CSE371 MIDTERM SOLUTIONS Spring 2024
(100pts + 10pts extra)

Midterm has 5 Questions. Extra Credit 10pts is included i the Total sum of 110 pts for the test.

QUESTION 1 (20 pts)]

Write the following natural language statement:

One likes to play bridge, or from the fact that the weather is good we conclude the following: one does not like to play bridge or one likes not to play bridge

as a formula of 2 different languages

1. (10pts) Formula $A_1 \in \mathcal{F}_1$ of a language $\mathcal{L}_{\{\neg,\, \mathbf{L},\, \cup,\, \Rightarrow\}}$, where $\mathbf{L}A$ represents statement "one likes A", "A is liked".

Use Propositional Variables a, b as consecutive statements

Solution

a denotes statement: *play bridge*,

b denotes statement: *the weather is good*

The formula $A_1 \in \mathcal{F}_1$ is

$$A_1 = (\mathbf{L}a \cup (b \Rightarrow (\neg \mathbf{I}a \cup \mathbf{L}\neg a)))$$

2. (10pts) Formula $A_2 \in \mathcal{F}_2$ of a language $\mathcal{L}_{\{\neg,\, \cup,\, \Rightarrow\}}$.

Use Propositional Variables a, b, c as consecutive statements

Solution

a denotes statement: *One likes to play bridge* ,

b denotes statement: *the weather is good*,

c denotes statement: *one likes not to play bridge*

The formula $A_2 \in \mathcal{F}_2$ is

$$A_2 = (a \cup (b \Rightarrow (\neg a \cup c)))$$

QUESTION 2 (20 pts)

Let A be a formula

$$((((a \cap \neg c) \Rightarrow \neg b) \cup a) \Rightarrow (c \cup b))$$

1. (5pts) A language $\mathcal{L}_{CON}$ to which the formula A belongs is:

Solution: The language is $\mathcal{L}_{\{\neg, \cap, \Rightarrow\}}$.

2. (5pts) Determine the degree of A and write down all its sub-formulas of the degree 2.

Solution: The degree of A is 7. There is only one sub-formula of the degree 2: $(a \cap \neg c)$.

3. (5pts) Determine whether $A \in \mathbf{T}$. Use "proof by contradiction" method and **shorthand** notation.

Solution: of the case $A \in \mathbf{T}$.

Assume $((((a \cap \neg c) \Rightarrow \neg b) \cup a) \Rightarrow (c \cup b)) = F$. This is possible if and only if $(((a \cap \neg c) \Rightarrow \neg b) \cup a) = T$ and $(c \cup b) = F$. This gives as that $c = F, b = F$. We evaluate $(((a \cap \neg F) \Rightarrow \neg F) \cup a) = T$. This is possible for $a = T$.

Any truth assignment such that $a = T, b = F, c = F$ is a counter-model for A, hence $A \notin \mathbf{T}$.

4. (5pts) Determine whether $A \in \mathbf{C}$. Use **shorthand** notation.

Solution: Any truth assignment such that $a = T,\ b = T,\ c = F$ is a model for A, hence $A \notin \mathbf{C}$. This is not the only model.

QUESTION 3 (20 pts)

1. (5pts) Given $\mathcal{L} = \mathcal{L}_{\{\neg, \Rightarrow, \cup, \cap\}}$ and **classical semantics**.

We **define:** A set $\mathcal{G} \subseteq \mathcal{F}$ is **consistent** if and only if there is a truth assignment v such that $v \models \mathcal{G}$

PROVE that the set

$$\mathcal{G} = \{((a \cap b) \Rightarrow b),\ (a \cup b),\ \neg b,\ (c \Rightarrow b)\}$$

is **consistent**. Use **shorthand** notation.

Solution We find a restricted model for $\mathcal{G}$ as follows

First observe that the formula $((a \cap b) \Rightarrow b)$, is a **tautology**, hence any v is its model. So we have only to see whether the other formulas have a common model. It means we check if it is possible to find v, such that $v^*(\neg b) = T$, $v^*((a \cup b)) = T$, and the $v^*((c \Rightarrow b)) = T$.

