}\} = v_2|\{b_1, ..., b_{n-2}\} \tag{34}$$

and $w_1(b_{n-1}) = T$ and $w_2(b_{n-1}) = F$.

As before we apply Main Lemma, Deduction Theorem, monotonicity, proper substitutions of the formula $((A \Rightarrow B) \Rightarrow ((\neg A \Rightarrow B) \Rightarrow B))$, and Modus Ponens twice and eliminate B_{n-1} just as we eliminated B_n.

After n steps, we finally obtain that

$$\vdash \ A.$$

This **ends** the proof of Completeness Theorem.

Observe that our proof of the fact that $\vdash A$ is a constructive one. Moreover, we have used in it only Main Lemma 4 and Deduction Theorem 3, and both of them have fully constructive proofs. So we can always reconstruct all steps in proofs which use the Main Lemma 4and Deduction Theorem 3 back to the original axioms of H_2. The same applies to the proofs that use the formulas proved in H_2 that are stated in Lemma 3.

It means that for any $A \in \mathcal{F}$, such that $\models \ A$, the set V_A of all v restricted to A provides us a method of a construction of the formal proof of A in H_2 from its axioms $A1$, $A2$, $A3$ only. .

2.1 Examples

Example 10

As an example of how the **Proof One** *of the Completeness Theorem works, we consider a following tautology*

$$\models (a \Rightarrow (\neg a \Rightarrow b))$$

and show how to construct its proof, i.e. to show that

$$\vdash (a \Rightarrow (\neg a \Rightarrow b)).$$

We apply the Main Lemma 4 to all possible variable assignments $v \in V_A$. We have 4 variable assignments to consider.

Case 1: $v(a) = T, v(b) = T$.

In this case $B_1 = a, B_2 = b$ and, as in all cases, $A' = A$ and by the Main Lemma 4

$$a, b \vdash (a \Rightarrow (\neg a \Rightarrow b)).$$

Case 2: $v(a) = T, v(b) = F$.

In this case $B_1 = a, B_2 = \neg b$ and by the Main Lemma 4

$$a, \neg b \vdash (a \Rightarrow (\neg a \Rightarrow b)).$$

Case 3: $v(a) = F, v(b) = T$.

In this case $B_1 = \neg a, B_2 = b$ and by the Main Lemma 4

$$\neg a, b \vdash (a \Rightarrow (\neg a \Rightarrow b)).$$

Case 4: $v(a) = F, v(b) = F$.

In this case $B_1 = \neg a, B_2 = \neg b$ and by the lemma 4

$$\neg a, \neg b \vdash (a \Rightarrow (\neg a \Rightarrow b)).$$

Applying the Deduction Theorem 3 to the cases above we have that

D1 (Cases 1 and 2)

$$a \vdash (b \Rightarrow (a \Rightarrow (\neg a \Rightarrow b))),$$

$$a \vdash (\neg b \Rightarrow (a \Rightarrow (\neg a \Rightarrow b))),$$

D2 (Cases 2 and 3)

$$\neg a \vdash (b \Rightarrow (a \Rightarrow (\neg a \Rightarrow b))),$$

$$\neg a \vdash (\neg b \Rightarrow (a \Rightarrow (\neg a \Rightarrow b))).$$

By the monotonicity and the proper substitution of formula

$$((A \Rightarrow B) \Rightarrow ((\neg A \Rightarrow B) \Rightarrow B))$$

the provable by Lemma 3, we have that

$$a \vdash ((b \Rightarrow (a \Rightarrow (\neg a \Rightarrow b))) \Rightarrow ((\neg b \Rightarrow (a \Rightarrow (\neg a \Rightarrow b))) \Rightarrow (a \Rightarrow (\neg a \Rightarrow b))),$$

$$\neg a \vdash ((b \Rightarrow (a \Rightarrow (\neg a \Rightarrow b))) \Rightarrow ((\neg b \Rightarrow (a \Rightarrow (\neg a \Rightarrow b))) \Rightarrow (a \Rightarrow (\neg a \Rightarrow b))).$$

Applying Modus Ponens twice to **D1**, **D2** and these above, respectively, gives us

$$a \vdash (a \Rightarrow (\neg a \Rightarrow b)) \; and$$

$$\neg a \vdash (a \Rightarrow (\neg a \Rightarrow b)).$$

Applying the Deduction Theorem 3 to the above we obtain

D3 $\vdash\ (a \Rightarrow (a \Rightarrow (\neg a \Rightarrow b)))$,

D4 $\vdash\ (\neg a \Rightarrow (a \Rightarrow (\neg a \Rightarrow b)))$.

We form now an appropriate form of the formula

$$((A \Rightarrow B) \Rightarrow ((\neg A \Rightarrow B) \Rightarrow B)), \tag{35}$$

provable by the Lemma 3. The appropriate form is

$$\Rightarrow ((\neg a \Rightarrow (a \Rightarrow (\neg a \Rightarrow b))) \Rightarrow (a \Rightarrow (\neg a \Rightarrow b)))). \tag{36}$$

We apply Modus Ponens twice to **D3** and **D4** and (58) and get finally the proof of $(a \Rightarrow (\neg a \Rightarrow b))$, i.e. we have proved that

$$\vdash\ (a \Rightarrow (\neg a \Rightarrow b)).$$

Example 11

The Proof One of Completeness Theorem defines a method of efficiently combining $v \in V_A$ as defined in (27), while constructing the proof of A. Let's consider the following tautology $A = A(a, b, c)$

$$((\neg a \Rightarrow b) \Rightarrow (\neg(\neg a \Rightarrow b) \Rightarrow c).$$

We present bellow all steps of Proof One as applied to A.

By the Main Lemma 4 and the assumption that $\models A(a, b, c)$ any $v \in V_A$ defines formulas B_a , B_b, B_c such that

$$B_a, B_b, B_c\ \vdash\ A. \tag{37}$$

The proof is based on a method of using all $v \in V_A$ (there is 16 of them) to define a process of elimination of all hypothesis B_a, B_b, B_c in (37) to construct the proof of A in H_2 i.e. to prove that $\vdash\ A$.

Step 1: elimination of B_c.

Observe that by definition 4, B_c is c or $\neg c$ depending on the choice of $v \in V_A$. We choose two truth assignments $v_1 \neq v_2 \in V_A$ such that

$$v_1|\{a, b\} = v_2|\{a, b\} \tag{38}$$

and $v_1(c) = T$ and $v_2(c) = F$.

Case 1: $v_1(c) = T$, by definition 4 $B_c = c$. By the property (38), assumption that $\models A$, and the Main Lemma 4 applied to v_1

$$B_a, B_b, c\ \vdash\ A.$$

By Deduction Theorem 3 we have that

$$B_a, B_b \vdash (c \Rightarrow A). \tag{39}$$

Case 2: $v_2(c) = F$ hence by definition 4 $B_c = \neg c$. By the property (38), assumption that $\models A$, and the Main Lemma 4 applied to v_2

$$B_a, B_b, \neg c \vdash A.$$

By the Deduction Theorem 3 we have that

$$B_a, B_b \vdash (\neg c \Rightarrow A). \tag{40}$$

By Lemma 3, i.e. provability of the formula (35) for $A = c, B = A$ we have that

$$\vdash ((c \Rightarrow A) \Rightarrow ((\neg c \Rightarrow A) \Rightarrow A)).$$

By monotonicity we have that

$$B_a, B_b \vdash ((c \Rightarrow A) \Rightarrow ((\neg c \Rightarrow A) \Rightarrow A)). \tag{41}$$

Applying Modus Ponens twice to the above property (41) and properties (39), (40) we get that

$$B_a, B_b \vdash A. \tag{42}$$

and hence we have eliminated B_c.

Step 2: elimination of B_b from (42). We repeat the Step 1.
As before we have 2 cases to consider: $B_b = b$ or $B_b = \neg b$. We choose from V_A two truth assignments $w_1 \neq w_2 \in V_A$ such that

$$w_1|\{a\} = w_2|\{a\} = v_1|\{a\} = v_2|\{a\} \tag{43}$$

and $w_1(b) = T$ and $w_2(b) = F$.

