CSE/MAT371 MIDTERM SOLUTIONS Fall 2022  
(75pts)

NAME: Math/CS

Please write carefully your solutions. NO PARTIAL CREDIT. Formulas must be fully correct for credit.

QUESTION 1 (10pts)

Write the following natural language statement:

One likes to play bridge, or from the fact that the weather is good we conclude the following: one does not like to play bridge or one likes not to play bridge

as a formula of 2 different languages

1. (5pts) Formula $A_1 \in \mathcal{F}_1$ of a language $\mathcal{L}(\neg, \lor, \Rightarrow)$, where $L_A$ represents statement "one likes A", "A is liked".

Solution  Propositional Variables are: (use a, b, ... and you must write which variables denote which sentences)

- $a$ denotes statement: play bridge,
- $b$ denotes a statement: the weather is good

Translation  $A_1 = (L_a \lor (b \Rightarrow (\neg L_a \lor L_{\neg a})))$

2. (5pts) Formula $A_2 \in \mathcal{F}_2$ of a language $\mathcal{L}(\neg, \lor, \Rightarrow)$.

Solution  Propositional Variables are: (use a, b, ..., and you must write which variables denote which sentences)

- $a$ denotes statement: One likes to play bridge,
- $b$ denotes a statement: the weather is good,
- $c$ denotes a statement: one likes not to play bridge

Translation  $A_2 = (a \lor (b \Rightarrow (\neg a \lor c)))$

QUESTION 2 (15 pts)

Here is a mathematical statement $S$:

For all rational numbers $x \in \mathbb{Q}$ the following holds: If $x = 0$, then there is a natural number $n \in \mathbb{N}$, such that $x + n = 0$

1. (5pts) Re-write $S$ as a symbolic mathematical statement $SM$ that only uses mathematical and logical symbols.

Solution  $S$ becomes a symbolic mathematical statement

$$SM : \forall x \in \mathbb{Q}(x = 0 \Rightarrow \exists n \in \mathbb{N} x + n = 0)$$

2. (5pts) Translate the symbolic statement $SM$ into a corresponding formula of the predicate language $\mathcal{L}$ with restricted quantifiers.

Use SYMBOLS: $Q(x)$ for $x \in \mathbb{Q}$, $N(y)$ for $y \in \mathbb{N}$, $c$ for the number 0. Use $E \in \mathbb{P}$ to denote the relation $=$ and $f \in \mathbb{F}$ to denote the function $+$

Solution  The statement $x = 0$ becomes an atomic formula $E(x, c)$. The statement $x + n = 0$ becomes an atomic formula $E(f(x,y), c)$. 

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The symbolic mathematical statement SM becomes a **restricted quantifiers** formula

\[ \forall_{Q|A}(E(x, c) \Rightarrow \exists_{N|f}E(f(x, y), c)) \]

3. (5pts) Translate your **restricted domain** quantifiers logical formula into a correct formula \( A \) of the language \( L \)

**Solution** We apply now the **transformation rules** and get a corresponding formula \( A \in \mathcal{F} : \)

\[ \forall_{x}(\exists_{N}(E(x, c) \Rightarrow \exists_{y}(N(y) \cap E(f(x, y), c)))) \]

**QUESTION 3 (15 pts)**

Let \( A \) be a formula \( (((a \cap \neg c) \Rightarrow \neg b) \cup a) \Rightarrow (c \cup b)) \).

1. (2pts) A language \( L_{CON} \) to which the formula \( A \) belongs is:

**Solution:** The language is \( L_{\{\neg,\cap,\cup,\Rightarrow\}} \).

2. (3pts) Determine the degree of \( A \) and write down all its sub-formulas of the degree 2.

**Solution:** The degree of \( A \) is 7. There is only one sub-formula of the degree 2: \( (a \cap \neg c) \).

3. (5pts) Determine whether \( A \in T \). Use "proof by contradiction" method and shorthand notation.

**Solution** This is a question about the knowledge of the "proof by contradiction" method.

You get 0pts if you do not use it.

Assume \( (((a \cap \neg c) \Rightarrow \neg b) \cup a) \Rightarrow (c \cup b)) = T \). This is possible if and only if \( ((((a \cap \neg c) \Rightarrow \neg b) \cup a) = T \) and \( (c \cup b) = T \). This gives as that \( c = F, b = F \). We evaluate \( (((a \cap \neg F) \Rightarrow \neg F) \cup a) = T \). This is possible for \( a = T \).

Any truth assignment such that \( a = T, b = F, c = F \) is a counter-model for \( A \), hence \( A \notin \mathcal{T} \).

4. (5pts) Determine whether \( A \in C \). Use shorthand notation.

**Solution:** Any truth assignment such that \( a = T, b = T, c = F \) is a model for \( A \), hence \( A \notin \mathcal{C} \). This is not the only model.

