Midterm has 5 Questions. Extra Credit 10pts is included in the Total sum of 110 pts for the test.

QUESTION 1 (20 pts)

Write the following natural language statement:
One likes to play bridge, or from the fact that the weather is good we conclude the following:
one does not like to play bridge or one likes not to play bridge
as a formula of 2 different languages

1. (10pts) Formula $A_1 \in F_1$ of a language $L_{\neg, \lor, \Rightarrow}$, where $LA$ represents statement “one likes A”,
"A is liked".

Use Propositional Variables $a, b$ as consecutive statements

Solution

$a$ denotes statement:  play bridge,

$b$ denotes statement:  the weather is good

The formula $A_1 \in F_1$ is

\[ A_1 = (La \lor (b \Rightarrow (\neg Ia \lor L\neg a))) \]

2. (10pts) Formula $A_2 \in F_2$ of a language $L_{\neg, \lor, \Rightarrow}$.

Use Propositional Variables $a, b, c$ as consecutive statements

Solution

$a$ denotes statement:  One likes to play bridge,

$b$ denotes statement:  the weather is good,

$c$ denotes statement:  one likes not to play bridge

The formula $A_2 \in F_2$ is

\[ A_2 = (a \lor (b \Rightarrow (\neg a \lor c))) \]

QUESTION 2 (20 pts)

Let $A$ be a formula

\[ (((a \cap \neg c) \Rightarrow \neg b) \lor a) \Rightarrow (c \lor b)) \]

1. (5pts) A language $L_{CON}$ to which the formula $A$ belongs is:

Solution:  The language is $L_{\neg, \lor, \Rightarrow}$.

2. (5pts) Determine the degree of $A$ and write down all its sub-formulas of the degree 2.

Solution:  The degree of $A$ is 7. There is only one sub-formula of the degree 2:  $(a \cap \neg c)$.

3. (5pts) Determine whether $A \in T$. Use ”proof by contradiction” method and shorthand notation.
Solution: of the case $A \in T$.
Assume $(((a \land \neg c) \Rightarrow \neg b) \lor a) \Rightarrow (c \lor b)) = F$. This is possible if and only if $(((a \land \neg c) \Rightarrow \neg b) \lor a) = T$ and $(c \lor b) = F$. This gives as that $c = F, b = F$. We evaluate $(((a \land \neg c) \Rightarrow \neg b) \lor a) = T$. This is possible for $a = T$.
Any truth assignment such that $a = T, b = F, c = F$ is a counter-model for $A$, hence $A \notin T$.

4. (5pts) Determine whether $A \in C$. Use shorthand notation.

Solution: Any truth assignment such that $a = T, b = T, c = F$ is a model for $A$, hence $A \notin C$. This is not the only model.

QUESTION 3 (20 pts)

1. (5pts) Given $\mathcal{L} = \mathcal{L}_{(\neg, \Rightarrow, \lor, \land)}$ and classical semantics.

We define: A set $\mathcal{G} \subseteq \mathcal{F}$ is consistent if and only if there is a truth assignment $v$ such that $v \models \mathcal{G}$

PROVE that the set $\mathcal{G} = \{((a \land b) \Rightarrow b), (a \lor b), \neg b, (c \Rightarrow b)\}$
is consistent. Use shorthand notation.

Solution We find a restricted model for $\mathcal{G}$ as follows

First observe that the formula $((a \land \Rightarrow b)$, is a tautology, hence any $v$ is its model. So we have only to see whether the other formulas have a common model. It means we check if it is possible to find $v$, such that $v^*(\neg b) = T, v^*(a \land b)) = T$, and the $v^*(c \Rightarrow b)) = T$.
We have that $\neg b = T$ if and only if $b = F$. We evaluate $(a \lor b) = (a \lor F) = T$ if and only if $a = T$. Consequently, $(c \Rightarrow b) = (c \Rightarrow F) = T$ if and only if $c = F$. Hence, any $v$, such that $a = T, b = F$, and $c = F$ is a model for $\mathcal{G}$.

2. (5pts) How many restricted MODELS does $\mathcal{G}$ have?

Solution

We proved that $a = T, b = F$, and $c = F$ is the only restricted model for $\mathcal{G}$.

3. (10pts) We define: a formula $A \in \mathcal{T}$ is called independent from a set $\mathcal{G} \subseteq \mathcal{T}$ if and only if when there are truth assignments $v_1, v_2$ such that $v_1 \models \mathcal{G} \cup \{A\}$ and $v_2 \models \mathcal{G} \cup \{\neg A\}$.

PROVE the a formula $A = (d \lor (b \Rightarrow \neg a))$ is independent of $\mathcal{G}$ defined in 1. Use shorthand notation.

Solution

We proved that $a = T, b = F$ and $c = F$ is the only restricted model for $\mathcal{G}$.

Any $v_1$ such that $a = T, b = F, c = F$, and $d = T$ is a MODEL for $\mathcal{G} \cup \{A\}$ because the main connective of $A = (d \lor (b \Rightarrow \neg a))$ is disjunction and $(T \lor (b \Rightarrow \neg a)) = T$ for any logical values of $a, b$, in particular for $a = T, b = F$

Any $v_1$ such that $a = T, b = F, c = F$, and $d = F$ is a MODEL for $\mathcal{G} \cup \{\neg A\}$ as $(T \lor (b \Rightarrow \neg a)) = T$
for any logical values of $a, b$, in particular for $a = T$, $b = F$.

**QUESTION 4 (20pts)**

We define a 3 valued extensional semantics $M$ for the language $L_{\neg, \cup, \Rightarrow}$ by defining the connectives its connectives on a set $\{F, \bot, T\}$ of logical values by the following truth tables.