Case 1: $w_1(b) = T$, by definition 4 $B_b = b$. By the property (43), assumption that $\models A$, and the Main Lemma 4 applied to w_1

$$B_a, b \vdash A.$$

By Deduction Theorem 3 we have that

$$B_a \vdash (b \Rightarrow A). \tag{44}$$

Case 2: $w_2(c) = F$ hence by definition 4 $B_b = \neg b$. By the property (3), assumption that $\models A$, and the Main Lemma 4 applied to w_2

$$B_a, \neg b \ \vdash \ A.$$

By the Deduction Theorem 3 we have that

$$B_a \ \vdash \ (\neg b \Rightarrow A). \tag{45}$$

By Lemma 3, i.e. provability of the formula (35) for $A = b, B = A$ we have that

$$\vdash ((b \Rightarrow A) \Rightarrow ((\neg b \Rightarrow A) \Rightarrow A)).$$

By monotonicity we have that

$$B_a \ \vdash \ ((b \Rightarrow A) \Rightarrow ((\neg b \Rightarrow A) \Rightarrow A)). \tag{46}$$

Applying Modus Ponens twice to the above property (46) and properties (44), (45) we get that

$$B_a \ \vdash \ A. \tag{47}$$

and hence we have eliminated B_b.

Step 3: elimination of B_a from (47). We repeat the Step 2.
As before we have 2 cases to consider: $B_a = a$ or $B_a = \neg a$. We choose from V_A two truth assignments $g_1 \neq g_2 \in V_A$ such that

$$g_1(a) = T \text{ and } g_2(a) = F. \tag{48}$$

Case 1: $g_1(a) = T$, by definition 4 $B_a = a$. By the property (48), assumption that $\models A$, and the Main Lemma 4 applied to g_1

$$a \ \vdash \ A.$$

By Deduction Theorem 3 we have that

$$\vdash \ (a \Rightarrow A). \tag{49}$$

Case 2: $g_2(a) = F$ hence by definition 4 $B_a = \neg a$. By the property (48), assumption that $\models A$, and the Main Lemma 4 applied to g_2

$$\neg a \ \vdash \ A.$$

By the Deduction Theorem 3 we have that

$$\vdash \ (\neg a \Rightarrow A). \tag{50}$$

By Lemma 3, i.e. provability of the formula (35) for $A = a, B = A$ we have that

$$\vdash ((a \Rightarrow A) \Rightarrow ((\neg a \Rightarrow A) \Rightarrow A)). \tag{51}$$

Applying Modus Ponens twice to the above property (51) and properties (49), (50) we get that

$$\vdash \ A. \tag{52}$$

and hence we have eliminated B_a, B_b and B_c and constructed the proof of A.

2.2 Homework Problems

For the formulas A_i and corresponding truth assignments v find formulas $B_1, ..B_k, A_i^{'}$ as described by the Main Lemma 4, i.e. such that

$$B_1, ...B_k \vdash A_i^{'}.$$

1. $A_1 = ((\neg(b \Rightarrow a) \Rightarrow \neg a) \Rightarrow ((\neg b \Rightarrow (a \Rightarrow \neg c)) \Rightarrow c))$
   $v(a) = T, v(b) = F, v(c) = T.$

2. $A_2 = ((a \Rightarrow (c \Rightarrow (\neg b \Rightarrow c))) \Rightarrow ((\neg d \Rightarrow (a \Rightarrow (\neg a \Rightarrow b))) \Rightarrow (a \Rightarrow (\neg a \Rightarrow b))))$
   $v(a) = F, v(b) = F, v(c) = T, v(d) = F$

3. $A_3 = (\neg b \Rightarrow (c \Rightarrow (\neg a \Rightarrow b)))$
   $v(a) = F, v(b) = F, v(c) = T$

4. $A_4 = (\neg a_1 \Rightarrow (a_2 \Rightarrow (\neg a_3 \Rightarrow a_1)))$
   $v(a_1) = F, v(a_2) = F, v(a_3) = T$

4. $A_5 = ((b \Rightarrow (a_1 \Rightarrow (\neg c \Rightarrow b))) \Rightarrow ((\neg b \Rightarrow (a_2 \Rightarrow (\neg a_1 \Rightarrow b))) \Rightarrow (c \Rightarrow (\neg a \Rightarrow b)))$
   $v(a) = F, v(b) = T, v(c) = F, v(a_1) = T, v(a_2) = F$

   For any of the formulas listed below construct their formal proofs, as described in the Proof One of the Completeness Theorem. Follow example 10, or example 11.

5. $A_1 = (\neg\neg b \Rightarrow b)$

6. $A_2 = ((a \Rightarrow b) \Rightarrow (\neg b \Rightarrow \neg a))$

7. $A_3 = (\neg(a \Rightarrow b) \Rightarrow \neg(\neg b \Rightarrow \neg a))$
8. $A_4 = (\neg(\neg(a \Rightarrow \neg b) \Rightarrow \neg c) \Rightarrow \neg(b \Rightarrow \neg c))$
9. $A_5 = ((a \Rightarrow (b \Rightarrow \neg a)) \Rightarrow (\neg(b \Rightarrow \neg a) \Rightarrow \neg a))$.

   Read carefully proofs of Deduction Theorem 3 and Completeness Theorem 6 and write careful answers to the following problems.

10. List all formulas that have to be provable in H_2, axioms included, that are are needed for the proof of Deduction Theorem 3. Write down each part of the proof that uses them.
11. List all formulas that have to be provable in H_2, axioms included, that are needed for the proof of Main Lemma 4.
12. List all formulas that have to be provable in H_2, axioms included, that are included in the Proof of Completeness Theorem part of the Proof One.
13. List all formulas that have to be provable in H_2, axioms included, that are needed to carry all of the Proof One of Completeness Theorem **??**.
14. We proved the Completeness Theorem for the proof system H_2 based on the language $\mathcal{L}_{\{\neg,\Rightarrow\}}$. Extend the H_2 proof system to a proof system S_1 based on a language $\mathcal{L}_{\{\neg,\Rightarrow,\cup\}}$ by adding new logical axioms, as we did in a case of H_1 and H_2 systems. The added logical axioms must be such that they allow to adopt the Proof One to S_1, i.e. such that it is a complete proof system with respect to classical semantics.
15. Repeat the same for the language $\mathcal{L}_{\{\neg,\Rightarrow,\cap\}}$. Call resulting proof system S_2.
16. Repeat the same for the language $\mathcal{L}_{\{\neg,\Rightarrow,\cap,\cup\}}$, i.e. extends systems S_1 or S_2 to a complete proof system S_3 based on the language $\mathcal{L}_{\{\neg,\Rightarrow,\cap,\cup\}}$.
17. Prove Completeness Theorem for the system S_3 from the previous problem.

3 Completeness Theorem: Proof Two

The Proof Two is much more complicated then the Proof One. Its strength and importance lies in a fact that the methods it uses can be applied in an extended version to the proof of completeness for classical predicate logic and even many of non-classical propositional and predicate logics. The main point of the proof is a presentation of a general, non- constructive method for proving existence of a counter-model for any non-provable A. The generality of the method makes it possible to adopt it for other cases of predicate and some non-classical logics. We call it a a *counter-model existence method.*

We prove now the completeness part of the Completeness Theorem 6 for H_2 by proving that the opposite implication:

$$\text{if} \quad \not\vdash A, \text{ then } \quad \not\models A \tag{53}$$

to the implication (20):

$$\text{if} \quad \models A, \text{then} \quad \vdash A$$

holds hat for all $A \in \mathcal{F}$.

