CSE371/math371 Practice Final Fall 2022
(15 pts - extra credit)

Final - December 13, 11:15 am -1:45pm in the classroom.

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PROBLEM 1

Given a set of formulas

$$\mathcal{G} = \{((a \Rightarrow (a \cup b)),\ (a \cup b),\ \neg b,\ (c \Rightarrow b)\}$$

1. Show that $\mathcal{G}$ is CONSISTENT under classical semantics. Use shorthand notation.

Solution

We find a restricted model for $\mathcal{G}$. The formula $((a \Rightarrow (a \cup b))$ is a tautology, hence any v is its model. $\neg b = T$ only if b=F. We evaluate $(a \cup b) = (a \cup F) = T$ only if a=T. Consequently, $(c \Rightarrow b) = (c \Rightarrow F) = T$ only if c=F. Hence, any v, such that a=T, b= T, and c= F is a model for $\mathcal{G}$.

2. Find a formula A that is iINDEPENDENT of $\mathcal{G}$. Must prove it. Use shorthand notation.

Solution

THIS IS MY SOLUTION. THERE ARE MANY OTHERS!

Let A be any atomic formula $d \in VAR - \{a, b, c\}$. Any v, such that a=T, b= T, and c= F, d= T is a model for $\mathcal{G} \cup \{A\}$. Any v, such that a=T, b= T, and c= F, d= F is a model for $\mathcal{G} \cup \{\neg A\}$.

3. Find an infinite number of formulas that are iINDEPENDENT of $\mathcal{G}$. Justify your answer.

Solution

There is countably infinitely many atomic formulas A=d where $d \in VAR - \{a, b, c\}$.

PROBLEM 2

Given a language $\mathcal{L} = \mathcal{L}_{\{\neg, \Rightarrow, \cup, \cap\}}$. We a define a $\mathbf{L}_4$ **semantics** as follows.

Logical values are $F,\ \bot_1,\ \bot_2,\ T$ and they are ordered: $F < \bot_1 < \bot_2 < T$.

The **connectives** are defined as follow

$\neg\bot_1 = \bot_1,\ \ \neg\bot_2 = \bot_2,\ \ \neg F = T,\ \ \neg T = F.$

For any $x, y \in \{F, \bot_1, \bot_2, T\}$, $x \cap y = min\{x,\ y\}$, $x \cup y = max\{x,\ y\}$, and

$$x \Rightarrow y = \begin{cases} \neg x \cup y & \text{if } x > y \\ T & \text{otherwise} \end{cases}$$

1. Write Truth Tables for **implication** and **negation**.

Solution

$\Rightarrow$	F	$\bot_1$	$\bot_2$	T
F	T	T	T	T
$\bot_1$	$\bot_1$	T	T	T
$\bot_2$	$\bot_2$	$\bot_2$	T	T
T	F	$\bot_1$	$\bot_2$	T

$\neg$	F	$\bot_1$	$\bot_2$	T
	T	$\bot_1$	$\bot_2$	F

2. Prove/disaprove: $\models_{\mathbf{L}_4}((a \Rightarrow b) \Rightarrow (\neg a \cup b))$. Use **shorthand** notation.

Solution

Let v be a truth assignment such that $v(a) = v(b) = \bot_1$.

We evaluate $v^*((a \Rightarrow b) \Rightarrow (\neg a \cup b)) = ((\bot_1 \Rightarrow \bot_1) \Rightarrow (\neg\bot_1 \cup \bot_1)) = (T \Rightarrow (\bot_1 \cup \bot_1)) = (T \Rightarrow \bot_1) = \bot_1$.

This proves that v is a **counter-model** for our formula and that $\not\models_{\mathbf{L}_4} ((a \Rightarrow b) \Rightarrow (\neg a \cup b))$.

