CSE/MAT371 MIDTERM SOLUTIONS Fall 2022
(75pts)

NAME **ID:** **Math/CS**

Please write carefully your solutions. NO PARTIAL CREDIT. Formulas must be fully correct for credit.

QUESTION 1 (10pts)

Write the following natural language statement:

One likes to play bridge, or from the fact that the weather is good we conclude the following: one does not like to play bridge or one likes not to play bridge

as a formula of 2 different languages

1. (5pts) Formula $A_1 \in \mathcal{F}_1$ of a language $\mathcal{L}_{\{\neg,\ \mathbf{L},\ \cup,\ \Rightarrow\}}$, where $\mathbf{L}A$ represents statement "one likes A", "A is liked".

Solution Propositional Variables are: (use a, b, ... and you must write which variables denote which sentences)
a denotes statement: *play bridge*,
b denotes a statement: *the weather is good*

Translation $A_1 = (\mathbf{L}a \cup (b \Rightarrow (\neg\mathbf{L}a \cup \mathbf{L}\neg a)))$

2. (5pts) Formula $A_2 \in \mathcal{F}_2$ of a language $\mathcal{L}_{\{\neg,\ \cup,\ \Rightarrow\}}$.

Solution Propositional Variables are: (use a, b, ... and you must write which variables denote which sentences)
a denotes statement: *One likes to play bridge* ,
b denotes a statement: *the weather is good*,
c denotes a statement: *one likes not to play bridge*

Translation $A_2 = (a \cup (b \Rightarrow (\neg a \cup c)))$

QUESTION 2 (15 pts)

Here is a mathematical statement **S**:

For all rational numbers $x \in Q$ the following holds: If $x = 0$, then there is a natural number $n \in N$, such that $x + n = 0$

1. (5pts). Re-write **S** as a symbolic mathematical statement **SM** that only uses mathematical and logical symbols.

Solution **S** becomes a symbolic mathematical statement

$$\mathbf{SM}: \quad \forall_{x\in Q}(x = 0 \Rightarrow \exists_{n\in N}\ x + n = 0)$$

2. (5pts) Translate the symbolic statement **SM** into to a corresponding formula of the predicate language $\mathcal{L}$ with **restricted quantifiers**.

Use SYMBOLS: Q(x) for $x \in Q$, N(y) for $y \in N$, c for the number 0. Use $E \in \mathbf{P}$ to denote the relation = and $f \in \mathbf{F}$ to denote the function +

Solution

The statement $x = 0$ becomes an **atomic formula** E(x, c). The statement $x + n = 0$ becomes an **atomic formula** E(f(x,y), c).

The symbolic mathematical statement **SM** becomes a **restricted quantifiers** formula

$$\forall_{Q(x)}(E(x,c) \Rightarrow \exists_{N(y)}E(f(x,y),c))$$

3. (5pts) Translate your **restricted domain** quantifiers logical formula into a correct formula A of the language $\mathcal{L}$

Solution We apply now the **transformation rules** and get a corresponding formula $A \in \mathcal{F}$:

$$\forall x(Q(x) \Rightarrow (E(x,c) \Rightarrow \exists y(N(y) \cap E(f(x,y),c))))$$

QUESTION 3 (15 pts)

Let A be a formula $((((a \cap \neg c) \Rightarrow \neg b) \cup a) \Rightarrow (c \cup b))$.

1. (2pts) A language $\mathcal{L}_{CON}$ to which the formula A belongs is:

Solution: The language is $\mathcal{L}_{\{\neg,\cap,\cup,\Rightarrow\}}$.

2. (3pts) Determine the degree of A and write down all its sub-formulas of the degree 2.

Solution: The degree of A is 7. There is only one sub-formula of the degree 2: $(a \cap \neg c)$.

3. (5pts) Determine whether $A \in \mathbf{T}$. Use "proof by contradiction" method and **shorthand** notation.

Solution This is a question about the knowledge of the **"proof by contradiction"** method.
You get 0pts if you do not use it.

Assume $((((a \cap \neg c) \Rightarrow \neg b) \cup a) \Rightarrow (c \cup b)) = F$. This is possible if and only if $(((a \cap \neg c) \Rightarrow \neg b) \cup a) = T$ and $(c \cup b) = F$. This gives as that $c = F, b = F$. We evaluate $(((a \cap \neg F) \Rightarrow \neg F) \cup a) = T$. This is possible for $a = T$. Any truth assignment such that $a = T, b = F, c = F$ is a counter-model for A, hence $A \notin \mathbf{T}$.

4. (5pts) Determine whether $A \in \mathbf{C}$. Use **shorthand** notation.

Solution: any truth assignment such that $a = T,\ b = T,\ c = F$ is a model for A, hence $A \notin \mathbf{C}$. This is not the only model.