We will show now how one can define of a counter-model for A from the fact that A is not provable. This means that we deduce that a formula A is not a tautology from the fact that it does not have a proof. We hence call it a *a counter-model existence method.*

The definition of the counter-model for any non-provable A is much more general (and less constructive) then in the case of the Proof One in section 2. It can be generalized to the case of predicate logic, and many of non-classical logics; propositional and predicate. It is hence a much more general method then the first one and this is the reason we present it here.

We remind that $\not\models \ A$ means that there is a truth assignment $v : VAR \longrightarrow \{T, F\}$, such that $v^*(A) \neq T$, i.e. in classical semantics, such that that $v^*(A) = F$. Such v is called a counter-model for A, hence the proof provides a counter-model construction method.

Since we assume in (53) that A does not have a proof in H_2 ($\not\vdash \ A$) the method uses this information in order to show that A is not a tautology, i.e. to define v such that $v^*(A) = F$. We also have to prove that all steps in that method are correct. This is done in the following steps.

Step 1: Definition of Δ^*

We use the information $\not\vdash \ A$ to define a special set $\Delta^* \subseteq \mathcal{F}$, such that $\neg A \in \Delta^*$.

Step 2: Counter - model definition

We define the truth assignment $v : VAR \longrightarrow \{T, F\}$ as follows:

$$v(a) = \begin{cases} T & \text{if } \Delta^* \vdash a \\ F & \text{if } \Delta^* \vdash \neg a. \end{cases}$$

Step 3: Prove that v is a counter-model

We first prove a more general property, namely we prove that the set Δ^* and v defined in the steps 1 and 2, respectively, are such that for every formula $B \in \mathcal{F}$,

$$v^*(B) = \begin{cases} T & \text{if } \Delta^* \vdash B \\ F & \text{if } \Delta^* \vdash \neg B. \end{cases}$$

Then we use the **Step 1** (definition of Δ^*) to prove that $v^*(A) = F$.

The definition and the properties of the set Δ^*, and hence the **Step 1**, are the most essential for the proof. The other steps have mainly technical character. The main notions involved in the **Step 1** (definition of Δ^*) are: *consistent set, complete set* and a *consistent complete extension of a set.* We are going now to introduce them and to prove some essential facts about them.

Consistent and Inconsistent Sets

There exist two definitions of consistency; semantical and syntactical. The **semantical** one uses definition the notion of a model and says, in plain English:

a set of formulas is consistent if it has a model.

The **syntactical** one uses the notion of provability and says:

a set of formulas is consistent if one can't prove a contradiction from it.

In our Proof Two of the Completeness Theorem we use assumption that a given formula A does not have a proof to deduce that A is not a tautology. We hence use the following syntactical definition of consistency.

Consistent set

We say that a set $\Delta \subseteq \mathcal{F}$ of formulas is **consistent** if and only if **there is no** a formula $A \in \mathcal{F}$ such that

$$\Delta \vdash A \quad \text{and} \quad \Delta \vdash \neg A. \tag{54}$$

Inconsistent set

A set $\Delta \subseteq \mathcal{F}$ is inconsistent if and only if **there is** a formula $A \in \mathcal{F}$ such that $\Delta \vdash A$ and $\Delta \vdash \neg A$.

The notion of consistency, as defined above, is characterized by the following lemma.

Lemma 5 (Consistency Condition)

For every set $\Delta \subseteq \mathcal{F}$ of formulas, the following conditions are equivalent:

(i) Δ *is* consistent*,*

(ii) *there is a formula $A \in \mathcal{F}$ such that $\Delta \nvdash A$.*

Proof The implications: **(i)** implies **(ii)** and vice-versa are proved by showing the corresponding opposite implications. I.e. to establish the equivalence of **(i)** and **(ii)**, we first show that **not (ii)** implies **not (i)**, and then that **not (i)** implies **not (ii)**.

Case 1

Assume that **not (ii)**. It means that for all formulas $A \in \mathcal{F}$ we have that $\Delta \vdash A$. In particular it is true for a certain $A = B$ and $A = \neg B$ and hence proves that Δ is inconsistent, i.e. **not (i)** holds.

Case 2

Assume that **not (i)**, i.e that Δ is inconsistent. Then there is a formula A such that $\Delta \vdash A$ and $\Delta \vdash \neg A$. Let B be any formula. Since $(\neg A \Rightarrow (A \Rightarrow B))$ is provable in H_2 by Lemma 3, hence by applying Modus Ponens twice and by detaching from it $\neg A$ first, and A next, we obtain a formal proof of B from the set Δ, so that $\Delta \vdash B$ for any formula B. Thus **not (ii)**.

The inconsistent sets are hence characterized by the following fact.

Lemma 6 (Inconsistency Condition)

For every set $\Delta \subseteq \mathcal{F}$ of formulas, the following conditions are equivalent:

(i) Δ *is* inconsistent*,*

(ii) *for all formulas $A \in \mathcal{F}$, $\Delta \vdash A$.*

We remind here the property of the finiteness of the consequence operation.

Lemma 7

For every set Δ of formulas and for every formula $A \in \mathcal{F}$, $\Delta \vdash A$ if and only if there is a finite subset $\Delta_0 \subseteq \Delta$ such that $\Delta_0 \vdash A$.

Proof
If $\Delta_0 \vdash A$ for a certain $\Delta_0 \subseteq \Delta$, then by the monotonicity of the consequence, also $\Delta \vdash A$. Assume now that $\Delta \vdash A$ and let $A_1, A_2, ..., A_n$ be a formal

proof of A from Δ. Let $\Delta_0 = \{A_1, A_2, ..., A_n\} \cap \Delta$. Obviously, Δ_0 is finite and $A_1, A_2, ..., A_n$ is a formal proof of A from Δ_0.

The following theorem is a simply corollary of the above Lemma 7.

Theorem 7 (Finite Inconsistency)

If a set Δ is inconsistent, then there is a finite subset $\Delta_0 \subseteq \Delta$ which is inconsistent. It follows therefore from that if every finite subset of a set Δ is consistent, then the set Δ is also consistent.

Proof
If Δ is inconsistent, then for some formula A, $\Delta \vdash A$ and $\Delta \vdash \neg A$. By above Lemma 7, there are finite subsets Δ_1 and Δ_2 of Δ such that $\Delta_1 \vdash A$ and $\Delta_2 \vdash \neg A$. By monotonicity, the union $\Delta_1 \cup \Delta_2$ is a finite subset of Δ, such that $\Delta_1 \cup \Delta_2 \vdash A$ and $\Delta_1 \cup \Delta_2 \vdash \neg A$. Hence $\Delta_1 \cup \Delta_2$ is a finite inconsistent subset of Δ. The second implication is the opposite to the one just proved and hence also holds.

The following lemma links the notion of non-provability and consistency. It will be used as an important step in our proof of the Completeness Theorem.

Lemma 8

For any formula $A \in \mathcal{F}$, if $\nvdash A$, then the set $\{\neg A\}$ is consistent.

Proof
If $\{\neg A\}$ is inconsistent, then by the Inconsistency Condition 6 we have $\{\neg A\} \vdash A$. This and the Deduction Theorem 3 imply $\vdash (\neg A \Rightarrow A)$. Applying the Modus Ponens rule to $\vdash (\neg A \Rightarrow A)$ a formula $((\neg A \Rightarrow A) \Rightarrow A)$, provable by LemmaH2lemma, we get that $\vdash A$, contrary to the assumption of the lemma.

Complete and Incomplete Sets

Another important notion, is that of *a complete* set of formulas. Complete sets, as defined here are sometimes called *maximal*, but we use the first name for them. They are defined as follows.

Complete set

A set Δ of formulas is called complete if **for every** formula $A \in \mathcal{F}$,

$$\Delta \vdash A \ \ or \ \ \Delta \vdash \neg A. \tag{55}$$

The complete sets are characterized by the following fact.

Lemma 9 (Complete set condition)

For every set $\Delta \subseteq \mathcal{F}$ of formulas, the following conditions are equivalent:

(i) Δ *is* complete,

(ii) *for every formula $A \in \mathcal{F}$, if $\Delta \nvdash A$, then the set $\Delta \cup \{A\}$ is inconsistent.*

Proof
We consider two cases. We show that **(i)** implies **(ii)** and vice-versa, that **(ii)** also implies **(i)**.