Observe that there are other counter-models. For example, v such that $v(a) = v(b) = \bot_2$ is also a counter model, as $v^*((a \Rightarrow b) \Rightarrow (\neg a \cup b)) = ((\bot_2 \Rightarrow \bot_2) \Rightarrow (\neg\bot_2 \cup \bot_2)) = (T \Rightarrow (\bot_2 \cup \bot_2)) = (T \Rightarrow \bot_2) = \bot_2$.

3. **Prove** that the equivalence defining $\cup$ in terms of negation and implication in classical logic **does not hold** under $\mathbf{L}_4$, i.e. prove that $(A \cup B) \not\equiv_{\mathbf{L}_4} (\neg A \Rightarrow B)$.

Solution

Any v such that $v^*(A) = \bot_2$ and $v^*(B) = \bot_1$ is a **counter- model**. This is not the only counter-model.

PROBLEM 3

Consider the Hilbert system $H1 = (\ \mathcal{L}_{\{\Rightarrow\}},\ \mathcal{F},\ \{A1, A2\},\ (MP)\ \frac{A\ ;\ (A\Rightarrow B)}{B}\ )$ where for any $A, B \in \mathcal{F}$

$A1$; $(A \Rightarrow (B \Rightarrow A))$, $A2$: $((A \Rightarrow (B \Rightarrow C)) \Rightarrow ((A \Rightarrow B) \Rightarrow (A \Rightarrow C)))$.

1. We have proved that the **Deduction Theorem** holds for $H1$.

Use **Deduction Theorem** to prove $(A \Rightarrow (C \Rightarrow B)) \vdash_H (C \Rightarrow (A \Rightarrow B))$.

Solution

We apply the **Deduction Theorem** twice, i.e. we get

$(A \Rightarrow (C \Rightarrow B)) \vdash_H (C \Rightarrow (A \Rightarrow B))$ if and only if

$(A \Rightarrow (C \Rightarrow B)),\ C \vdash_H (A \Rightarrow B)$ if and only if

$(A \Rightarrow (C \Rightarrow B)),\ C,\ A \vdash_H B$

We now construct a proof of $(A \Rightarrow (C \Rightarrow B)), C, A \vdash_H B$ as follows

B_1 : $(A \Rightarrow (C \Rightarrow B))$ hypothesis

B_2 : C hypothesis

B_3 : A hypothesis

B_4 : $(C \Rightarrow B)$ B_1 , B_3 and (MP)

B_5 : C B_2 , B_4 and (MP)

2. Explain why **1.** proves that $(\neg a \Rightarrow ((b \Rightarrow \neg a) \Rightarrow b)) \vdash_H ((b \Rightarrow \neg a) \Rightarrow (\neg a \Rightarrow b))$.

Solution

This is **1.** for $A = \neg a$, $C = (b \Rightarrow \neg a)$, and $B = b$.

3. $H1$ **is sound** under classical semantics. Explain why H1 **is not complete**.

Solution

The system S **is not complete** under classical semantics means that not all classical tautologies have a proof in S. We have proved that one needs negation and one of other connectives $\cup, \cap, \Rightarrow$ to express all classical connectives, and hence all classical tautologies. Our language contains only implication and one can't express negation in terms of implication alone and hence we can't provide a proof of any tautology i.e. its logically equivalent form in our language $\mathcal{L}_{\{\Rightarrow\}}$.

4. Let $H2$ be the proof system obtained from the system $H1$ by **extending the language** to contain the negation $\neg$ and **adding** one additional axiom:

A3 $((\neg B \Rightarrow \neg A) \Rightarrow ((\neg B \Rightarrow A) \Rightarrow B)))$.

Explain shortly why **Deduction Theorem** holds for H2.

Solution

The proof of the Deduction Theorem for H1 used only axioms **A1, A2** so Adding axiom **A3** (and adding $\neg$ to the language) does not change anything in the proof. Hence **Deduction Theorem** holds for H2.

5. We know that $H2$ is **complete**.

Let $H3$ be the proof system obtained from the system $H2$ **adding** additional axiom

A4 $(\neg(A \Rightarrow B) \Rightarrow \neg(A \Rightarrow \neg B))$

Does **Deduction Theorem** holds for $H3$? Justify.