<table>
<thead>
<tr>
<th>L Connective</th>
<th>Negation :</th>
<th>Disjunction :</th>
</tr>
</thead>
<tbody>
<tr>
<td>$L$</td>
<td>$T$</td>
<td>$\bot$</td>
</tr>
<tr>
<td>$F$</td>
<td>$T$</td>
<td>$T$</td>
</tr>
<tr>
<td>$\bot$</td>
<td>$T$</td>
<td>$T$</td>
</tr>
<tr>
<td>$T$</td>
<td>$T$</td>
<td>$T$</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Implication</th>
<th>$\Rightarrow$</th>
<th>$\cup$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$F$</td>
<td>$T$</td>
<td>$T$</td>
</tr>
<tr>
<td>$\bot$</td>
<td>$T$</td>
<td>$T$</td>
</tr>
<tr>
<td>$T$</td>
<td>$T$</td>
<td>$T$</td>
</tr>
</tbody>
</table>

1. (10pts) Verify whether $\models_M (L \cup \neg L)$. You can use shorthand notation.

**Solution**

We verify all possible logical values for the formula $A$.

$L_T \cup \neg L_T = T \cup F = T$, $L_\bot \cup \neg L_\bot = F \cup \neg F = F \cup T = T$, $L_F \cup \neg L_F = F \cup \neg F = T$

2. (5pts) Verify whether the formula

$$(L_a \cup (b \Rightarrow (\neg L_a \cup L_\neg a)))$$

has a model under the semantics $M$. Use shorthand notation.

**Solution**

Any $v$, such that $v(a) = T$ is a $M$ model for $A$ directly from the definition of $\cup$ and $L$.

We evaluate $$(L_T \cup (b \Rightarrow (\neg L_T \cup L_\neg T))) = (T \cup (b \Rightarrow (\neg L_T \cup L_\neg T))) = T$$

for any logical value of $b$ and $(b \Rightarrow (\neg L_T \cup L_\neg T))$

3. (5pts) Verify whether the following set $G$ is $M$-consistent. Use shorthand notation

$G = \{ L_a, (a \cup \neg L_b), (a \Rightarrow b), b \}$

**Solution**

Any $v$, such that $v(a) = T$, $v(b) = T$ is a $M$ model for $G$ as

$L_T = T$, $(T \cup \neg L_T) = T$, $(T \Rightarrow T) = T$, $b = T$. 

3
QUESTION 5  (30pts)

1. (10pts) Given a formula \( A = ((a \cap \neg c) \Rightarrow (c \cup b)) \) of a language \( L_{\{\neg, \cap, \cup, \Rightarrow\}} \).

Transform \( A \) to a formula \( B \) of a language \( L_{\{\neg, \Rightarrow\}} \), such that \( A \equiv B \).

Solution
\[
((a \cap \neg c) \Rightarrow (c \cup b)) \equiv (\neg(a \Rightarrow \neg\neg c) \Rightarrow (\neg c \Rightarrow b))
\]

List all proper logical defining \( \cup, \cap \) connectives in terms of \( \neg, \Rightarrow \)
\[
(A \cap B) \equiv \neg(A \Rightarrow \neg B) \quad \text{and} \quad (A \cup B) \equiv (\neg A \Rightarrow B)
\]

Plus Plus Substitution Theorem.

2. (10pts) Given a formula \( A = ((a \cap \neg c) \Rightarrow (c \cup b)) \) of a language \( L_{\{\neg, \cap, \cup, \Rightarrow\}} \).

Transform it to a formula \( B \) of a language \( L_{\{\neg, \cap, \cup\}} \), such that \( A \equiv B \).

Solution
\[
((a \cap \neg c) \Rightarrow (c \cup b)) \equiv (\neg(a \cap \neg c) \cup (c \cup b)) \quad \text{or} \quad \neg((a \cap \neg c) \cap (c \cup b))
\]

List all proper logical equivalences defining \( \Rightarrow \) in terms of \( \neg, \cup \) or \( \neg \cap \), respectively.

List all proper logical equivalences defining \( \Rightarrow \) in terms of \( \neg, \cup, \cap \), respectively.
\[
(A \Rightarrow B) \equiv (\neg A \cup B) \quad \text{or} \quad (A \Rightarrow B) \equiv (\neg A \Rightarrow \neg B)
\]

3. (10pts) Prove that \( L_{\{\neg, \cap, \cup, \Rightarrow\}} \equiv L_{\{\neg, \Rightarrow\}} \).

Solution

We have to prove that \( L_{\{\neg, \Rightarrow\}} \equiv L_{\{\neg, \cap, \cup, \Rightarrow\}} \).

Condition C1 holds because \( \{\neg, \Rightarrow\} \subseteq \{\neg, \cap, \cup, \Rightarrow\} \).

Condition C2 holds because of the Substitution Theorem and because of the following logical equivalences defining \( \cup, \cap \) in terms of \( \neg, \Rightarrow \).
\[
(A \cap B) \equiv \neg(A \Rightarrow \neg B) \quad \text{and} \quad (A \cup B) \equiv (\neg A \Rightarrow B)
\]

Reminder

We define the equivalence of languages as follows:

Given two languages: \( L_1 = L_{CON_1} \) and \( L_2 = L_{CON_2} \), for \( CON_1 \neq CON_2 \), we say that they are logically equivalent, i.e. \( L_1 \equiv L_2 \) if and only if the following conditions C1, C2 hold.

C1: For every formula \( A \) of \( L_1 \), there is a formula \( B \) of \( L_2 \), such that \( A \equiv B \).

C2: For every formula \( C \) of \( L_2 \), there is a formula \( D \) of \( L_1 \), such that \( C \equiv D \).