Case 1

Assume that **(i)** and that for every formula $A \in \mathcal{F}$, $\Delta \nvdash A$, we have to show that in this case $\Delta \cup \{A\}$ is inconsistent. But if $\Delta \nvdash A$, then from the definition of complete set and assumption that Δ is complete set, we get that $\Delta \vdash \neg A$. By the monotonicity of the consequence we have that $\Delta \cup \{A\} \vdash \neg A$ as well. Since, by formula **??** we have $\vdash (A \Rightarrow A)$, by monotonicity $\Delta \vdash (A \Rightarrow A)$ and by Deduction Theorem $\Delta \cup \{A\} \vdash A$. This proves that $\Delta \cup \{A\}$ is inconsistent. Hence **(ii)** holds.

Case 2

Assume that **(ii)**. Let A be any formula. We want to show that the condition: $\Delta \vdash A$ or $\Delta \vdash \neg A$ is satisfied. If $\Delta \vdash \neg A$, then the condition is obviously satisfied.

If, on other hand, $\Delta \nvdash \neg A$, then we are going to show now that it must be , under the assumption of **(ii)**, that $\Delta \vdash A$, i.e. that **(i)** holds.

Assume that $\Delta \nvdash \neg A$, then by **(ii)**, the set $\Delta \cup \{\neg A\}$ is inconsistent. It means, by the Consistency Condition 5, that $\Delta \cup \{\neg A\} \vdash A$. By the Deduction Theorem 3, this implies that $\Delta \vdash (\neg A \Rightarrow A)$. Since $((\neg A \Rightarrow A) \Rightarrow A)$ is provable in H_2 (Lemma 3), by monotonicity $\Delta \vdash ((\neg A \Rightarrow A) \Rightarrow A)$. Detaching $(\neg A \Rightarrow A)$, we obtain that $\Delta \vdash A$, what ends the proof that **(i)** holds.

Incomplete set

A set Δ of formulas is called incomplete if it is not complete, i.e. if **there exists** a formula $A \in \mathcal{F}$ such that

$$\Delta \nvdash A \textit{ and } \Delta \nvdash \neg A. \tag{56}$$

We get as a direct consequence of the lemma 9 the following characterization of incomplete sets.

Lemma 10 (Incomplete Set Condition)

For every set $\Delta \subseteq \mathcal{F}$ of formulas, the following conditions are equivalent:

(i) Δ *is* incomplete*,*

(ii) *there is formula $A \in \mathcal{F}$ such that $\Delta \not\vdash A$ and the set $\Delta \cup \{A\}$ is consistent.*

Main Lemma: Complete Consistent Extension

Now we are going to prove a lemma that is essential to the construction of the special set Δ^* mentioned in the **Step 1** of the proof of the Completeness Theorem, and hence to the proof of the theorem itself. Let's first introduce one more notion.

Extensions

A set Δ^* of formulas is called an **extension** of a set Δ of formulas if the following condition holds.

$$\{A \in \mathcal{F} : \ \Delta \vdash A\} \subseteq \{A \in \mathcal{F} : \ \Delta^* \vdash A\}. \tag{57}$$

In this case we say also that Δ **extends** to the set of formulas Δ^*.

The Main Lemma states as follows.

Lemma 11 (Complete Consistent Extension)

Every consistent set Δ of formulas can be extended to a complete consistent set Δ^* of formulas.

Proof

Assume that the lemma does not hold, i.e. that there is a consistent set Δ, such that all its consistent extensions are not complete. In particular, as Δ is an consistent extension of itself, we have that Δ is not complete.

The proof consists of a construction of a particular set Δ^* and proving that it forms a complete consistent extension of Δ, contrary to the assumption that all its consistent extensions are not complete.

Construction of Δ^*.

As we know, the set $\mathcal{F}$ of all formulas is enumerable. They can hence be put in an infinite sequence

$$A_1, A_2, ..., A_n, ... \tag{58}$$

such that every formula of $\mathcal{F}$ occurs in that sequence exactly once.

We define now, as the first step in the construction of Δ^*, an infinite sequence $\{\Delta_n\}_{n \in N}$ of consistent subsets of formulas together with a sequence $\{B\}_{n \in N}$ of formulas as follows.

Initial Step

In this step we define the sets Δ_1, Δ_2 and the formula B_1. We prove that Δ_1 and Δ_2 are consistent, incomplete extensions of Δ.

We take, as the first set, the set Δ, i.e. we define

$$\Delta_1 = \Delta. \tag{59}$$

Since, by assumption, the set Δ, and hence also Δ_1 is not complete, it follows from the Incomplete Set Condition Lemma 10, that there is a formula $B \in \mathcal{F}$ such that $\Delta_1 \nvdash B$, then and the set $\Delta_1 \cup \{B\}$ is consistent.

Let

$$B_1$$

be the first formula with this property in the sequence (58) of all formulas; we then define

$$\Delta_2 = \Delta_1 \cup \{B_1\}. \tag{60}$$

The set Δ_2 is consistent and $\Delta_1 = \Delta \subseteq \Delta_2$, so by the monotonicity, Δ_2 is a consistent extension of Δ. Hence Δ_2 cannot be complete.

Inductive Step

Suppose that we have defined a sequence

$$\Delta_1, \Delta_2, ..., \Delta_n$$

of incomplete, consistent extensions of Δ, and a sequence

$$B_1, B_2, ...B_{n-1}$$

of formulas, for $n \geq 2$.

Since Δ_n is incomplete, it follows from the Incomplete Set Condition Lemma 10, that there is a formula $B \in \mathcal{F}$ such that $\Delta_n \nvdash B$ and the set $\Delta_n \cup \{B\}$ is consistent.

Let

$$B_n$$

be the first formula with this property in the sequence (58) of all formulas.

We then define

$$\Delta_{n+1} = \Delta_n \cup \{B_n\}. \tag{61}$$

By the definition, $\Delta \subseteq \Delta_n \subseteq \Delta_{n+1}$ and the set Δ_{n+1} is consistent. Hence Δ_{n+1} is an incomplete consistent extension of Δ.

By the principle of mathematical induction we have defined an infinite sequence

$$\Delta = \Delta_1 \subseteq \Delta_2 \subseteq ..., \subseteq \Delta_n \subseteq \Delta_{n+1} \subseteq .... \tag{62}$$

such that for all $n \in N$, Δ_n is consistent, and moreover, it is an incomplete consistent extension of Δ.

Moreover, we have also defined a sequence

$$B_1, B_2, ..., B_n, ... \tag{63}$$

of formulas, such that for all $n \in N$, $\Delta_n \nvdash B_n$, and the set $\Delta_n \cup \{B_n\}$ is consistent.
Observe that $B_n \in \Delta_{n+1}$ for all $n \geq 1$.

Definition of Δ^*

Now we are ready to define Δ^*, i.e. we define:

$$\Delta^* = \bigcup_{n \in N} \Delta_n. \tag{64}$$

To complete the proof our theorem we have now to prove that Δ^* is a complete consistent extension of Δ. Obviously, by the definition, Δ^* is an extension of Δ. Now we prove (by contradiction) the following.

Fact 5

The set Δ^* is consistent.

Proof
Assume that Δ^* is inconsistent. By the Finite Inconsistency Theorem 7 there is a finite subset Δ_0 of Δ^* that is inconsistent. By Definition 64 have that

$$\Delta_0 = \{C_1, ..., C_n\} \subseteq \bigcup_{n \in N} \Delta_n.$$

By the definition, $C_i \in \Delta_{k_i}$ for certain Δ_{k_i} in the sequence (62) and $1 \leq i \geq n$. Hence $\Delta_0 \subseteq \Delta_m$ for $m = max\{k_1, k_2, ..k_n\}$. But all sets of the sequence (62) are consistent. This contradicts the fact that Δ_m is inconsistent, as it contains an inconsistent subset Δ_0. Hence Δ^* must be consistent.