**QUESTION 4 (10pts)**

We define: A set \( G \subseteq \mathcal{F} \) is **consistent** if and only if there is a truth assignment \( v \) such that \( v \models G \)

**Prove** that the set \( G \) below is consistent under classical semantics. Use shorthand notation.

\[ G = \{(a \Rightarrow (a \cup b)), (a \cup b), \neg b, (c \Rightarrow b)\} \]

**Solution:** We find a restricted model for \( G \) as follows

First observe that the formula \( (a \Rightarrow a \cup b) \), is a tautology, hence any \( v \) is its model. So we have only to see whether three other formulas have a common model. It means we check if it is possible to find \( v \), such that

\[ v'(-b) = T, v'((a \cup b)) = T, \text{ and } v'((c \Rightarrow b)) = T. \]

We have that \( \neg b = T \) if and only if \( b = F \).

We evaluate \( a \cup b = (a \cup F) = T \) if and only if \( a = T \).

Consequently, \( (c \Rightarrow b) = (c \Rightarrow F) = T \) if and only if \( c = F \).

Hence, any \( v \), such that \( a = T, b = T, \text{ and } c = F \) is a model for \( G \).

**Observe** that \( a = T, b = T, \text{ and } c = F \) is the **only restricted model** for \( G \).
QUESTION 5  (20 pts)
We define a 3 valued extensional semantics \( \mathcal{M} \) for the language \( \mathcal{L}_{\neg, \land, \lor, \Rightarrow} \) by defining the connectives \( \neg, \land, \lor, \Rightarrow \) on a set \( \{ F, \bot, T \} \) of logical values by the following FUNCTIONS.

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<th>L Connective</th>
<th>Negation</th>
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1. (5pts) Verify whether \( \models_\mathcal{M} (L\land \neg L\land A) \). You can use shorthand notation.

Solution  We verify

\[
\begin{align*}
\mathcal{L}T \lor \neg \mathcal{L}T &= T \lor F = T, \\
\mathcal{L} \land \neg \mathcal{L} \land \neg \mathcal{L} &= F \lor \neg F = F \lor T = T, \\
\mathcal{L}F \lor \neg \mathcal{L}F &= F \lor \neg F = T
\end{align*}
\]

2. (10pts) Verify whether the formulas \( A_1 \) and \( A_2 \) from Question 1 have a model/ counter model under the semantics \( \mathcal{M} \). You can use shorthand notation.

Solution  The formulas are: \( A_1 = (L\land \land \neg \mathcal{L} \land (b \Rightarrow (\neg \mathcal{L} \land \neg \mathcal{L}))) \), and \( A_2 = (a \land (b \Rightarrow (\neg \mathcal{L} \land \neg \mathcal{L}))) \).

Any \( v \), such that \( v(a) = T \) is a \( \mathcal{M} \) model for \( A_1 \) and for \( A_2 \) directly from the definition of \( \land \) and \( \mathcal{L} \), as \( \mathcal{L}T = T \). There may be other models- this is the most obvious one.

3. (5pts) Verify whether the following set \( G \) is \( \mathcal{M} \)-consistent. You can use shorthand notation

\[
G = \{ L\land, (a \land \neg \mathcal{L}b), (a \Rightarrow b), b \}
\]

Solution  Any \( v \), such that \( v(a) = T \), \( v(b) = T \) is a \( \mathcal{M} \) model for \( G \) as

\[
\mathcal{L}T = T, \quad (T \lor \neg \mathcal{L}T) = T, \quad (T \Rightarrow T) = T, \quad b = T.
\]

QUESTION 6 (5pts )

Given a formula \( A = ((a \land \neg \mathcal{L}c) \Rightarrow (\neg a \land b)) \) of a language \( \mathcal{L}_{\neg, \land, \lor, \Rightarrow} \).

Find a formula \( B \) of a language \( \mathcal{L}_{\neg, \land, \lor, \Rightarrow} \), such that \( A \equiv B \). List all proper logical equivalences used at each step.

Solution  We evaluate

\[
A = ((a \land \neg \mathcal{L}c) \Rightarrow (\neg a \lor b)) \equiv ((a \land \neg \mathcal{L}c) \Rightarrow (a \Rightarrow b)) \equiv (\neg (a \Rightarrow a \lor c) \Rightarrow (a \Rightarrow b)) \equiv B
\]

Equivalences used: 1. \( (\neg A \land B) \equiv (A \Rightarrow B) \), 2. \( (A \land B) \equiv \neg (A \lor \neg B) \)

or if the equivalence 3. \( \neg \neg A \equiv A \) is also used we get

\[
A = ((a \land \neg \mathcal{L}c) \Rightarrow (\neg a \land b)) \equiv ((a \land \neg \mathcal{L}c) \Rightarrow (a \Rightarrow b)) \equiv (\neg (a \Rightarrow \neg (a \land \neg c) \Rightarrow (a \Rightarrow b)) \equiv (\neg (a \Rightarrow c) \Rightarrow (a \Rightarrow b)) \equiv B
\]