QUESTION 4 (10pts)

We define: A set $\mathcal{G} \subseteq \mathcal{F}$ is **consistent** if and only if there is a truth assignment v such that $v \models \mathcal{G}$

Prove that the set $\mathcal{G}$ below is **consistent** under classical semantics. Use **shorthand** notation.

$$\mathcal{G} = \{(a \Rightarrow (a \cup b)),\ (a \cup b),\ \neg b,\ (c \Rightarrow b)\}$$

Solution: We find a restricted model for $\mathcal{G}$ as follows

First observe that the formula $((a \Rightarrow a \cup b))$, is a tautology, hence any v is its model. So we have only to see whether

three other formulas have a common model. It means we check if it is possible to find v, such that

$v^*(\neg b) = T$, $v^*((a \cup b)) = T$, and $v^*((c \Rightarrow b)) = T$.

We have that $\neg b = T$ if and only if $b = F$.

We evaluate $(a \cup b) = (a \cup F) = T$ if and only if $a = T$.

Consequently, $(c \Rightarrow b) = (c \Rightarrow F) = T$ if and only if $c = F$.

Hence, any v, such that $a = T,\ b = T,$ and $c = F$ is a model for $\mathcal{G}$.

Observe that $a = T,\ b = T,$ and $c = F$ is the **only restricted model** for $\mathcal{G}$.

QUESTION 5 (20 pts)

We define a 3 valued extensional semantics **M** for the language $\mathcal{L}_{\{\neg,\ \mathbf{L},\ \cup,\ \Rightarrow\}}$ by **defining the connectives** $\neg$, **L**, $\cup$, $\Rightarrow$ on a set $\{F, \perp, T\}$ of logical values by the following FUNCTIONS.

L Connective

L	F	$\perp$	T
	F	*F*	T

Negation :

$\neg$	F	$\perp$	T
	T	*F*	F

Implication

$\Rightarrow$	F	$\perp$	T
F	T	T	T
$\perp$	*T*	$\perp$	T
T	F	*F*	T

Disjunction :

$\cup$	F	$\perp$	T
F	F	$\perp$	T
$\perp$	$\perp$	T	T
T	T	*T*	T

1. (5pts) Verify whether $\models_{\mathbf{M}} (\mathbf{L}A \cup \neg\mathbf{L}A)$. You can use **shorthand notation**.

Solution We verify

$\mathbf{L}T \cup \neg\mathbf{L}T = T \cup F = T$, $\quad \mathbf{L} \perp \cup\neg\mathbf{L} \perp = F \cup \neg F = F \cup T = T$, $\quad \mathbf{L}F \cup \neg\mathbf{L}F = F \cup \neg F = T$

2. (10pts) Verify whether the formulas A_1 and A_2 from Question 1 have a model/ counter model under the semantics **M**. You can use **shorthand notation**.

Solution

The formulas are: $A_1 = (\mathbf{L}a \cup (b \Rightarrow (\neg\mathbf{I}a \cup \mathbf{L}\neg a)))$, and $A_2 = (a \cup (b \Rightarrow (\neg a \cup c)))$.

Any v, such that $v(a) = T$ is a **M model** for A_1 and for A_2 directly from the definition of $\cup$ and **L**, as $\mathbf{L}T = T$. There may be other models- this is the most obvious one.

3. (5pts) Verify whether the following set **G** is **M**-consistent. You can use **shorthand notation**

$$\mathbf{G} = \{\ \mathbf{L}a,\ \ (a \cup \neg\mathbf{L}b),\ \ (a \Rightarrow b),\ \ b\ \}$$

Solution Any v, such that $v(a) = T$, $v(b) = T$ is a **M model** for **G** as

$$\mathbf{L}T = T, \quad (T \cup \neg\mathbf{L}T) = T, \quad (T \Rightarrow T) = T, \quad b = T.$$

QUESTION 6 (5pts)

Given a formula $A = ((a \cap \neg c) \Rightarrow (\neg a \cup b))$ of a language $\mathcal{L}_{\{\neg,\cap,\cup,\Rightarrow\}}$.

Find a formula B of a language $\mathcal{L}_{\{\neg,\Rightarrow\}}$, such that $A \equiv B$. **List** all proper logical equivalences used at at each step.

Solution We evaluate

$A = ((a \cap \neg c) \Rightarrow (\neg a \cup b)) \equiv ((a \cap \neg c) \Rightarrow (a \Rightarrow b)) \equiv (\neg(a \Rightarrow \neg\neg c) \Rightarrow (a \Rightarrow b)) = B$
Equivalences used: **1.** $(\neg A \cup B) \equiv (A \Rightarrow B)$, **2.** $(A \cap B) \equiv \neg(A \Rightarrow \neg B)$
or if the equivalence **3.** $\neg\neg A \equiv A$ is also used we get

$A = ((a \cap \neg c) \Rightarrow (\neg a \cup b)) \equiv ((a \cap \neg c) \Rightarrow (a \Rightarrow b)) \equiv (\neg(a \Rightarrow \neg\neg c) \Rightarrow (a \Rightarrow b)) \equiv (\neg(a \Rightarrow c) \Rightarrow (a \Rightarrow b)) = B$