Fact 6

The set Δ^* is complete.

Proof
Assume that Δ^* is not complete. By the Incomplete Set Condition Lemma 10, there is a formula $B \in \mathcal{F}$ such that $\Delta^* \nvdash B$ and the set $\Delta^* \cup \{B\}$ is consistent.

But, by definition (64) of Δ^*, the above condition means that for every $n \in N$, $\Delta_n \nvdash B$ holds and the set $\Delta_n \cup \{B\}$ is consistent.

Since the formula B is one of the formulas of the sequence (58) and it would have to be one of the formulas of the sequence (63), i.e. $B = B_j$ for certain j. Since $B_j \in \Delta_{j+1}$, it proves that $B \in \Delta^* = \bigcup_{n \in N}$. But this means that $\Delta^* \vdash B$, contrary to the assumption. This proves that Δ^* is a complete consistent extension of Δ and **ends** the proof out our lemma.

Now we are ready to prove the completeness theorem for the system H_2.

Proof of the Completeness Theorem

As by assumption our system H_2 is sound, we have to prove only the Completeness part of the Completeness Theorem 6, i.e we have to show the implication

$$\text{if} \quad \models A, \quad \text{then} \quad \vdash A$$

for any formula A. We prove it by proving the logically equivalent opposite implication

$$\text{if} \quad \nvdash A, \quad \text{then} \quad \not\models A.$$

We remind that $\not\models A$ means that there is a variable assignment $v : VAR \longrightarrow \{T, F\}$, such that $v^*(A) \neq T$. In classical case it means that $v^*(A) = F$, i.e. that there is a variable assignment that falsifies A. Such v is also called a **counter-model** for A.

Assume that A doesn't have a proof in S, we want to define a **counter-model** for A. But if $\nvdash A$, then by the lemma 8, the set $\{\neg A\}$ is consistent. By the Main Lemma 11 there is a complete, consistent extension of the set $\{\neg A\}$, i.e. there is a set set Δ^* such that $\{\neg A\} \subseteq \Delta^*$, i.e.

$$\neg A \in \Delta^*. \tag{65}$$

Since Δ^* is a consistent, complete set, it satisfies the following form of the consistency condition 54, which says that for any A, $\Delta^* \nvdash A$ or $\Delta^* \nvdash \neg A$. It also satisfies the completeness condition (55), which says that for any A, $\Delta^* \vdash A$ or $\Delta^* \vdash \neg A$. This means that for any A, exactly one of the following conditions is satisfied: $\Delta^* \vdash A$, $\Delta^* \vdash \neg A$. In particular, for every propositional variable $a \in VAR$ exactly one of the following conditions is satisfied: $\Delta^* \vdash a$, $\Delta^* \vdash \neg a$. This justifies the correctness of the following definition.

Definition of v

We define the variable assignment

$$v : VAR \longrightarrow \{T, F\} \tag{66}$$

as follows:

$$v(a) = \begin{cases} T & \text{if } \Delta^* \vdash a \\ F & \text{if } \Delta^* \vdash \neg a. \end{cases}$$

We show, as a separate lemma below, that such defined variable assignment v has the following property.

Lemma 12 (Property of v)

Let v be the variable assignment defined by (66) and v^* its extension to the set $\mathcal{F}$ of all formulas. Then for every formula $B \in \mathcal{F}$,

$$v^*(B) = \begin{cases} T & \textit{if } \Delta^* \vdash B \\ F & \textit{if } \Delta^* \vdash \neg B. \end{cases} \tag{67}$$

Given the above property (67) of v (still to be proven), we prove that the v is in fact, a counter model for any formula A, such that $\not\vdash A$ as follows. Let A be such that $\not\vdash A$. By (65), $\neg A \in \Delta^*$ and obviously, $\Delta^* \vdash \neg A$. Hence, by the property (67) of v, $v^*(A) = F$, what proves that v is a counter-model for A and hence ends the proof of the completeness theorem. In order to really complete the proof we still have to write a proof of the Lemma 12.

Proof of the Lemma 12

The proof is conducted by the induction on the degree of the formula A.

If A is a propositional variable, then the lemma is true holds by (66), i.e. by the definition of v.

If A is not a propositional variable, then A is of the form $\neg C$ or $(C \Rightarrow D)$, for certain formulas C, D. By the inductive assumption the lemma, i.e. the property (67) holds for the formulas C and D.

Case $A = \neg C$**.** We have to consider two possibilities: $\Delta^* \vdash A$ and $\Delta^* \vdash \neg A$.

Assume $\Delta^* \vdash A$. It means that $\Delta^* \vdash \neg C$. Then from the fact that Δ^* is *consistent* it must be that $\Delta^* \not\vdash C$. This means, by the inductive assumption, that $v^*(C) = F$, and accordingly

$$v^*(A) = v^*(\neg C) = \neg v^*(C) = \neg F = T.$$

Assume now that $\Delta^* \vdash \neg A$. Then from the fact that Δ^* is *consistent* it must be that $\Delta^* \nvdash A$. I.e. $\Delta^* \nvdash \neg C$. If so, then $\Delta^* \vdash C$, as the set Δ^* is *complete.* Hence by the inductive assumption, that $v^*(C) = T$, and accordingly

$$v^*(A) = v^*(\neg C) = \neg v^*(C) = \neg T = F.$$

Thus A satisfies the property (67).

Case $A = (C \Rightarrow D)$**.** As in the previous case, we assume that the lemma, i.e. the property (67) holds for the formulas C, D and we consider two possibilities: $\Delta^* \vdash A$ and $\Delta^* \vdash \neg A$.

Assume $\Delta^* \vdash A$. It means that $\Delta^* \vdash (C \Rightarrow D)$. If at the same time $\Delta^* \nvdash C$, then $v^*(C) = F$, and accordingly

$$v^*(A) = v^*(C \Rightarrow D) = v^*(C) \Rightarrow v^*(D) = F \Rightarrow v^*(D) = T.$$

If at the same time $\Delta^* \vdash C$, then, since $\Delta^* \vdash (C \Rightarrow D)$, we infer, by Modus Ponens, that $\Delta^* \vdash D$. If so, then

$$v^*(C) = v^*(D) = T,$$

and accordingly

$$v^*(A) = v^*(C \Rightarrow D) = v^*(C) \Rightarrow v^*(D) = T \Rightarrow T = T.$$

Thus, if $\Delta^* \vdash A$, then $v^*(A) = T$.

Assume now, as before, that $\Delta^* \vdash \neg A$. Then from the fact that Δ^* is *consistent* it must be that $\Delta^* \nvdash A$, i.e.,

$$\Delta^* \nvdash (C \Rightarrow D).$$

It follows from this that

$$\Delta^* \nvdash D,$$

for if $\Delta^* \vdash D$, then, as $(D \Rightarrow (C \Rightarrow D))$ is provable (Lemma 4), by monotonicity

$$\Delta^* \vdash (D \Rightarrow (C \Rightarrow D)).$$

Applying Modus Ponens we obtain $\Delta^* \vdash (C \Rightarrow D)$, which is contrary to the assumption.

Also we must have

$$\Delta^* \vdash C,$$

for otherwise, by the fact that Δ^* we would have

$$\Delta^* \vdash \neg C.$$

But this is impossible, since the formula $(\neg C \Rightarrow (C \Rightarrow D))$ is provable provable (Lemma 4) and by monotonicity

$$\Delta^* \vdash \ (\neg C \Rightarrow (C \Rightarrow D)).$$

Applying Modus Ponens we would get $\Delta^* \vdash (C \Rightarrow D)$, which is contrary to the assumption. This **ends** the proof of the Lemma 12 and the **Proof Two** of the Completeness Theorem 6.

4 Some Other Axiomatizations

We present here some of most known, and historically important axiomatizations of classical propositional logic, i.e. the following Hilbert proof systems.