We have that $\neg b = T$ if and only if $b = F$.

We evaluate $(a \cup b) = (a \cup F) = T$ if and only if $a = T$.

Consequently, $(c \Rightarrow b) = (c \Rightarrow F) = T$ if and only if $c = F$.

Hence, any v, such that $a = T,\ b = F,$ and $c = F$ is a model for $\mathcal{G}$.

2. (5pts) How many restricted MODELS does $\mathcal{G}$ have?

Solution

We proved that $a = T,\ b = F,$ and $c = F$ is the **only restricted model** for $\mathcal{G}$.

3. (10pts) We **define:** a formula $A \in \mathcal{F}$ is called **independent** from a set $\mathcal{G} \subseteq \mathcal{F}$ if and only if when there are truth assignments $v_1,\ v_2$ such that $v_1 \models \mathcal{G} \cup \{A\}$ and $v_2 \models \mathcal{G} \cup \{\neg A\}$.

PROVE the a formula $A = (d \cup (b \Rightarrow \neg a))$ is **independent** of $\mathcal{G}$ defined in **1.** Use **shorthand** notation.

Solution

We proved that $a = T,\ b = F$ and $c = F$ is the **only restricted model** for $\mathcal{G}$.

Any v_1 such that $a = T,\ b = F, c = F$, and $d = T$ is a MODEL for $\mathcal{G} \cup \{A\}$ because the **main connective** of $A = (d \cup (b \Rightarrow \neg a))$ is **disjunction** and $(T \cup (b \Rightarrow \neg a)) = T$ for any logical values of a, b, in particular for $a = T,\ b = F$

Any v_1 such that $a = T,\ b = F,\ c = F$, and $d = F$ is a MODEL for $\mathcal{G} \cup \{\neg A\}$ as $(T \cup (b \Rightarrow \neg a)) = T$

for any logical values of a, b, in particular for $a = T,\ b = F$.

QUESTION 4 (20pts)

We define a 3 valued extensional semantics **M** for the language $\mathcal{L}_{\{\neg,\ \mathbf{L},\ \cup,\ \Rightarrow\}}$ by **defining the connectives** its connectives on on a set $\{F, \bot, T\}$ of logical values by the following truth tables.

L Connective

L	F	⊥	T
	F	*F*	T

Negation :

¬	F	⊥	T
	T	*F*	F

Implication

⇒	F	⊥	T
F	T	T	T
⊥	*T*	⊥	T
T	F	*F*	T

Disjunction :

∪	F	⊥	T
F	F	⊥	T
⊥	⊥	T	T
T	T	*T*	T

1. (10pts) Verify whether $\models_{\mathbf{M}} (\mathbf{L}A \cup \neg\mathbf{L}A)$. You can use **shorthand notation**.

Solution

We verify all possible logical values for the formula A.

$\mathbf{L}T \cup \neg\mathbf{L}T = T \cup F = T, \quad \mathbf{L} \bot \cup \neg\mathbf{L} \bot = F \cup \neg F = F \cup T = T, \quad \mathbf{L}F \cup \neg\mathbf{L}F = F \cup \neg F = T$

2. (5pts) Verify whether the formula

$$(\mathbf{L}a \cup (b \Rightarrow (\neg\mathbf{L}a \cup \mathbf{L}\neg a)))$$

has a model under the semantics **M**. Use **shorthand notation**.

Solution

Any v, such that $v(a) = T$ is a **M model** for A directly from the definition of $\cup$ and **L** .