Łukasiewicz (1929)

$$Ł = (\ \mathcal{L}_{\{\neg,\ \Rightarrow\}},\ \ \mathcal{F},\ \ A1, A2, A3,\ \ MP\ ), \tag{68}$$

where

A1 $((\neg A \Rightarrow A) \Rightarrow A)$,

A2 $(A \Rightarrow (\neg A \Rightarrow B))$,

A3 $((A \Rightarrow B) \Rightarrow ((B \Rightarrow C) \Rightarrow (A \Rightarrow C)))$
for any $A, B, C \in \mathcal{F}$.

2. Hilbert and Ackermann (1928)

$$HA = (\ \mathcal{L}_{\{\neg,\cup\}},\ \ \mathcal{F},\ \ A1 - A4,\ \ MP\ ), \tag{69}$$

where

A1 $(\neg(A \cup A) \cup A)$,

A2 $(\neg A \cup (A \cup B))$,

A3 $(\neg(A \cup B) \cup (B \cup A))$,

A4 $(\neg(\neg B \cup C) \cup (\neg(A \cup B) \cup (A \cup C)))$,

for any $A, B, C \in \mathcal{F}$.

Modus Ponens rule in the language $\mathcal{L}_{\{\neg,\cup\}}$ has a form

$$(MP)\ \ \frac{A\ ;\ \ (\neg A \cup B)}{B}.$$

Observe that also the Deduction Theorem is now formulated as follow.

Theorem 8 (Deduction Theorem for HA)

For any subset Γ of the set of formulas $\mathcal{F}$ of HA and for any formulas $A, B \in \mathcal{F}$,

$$\Gamma,\ A \vdash_{HA} B \ \ if\ and\ only\ if\ \ \Gamma \vdash_{HA} (\neg A \cup B).$$

In particular,

$$A \vdash_{HA} B \ \ if\ and\ only\ if\ \ \vdash_{HA} (\neg A \cup B).$$

2. Hilbert (1928)

$$H = (\ \mathcal{L}_{\{\neg,\cup,\cap,\Rightarrow\}},\ \ \mathcal{F},\ \ A1 - A15,\ \ MP\ ), \tag{70}$$

where

A1 $(A \Rightarrow A)$,

A2 $(A \Rightarrow (B \Rightarrow A))$,

A3 $((A \Rightarrow B) \Rightarrow ((B \Rightarrow C) \Rightarrow (A \Rightarrow C)))$,

A4 $((A \Rightarrow (A \Rightarrow B)) \Rightarrow (A \Rightarrow B))$,

A5 $((A \Rightarrow (B \Rightarrow C)) \Rightarrow (B \Rightarrow (A \Rightarrow C)))$,

A6 $((A \Rightarrow B) \Rightarrow ((B \Rightarrow C) \Rightarrow (A \Rightarrow C)))$,

A7 $((A \cap B) \Rightarrow A)$,

A8 $((A \cap B) \Rightarrow B)$,

A9 $((A \Rightarrow B) \Rightarrow ((A \Rightarrow C) \Rightarrow (A \Rightarrow (B \cap C)))$,

A10 $(A \Rightarrow (A \cup B))$,

A11 $(B \Rightarrow (A \cup B))$,

A12 $((A \Rightarrow C) \Rightarrow ((B \Rightarrow C) \Rightarrow ((A \cup B) \Rightarrow C)))$,

A13 $((A \Rightarrow B) \Rightarrow ((A \Rightarrow \neg B) \Rightarrow \neg A))$,

A14 $(\neg A \Rightarrow (A \Rightarrow B))$,

A15 $(A \cup \neg A)$,

for any $A, B, C \in \mathcal{F}$.

Kleene (1952)

$$K = (\ \mathcal{L}_{\{\neg,\cup,\cap,\Rightarrow\}},\ \ \mathcal{F},\ \ A1 - A10,\ \ MP\ ), \tag{71}$$

where

A1 $(A \Rightarrow (B \Rightarrow A))$,

A2 $((A \Rightarrow (B \Rightarrow C)) \Rightarrow (B \Rightarrow (A \Rightarrow C)))$, A3 $((A \cap B) \Rightarrow A)$,

A4 $((A \cap B) \Rightarrow B)$,

A5 $(A \Rightarrow (B \Rightarrow (A \cap B)))$,

A6 $(A \Rightarrow (A \cup B))$,

A7 $(B \Rightarrow (A \cup B))$,

A8 $((A \Rightarrow C) \Rightarrow ((B \Rightarrow C) \Rightarrow ((A \cup B) \Rightarrow C)))$,

A9 $((A \Rightarrow B) \Rightarrow ((A \Rightarrow \neg B) \Rightarrow \neg A))$,

A10 $(\neg\neg A \Rightarrow A)$

for any $A, B, C \in \mathcal{F}$.

Rasiowa-Sikorski (1950)

$$RS = (\ \mathcal{L}_{\{\neg,\cup,\cap,\Rightarrow\}},\quad \mathcal{F},\quad A1 - A12,\quad MP\ ), \tag{72}$$

where

A1 $((A \Rightarrow B) \Rightarrow ((B \Rightarrow C) \Rightarrow (A \Rightarrow C)))$,

A2 $(A \Rightarrow (A \cup B))$,

A3 $(B \Rightarrow (A \cup B))$,

A4 $((A \Rightarrow C) \Rightarrow ((B \Rightarrow C) \Rightarrow ((A \cup B) \Rightarrow C)))$,

A5 $((A \cap B) \Rightarrow A)$,

A6 $((A \cap B) \Rightarrow B)$,

A7 $((C \Rightarrow A) \Rightarrow ((C \Rightarrow B) \Rightarrow (C \Rightarrow (A \cap B)))$,

A8 $((A \Rightarrow (B \Rightarrow C)) \Rightarrow ((A \cap B) \Rightarrow C))$,

A9 $(((A \cap B) \Rightarrow C) \Rightarrow (A \Rightarrow (B \Rightarrow C))$,

A10 $(A \cap \neg A) \Rightarrow B)$,

A11 $((A \Rightarrow (A \cap \neg A)) \Rightarrow \neg A)$,

A12 $(A \cup \neg A)$,

for any $A, B, C \in \mathcal{F}$.

Here is the shortest axiomatization for the language $\mathcal{L}_{\{\neg,\ \Rightarrow\}}$. It contains just one axiom.

Meredith (1953)

$$\text{Ł} = (\ \mathcal{L}_{\{\neg,\ \Rightarrow\}},\quad \mathcal{F},\quad A1\quad MP\ ), \tag{73}$$

where

A1 $((((((A \Rightarrow B) \Rightarrow (\neg C \Rightarrow \neg D)) \Rightarrow C) \Rightarrow E)) \Rightarrow ((E \Rightarrow A) \Rightarrow (D \Rightarrow A)))$.

We have proved in chapter **??** that

$$\mathcal{L}_{\{\neg,\cup,\cap,\Rightarrow\}} \equiv \mathcal{L}_{\{\uparrow\}} \equiv \mathcal{L}_{\{\downarrow\}}.$$

Here is another axiomatization that uses only one axiom,

Nicod (1917)

$$N = (\ \mathcal{L}_{\{\uparrow\}},\ \ \mathcal{F},\ \ A1,\ \ (r)\ ), \tag{74}$$

where

A1 $(((A \uparrow (B \uparrow C)) \uparrow ((D \uparrow (D \uparrow D)) \uparrow ((E \uparrow B) \uparrow ((A \uparrow E) \uparrow (A \uparrow E))))))$.
The rule of inference is (r) is expressed in the language $\mathcal{L}_{\{\uparrow\}}$ as

$$\frac{A \uparrow (B \uparrow C)}{A}.$$

5 Exercises

Here are few exercises designed to help the readers with understanding the notions of completeness, monotonicity of the consequence operation, the role of the deduction theorem and importance of some basic tautologies.
Let S be any Hilbert proof system

$$S = (\mathcal{L}_{\{\cap,\cup,\Rightarrow,\neg\}},\ \mathcal{F},\ LA,\ (MP)\frac{A, (A \Rightarrow B)}{B}) \tag{75}$$

with its set LA of logical axioms such that S is **complete** under classical semantics.

Let $X \subseteq \mathcal{F}$ be any subset of the set $\mathcal{F}$ of formulas of the language $\mathcal{L}_{\{\cap,\cup,\Rightarrow,\neg\}}$ of S. We define, as we did in chapter 4, a set $Cn(X)$ of all **consequences** of the set X as

$$Cn(X) = \{A \in \mathcal{F} :\ \ X \vdash_S A\}. \tag{76}$$

Plainly speaking, the set $Cn(X)$ of all **consequences** of the set X is the set of all formulas that can be proved in S from the set $(LA \cup X)$.

All exercises 9 - 13 concern the system S defined by (75).

Exercise 9

1. Prove that for any subsets X, Y of the set $\mathcal{F}$ of formulas the following ***monotonicity property*** *holds.*

$$\textit{If } X \subseteq Y, \textit{ then } \ Cn(X) \subseteq Cn(Y). \tag{77}$$

2. Do we need the **completeness** *of S to prove that the monotonicity property holds for S?*

Solution
1. Let $A \in \mathcal{F}$ be any formula such that $A \in Cn(X)$. By (76), we have that $X \vdash_S A$. This means that A has a formal proof from the set $X \cup LA$. But $X \subseteq Y$, hence this proof is also a proof from $Y \cup LA$, i.e . $Y \vdash_S A$, and hence $A \in Cn(Y)$. This proves that $Cn(X) \subseteq Cn(Y)$.

2. No, we do not need the completeness of S for the monotonicity property to hold. We have used only the definition of a formal proof from the hypothesis X and the definition of the consequence operation.

Exercise 10

Prove that for any set $X \subseteq \mathcal{F}$, the set $\mathbf{T} \subseteq \mathcal{F}$ of all propositional classical tautologies of the language $\mathcal{L}_{\{\cap,\cup,\Rightarrow,\neg\}}$ of the system S is a subset of $Cn(X)$, i.e. prove that

$$\mathbf{T} \subseteq Cn(X). \tag{78}$$

2. Do we need the **completeness** *of S to prove that the property (78) holds for S?*

Solution
1. The proof system S is complete, so by the completeness theorem we have that

$$\mathbf{T} = \{\in \mathcal{F} : \vdash_S A\}. \tag{79}$$

By definition (76) of the consequence,

$$\{A \in \mathcal{F} : \vdash_S A\} = Cn(\emptyset)$$

and hence $Cn(\emptyset) = \mathbf{T}$. But $\emptyset \subseteq X$ for any set X, so by monotonicity property (77),

$$\mathbf{T} \subseteq Cn(X).$$

2. Yes, the completeness (79) of S in the main property used. The next one is the monotonicity property (77).

Exercise 11

Prove that for any formulas $A, B \in \mathcal{F}$, and for any set $X \subseteq \mathcal{F}$,

$$(A \cap B) \in Cn(X) \ \text{ if and only if } \ A \in Cn(X) \ \text{ and } \ B \in Cn(X). \tag{80}$$

List all properties essential to the proof.

1. **Proof of the implication:**

$$\text{if } \ (A \cap B) \in Cn(X), \ \text{then } A \in Cn(X) \ \text{ and } B \in Cn(X).$$

Assume $(A \cap B) \in Cn(X)$, i.e.

$$X \vdash_S (A \cap B). \tag{81}$$

From monotonicity property proved in exercise 9, completeness of S, and the fact that

$$\models ((A \cap B) \Rightarrow A) \text{ and } \ \models ((A \cap B) \Rightarrow B) \tag{82}$$

we get that

$$X \vdash_S ((A \cap B) \Rightarrow A), \text{ and } \ X \vdash_S ((A \cap B) \Rightarrow B). \tag{83}$$

By the assumption (81) we have that $X \vdash_S (A \cap B)$, by (83), $X \vdash_S ((A \cap B) \Rightarrow A)$, and so we get $X \vdash_S A$ by Modus Ponens.

Similarly, $X \vdash_S (A \cap B)$, by the assumption (81), $X \vdash_S ((A \cap B) \Rightarrow B)$ by by (**??**), and so we get $X \vdash_S B$ by MP. This proves that $A \in Cn(X)$ and $B \in Cn(X)$ and **ends** the proof of the implication 1.

2. **Proof of the implication:**

$$\text{if } \ A \in Cn(X) \text{ and } B \in Cn(X), \ \text{then } (A \cap B) \in Cn(X).$$

Assume now that $A \in Cn(X)$ and $B \in Cn(X)$, i.e.

$$X \vdash_S A, \text{ and } X \vdash_S B. \tag{84}$$

By the monotonicity property, completeness of S, and a tautology $(A \Rightarrow (B \Rightarrow (A \cap B)))$, we get that

$$X \vdash_S (A \Rightarrow (B \Rightarrow (A \cap B))). \tag{85}$$

By the assumption (84) we have that $X \vdash_S A$, $X \vdash_S B$, by (85), $X \vdash_S (A \Rightarrow (B \Rightarrow (A \cap B)))$, so we get $X \vdash_S (B \Rightarrow (A \cap B))$ by Modus Ponens. Applying Modus Ponens again we obtain $X \vdash_S (A \cap B)$. This proves that $(A \cap B) \in Cn(X)$ and ends the proof and the implication 2, and the completes the proof of (80).

Exercise 12

Let S be the proof system (75). Prove that the Deduction Theorem holds for S, i.e. prove the following.

For any subset Γ of the set of formulas $\mathcal{F}$ of S and for any formulas $A, B \in \mathcal{F}$,

$$\Gamma, \ A \vdash_S B \quad \textit{if and only if} \ \ \Gamma \vdash_S (A \Rightarrow B). \tag{86}$$

Solution
The formulas $A1 = (A \Rightarrow (B \Rightarrow A))$ and $A2 = ((A \Rightarrow (B \Rightarrow C)) \Rightarrow ((A \Rightarrow B) \Rightarrow (A \Rightarrow C)))$ are basic propositional tautologies. By the completeness of S we have that

$$\vdash_S (A \Rightarrow (B \Rightarrow A)) \text{ and } \vdash_S ((A \Rightarrow (B \Rightarrow C)) \Rightarrow ((A \Rightarrow B) \Rightarrow (A \Rightarrow C))). \tag{87}$$

The formulas $A1, A2$ are axioms of the Hilbert system H_1 defined by (1). By (87) both axioms $A1, A2$ of H_1 are provable in S. These axioms were sufficient for the proof of the Deduction Theorem 3 for H_1 and its proof now can be repeated for the system S.

Exercise 13

Prove that for any $A, B \in \mathcal{F}$,

$$Cn(\{A, B\}) = Cn(\{(A \cap B)\})$$

1. **Proof of the inclusion:** $Cn(\{A, B\}) \subseteq Cn(\{(A \cap B)\})$.

Assume $C \in Cn(\{A, B\})$, i.e. $\{A, B\} \vdash_S C$, what we usually write as $A,\ B \vdash_S C$. Observe that by exercise 12 the Deduction Theorem (theorem 3) holds for S. We apply Deduction Theorem to the assumption $A,\ B \vdash_S C$ twice we get that the assumption is equivalent to

$$\vdash_S (A \Rightarrow (B \Rightarrow C)). \tag{88}$$

We use completeness of S, the fact that the formula $(((A \Rightarrow (B \Rightarrow C)) \Rightarrow ((A \cap B) \Rightarrow C)))$ is a tautology, and by monotonicity and get that

$$\vdash_S (((A \Rightarrow (B \Rightarrow C)) \Rightarrow ((A \cap B) \Rightarrow C))). \tag{89}$$

Applying Modus Ponens to the assumption (88) and (89) we get $\vdash_S ((A \cap B) \Rightarrow C)$. This is equivalent to $(A \cap B) \vdash_S C$ by Deduction Theorem. We have proved that $C \in Cn(\{(A \cap B)\})$.