We evaluate

$$(\mathbf{L}T \cup (b \Rightarrow (\neg\mathbf{L}T \cup \mathbf{L}\neg T))) = (T \cup (b \Rightarrow (\neg\mathbf{L}T \cup \mathbf{L}\neg T))) = T$$

for any logical value of b and $(b \Rightarrow (\neg\mathbf{L}T \cup \mathbf{L}\neg T))$

3. (5pts) Verify whether the following set **G** is **M -consistent**. Use **shorthand notation**

$$\mathbf{G} = \{\ \mathbf{L}a,\ \ (a \cup \neg\mathbf{L}b),\ \ (a \Rightarrow b),\ \ b\ \}$$

Solution

Any v, such that $v(a) = T,\ v(b) = T$ is a **M model** for **G** as

$$\mathbf{L}T = T, \quad (T \cup \neg\mathbf{L}T) = T, \quad (T \Rightarrow T) = T, \quad b = T.$$

QUESTION 5 (30pts

1. (10pts) Given a formula $A = ((a \cap \neg c) \Rightarrow (c \cup b))$ of a language $\mathcal{L}_{\{\neg,\cap,\cup,\Rightarrow\}}$.

Transform A to a formula B of a language $\mathcal{L}_{\{\neg,\Rightarrow\}}$, such that $A \equiv B$.

Solution

$$((a \cap \neg c) \Rightarrow (c \cup b)) \equiv (\neg(a \Rightarrow \neg\neg c) \Rightarrow (\neg c \Rightarrow b))$$

List all proper logical defining $\cup$, $\cap$ connectives in terms of $\neg, \Rightarrow$

$$(A \cap B) \equiv \neg(A \Rightarrow \neg B) \quad \text{and} \quad (A \cup B) \equiv (\neg A \Rightarrow B)$$

Plus Plus Substitution Theorem.

2. (10pts) Given a formula $A = ((a \cap \neg c) \Rightarrow (c \cup b))$ of a language $\mathcal{L}_{\{\neg,\cap,\cup,\Rightarrow\}}$.

Transform it to a formula B of a language $\mathcal{L}_{\{\neg,\cap,\cup\}}$, such that $A \equiv B$.

Solution

$$((a \cap \neg c) \Rightarrow (c \cup b)) \equiv (\neg(a \cap \neg c) \cup (c \cup b)) \quad \text{or} \quad \neg((a \cap \neg c) \cap \neg(c \cup b))$$

List all proper logical equivalences defining $\Rightarrow$ in terms of $\neg, \cup$ or $\neg\cap$, respectively.

List all proper logical equivalences defining $\Rightarrow$ in terms of $\neg, \cup, \cap$, respectively.

$$(A \Rightarrow B) \equiv (\neg A \cup B) \quad \text{or} \quad (A \Rightarrow B) \equiv \neg(A \cap \neg B)$$

3. (10pts) Prove that $\mathcal{L}_{\{\neg,\cap,\cup,\Rightarrow\}} \equiv \mathcal{L}_{\{\neg,\Rightarrow\}}$.

Solution

We have to prove that $\mathcal{L}_{\{\neg,\Rightarrow\}} \equiv \mathcal{L}_{\{\neg,\cap,\cup,\Rightarrow\}}$.

Condition **C1** holds because $\{\neg, \Rightarrow\} \subseteq \{\neg, \cap, \cup, \Rightarrow\}$.

Condition **C2** holds because of the **Substitution Theorem** and because of the following **logical equivalences** defining $\cup, \cap$ in terms of $\neg, \Rightarrow$.

$$(A \cap B) \equiv \neg(A \Rightarrow \neg B) \quad \text{and} \quad (A \cup B) \equiv (\neg A \Rightarrow B)$$

Reminder

We define the **equivalence of languages** as follows:

Given two languages: $\mathcal{L}_1 = \mathcal{L}_{CON_1}$ and $\mathcal{L}_2 = \mathcal{L}_{CON_2}$, for $CON_1 \neq CON_2$, we say that they are **logically equivalent**, i.e. $\mathcal{L}_1 \equiv \mathcal{L}_2$ if and only if the following conditions **C1, C2** hold.

C1: For every formula A of $\mathcal{L}_1$, there is a formula B of $\mathcal{L}_2$, such that $A \equiv B$,

C2: For every formula C of $\mathcal{L}_2$, there is a formula D of $\mathcal{L}_1$, such that $C \equiv D$.