2. **Proof of the inclusion:** $Cn(\{(A \cap B)\}) \subseteq Cn(\{A, B\})\})$.

Assume that $C \in Cn(\{(A \cap B)\})$, i.e. $(A \cap B) \vdash_S C$. By Deduction Theorem,

$$\vdash_S ((A \cap B) \Rightarrow C). \tag{90}$$

We want to prove that $C \in Cn(\{A, B\})$. This is equivalent, by the Deduction Theorem applied twice to proving that

$$\vdash_S (A \Rightarrow (B \Rightarrow C)).$$

The proof as similar to the previous case. We use completeness of S, the fact that the formula $(((A \cap B) \Rightarrow C) \Rightarrow (A \Rightarrow (B \Rightarrow C)))$ is a tautology and by monotonicity and get that

$$\vdash_S (((A \cap B) \Rightarrow C) \Rightarrow (A \Rightarrow (B \Rightarrow C)) \tag{91}$$

Applying Modus Ponens to the assumption (88) and (89) we get $\vdash_S (A \Rightarrow (B \Rightarrow C))$ what **ends** the proof.

6 Homework Problems

Completeness Proof Two Problems

1. List all formulas that have to be provable in H_2, axioms included, that are are needed for the Proof Two if the Completeness Theorem 6.
2. We proved the Completeness Theorem 6 for the proof system H_2 based on the language $\mathcal{L}_{\{\neg,\Rightarrow\}}$. Extend the H_2 proof system to a proof system S_1 based on a language $\mathcal{L}_{\{\neg,\Rightarrow,\cup\}}$ by adding new logical axioms, as we did in a case of H_1 and H_2 systems. The added logical axioms must be such that they allow to adopt the Proof Two to S_1, i.e. such that it is a complete proof system with respect to classical semantics.
3. Extend the H_2 proof system to a proof system based on a language $\mathcal{L}_{\{\neg,\Rightarrow,\cap\}}$ by adding new logical axioms. Call resulting proof system S_2. The added logical axioms must be such that they allow to adopt the Proof Two to S_2, i.e. such that it is a complete proof system with respect to classical semantics.
4. Repeat the same for the language $\mathcal{L}_{\{\neg,\Rightarrow,\cap,\cup\}}$, i.e. extends systems S_1 or S_2 to a complete proof system S_3 based on the language $\mathcal{L}_{\{\neg,\Rightarrow,\cap,\cup\}}$.
5. Conduct appropriate version of Proof Two of the Completeness Theorem 6 for the system S_3 from the previous problem.

Axiomatizations Problems

1. Let HA be Hilbert and Ackermann proof system (69). We use abbreviation $(A \Rightarrow B)$ for $(\not{A} \cup B)$.

   (i) Prove $\vdash_{HA} (A \Rightarrow A)$, for any $A \in \mathcal{F}$.

   (ii) Prove $\vdash_{HA} (A \Rightarrow (B \Rightarrow A))$, for any $A, B \in \mathcal{F}$.

   (iii) Prove $\vdash_{HA} ((A \Rightarrow (B \Rightarrow C)) \Rightarrow ((A \Rightarrow B) \Rightarrow (A \Rightarrow C)))$, for any $A, B, C \in \mathcal{F}$.

   (iv) Prove $(A \Rightarrow B), (B \Rightarrow C) \vdash_{HA} (A \Rightarrow C)$, for any $A, B, C \in \mathcal{F}$

   (v) Prove Deduction Theorem 8.

   (vi) Prove $\vdash_{HA} A$ if and only if $\models A$, for any $A \in \mathcal{F}$.

2. Let H be Hilbert proof system (**??**).

   (i) Prove $\vdash_{HA} ((A \Rightarrow (B \Rightarrow C)) \Rightarrow ((A \Rightarrow B) \Rightarrow (A \Rightarrow C)))$, for any $A, B, C \in \mathcal{F}$.

   (ii) Prove Deduction Theorem for H.

   (ii) Prove Completeness Theorem for H.

3. Let K be Kleene proof system (71).

   (i) Prove $\vdash_K (A \Rightarrow A)$, for any $A \in \mathcal{F}$.

   (ii) Prove the following.

   For any subset Γ of the set of formulas $\mathcal{F}$ of K and for any formulas $A, B \in \mathcal{F}$, $\Gamma,\ A \vdash_K B\ \ if\ and\ only\ if\ \ \Gamma \vdash_K ((A \Rightarrow B))$.

Completeness General Problems

1. Let RS be Rasiowa-Sikorski proof system (72).

   The set $\mathcal{F}$ of formulas of $\mathcal{L}$ determines an abstract algebra

   $$F \ = \ (\ \mathcal{F},\ \cup,\ \cap,\ \Rightarrow,\ \neg\ ), \tag{92}$$

   where by performing an operation on a formula (two formulas) means writing the formula having this operation as a main connective. For example $\cap(A, B) = (A \cup B)$. We define an binary relations $\leq$ and $\approx$ in the algebra F of formulas of $\mathcal{L}$ as follows. For any $A, B \in \mathcal{F}$,

   $$A \leq B\ \ if\ and\ \ only\ if\ \ \vdash_{RS} (A \Rightarrow B), \tag{93}$$

   $$A \approx B\ \ if\ and\ \ only\ if\ \ \vdash_{RS} (A \Rightarrow B)\ \ and\ \ \vdash\ (B \Rightarrow A). \tag{94}$$

   (i) Prove that the relation $\leq$ defined by (93) is a quasi-ordering in $\mathcal{F}$.

   (ii) Prove that the relation $\approx$ defined by (93) is an equivalence relation in $\mathcal{F}$. The equivalence class containing a formula A is denoted by $\|A\|$.

   (iii) The quasi ordering $\leq$ in $\mathcal{F}$ as defined by (93) induces a relation $\leq$ in $\mathcal{F}/\approx$ defined as follows:

   $$\|A\| \leq \|B\|\ \ \text{if and only if}\ \ A \leq B,\ \text{i.e.}$$

   $$\|A\| \leq \|B\|\ \ if\ and\ only\ if\ \ \vdash_{RS} (A \Rightarrow B). \tag{95}$$

   Prove that the relation (95) is an order relation in $\mathcal{F}/\approx$

   (iv) Prove that the relation $\approx$ defined by (94) is a congruence in the algebra $\mathcal{F}$ of formulas defined by (92).

2. The algebra $LT = (\mathcal{F}/\approx, \ \cup, \ \cap, \ \Rightarrow, \ \neg)$, where the operations $\cup, \ \cap, \ \Rightarrow$ and $\neg$ are determined by the congruence relation (94) i.e.

$$\|A\| \cup \|B\| = \|(A \cup B)\|,$$

$$\|A\| \cap \|B\| = \|(A \cap B)\|,$$

$$\|A\| \Rightarrow \|B\| = \|(A \Rightarrow B)\|,$$

$$\neg\|A\| = \|\neg A\|,$$

   is called a Lindenbaum-Tarski algebra of RS.

   Prove that the Lindenbaum-Tarski algebra of RS as defined by (72) is a Boolean algebra. The unit element is the greatest element in $(\mathcal{F}/\approx, \leq)$, where the order relation $\leq$ is defined by (95).

3. Formulate and prove the Deduction Theorem for Hilbert and Ackermann system (69).

4. Formulate and prove the Deduction Theorem for Lukasiewicz system (68).

5. Formulate and prove the Deduction Theorem Kleene system (71).

6. Formulate and prove the Deduction Theorem Rasiowa-Sikorski system (72)

7. Let HS be any Hilbert proof system based on a language $\mathcal{L}_{HS}$. Prove that if HS is complete under classical semantic, them the Deduction Theorem appropriately expressed in the language $\mathcal{L}_{HS}$ holds for HS.