Solution

Does **Completeness Theorem** holds for $H3$? Justify.

Solution

No, **it does't**. The system H3 is **not sound**. Axiom **A4** is not a tautology.

Any v such that A=T and B=F is a **counter model** for $(\neg(A \Rightarrow B) \Rightarrow \neg(A \Rightarrow \neg B))$.

PROBLEM 4

Let **GL** be the Gentzen style proof system for classical logic defined in chapter 6.

Prove, by constructing a proper decomposition tree that

$$\vdash_{\mathbf{GL}}((\neg a \Rightarrow \neg\neg b) \Rightarrow (\neg b \Rightarrow a))$$

Solution

By definition we have that

$$\vdash_{\mathbf{GL}}((\neg a \Rightarrow \neg\neg b) \Rightarrow (\neg b \Rightarrow a)) \quad \text{if and only if} \quad \vdash_{\mathbf{GL}} \longrightarrow ((\neg a \Rightarrow \neg\neg b) \Rightarrow (\neg b \Rightarrow a)).$$

$$\mathbf{T}_{\to A}$$

$$\longrightarrow ((\neg a \Rightarrow \neg\neg b) \Rightarrow (\neg b \Rightarrow a))$$

$$\mid (\rightarrow\Rightarrow)$$

$$(\neg a \Rightarrow \neg\neg b) \longrightarrow (\neg b \Rightarrow a)$$

$$\mid (\rightarrow\Rightarrow)$$

$$\neg b, (\neg a \Rightarrow \neg\neg b) \longrightarrow a$$

$$\mid (\rightarrow \neg)$$

$$(\neg a \Rightarrow \neg\neg b) \longrightarrow b, a$$

$$\bigwedge(\Rightarrow\longrightarrow)$$

$\longrightarrow \neg a, b, a$	$\neg\neg b \longrightarrow b, a$
$\mid (\rightarrow \neg)$	$\mid (\neg \rightarrow)$
$a \longrightarrow b, a$	$\longrightarrow \neg b, b, a$
axiom	$\mid (\rightarrow \neg)$
	$b \longrightarrow b, a$
	axiom

All leaves of the tree are axioms, hence we have found the proof of A in **GL**.

PROBLEM 5

Prove, by **constructing** proper decomposition trees that

$$\nvdash_{\mathbf{GL}} ((b \Rightarrow a) \Rightarrow (\neg a \Rightarrow b))$$

Solution

Observe that for any formula A, its decomposition tree $\mathbf{T}_{\rightarrow A}$ in **GL** is not unique. Hence when constructing decomposition trees we have to cover all possible cases.

We construct the decomposition tree for $\longrightarrow A$ as follows.

$$\mathbf{T}_{\mathbf{1}\rightarrow A}$$

$$\longrightarrow ((b \Rightarrow a) \Rightarrow (\neg a \Rightarrow b))$$

$$\mid (\rightarrow \Rightarrow)$$

$$(one\ choice)$$

$$(b \Rightarrow a)\ \longrightarrow\ (\neg a \Rightarrow b)$$

$$\mid (\rightarrow \Rightarrow)$$

$$(first\ of\ two\ choices)$$

$$\neg a, (b \Rightarrow a)\ \longrightarrow\ b$$

$$\mid (\neg \rightarrow)$$

$$(one\ choice)$$

$$(b \Rightarrow a)\ \longrightarrow\ a, b$$

$$\bigwedge (\Rightarrow \longrightarrow)$$

$$(one\ choice)$$

$$\longrightarrow\ b, a, b \qquad\qquad a \longrightarrow a, b$$

$$non - axiom \qquad\qquad axiom$$

The tree contains a non- axiom leaf $\longrightarrow\ b, a, b$, hence it is not a proof in **GL**.

We have only one more tree to construct. Here it is.

$$\mathbf{T}_{\mathbf{2}\rightarrow A}$$

$$\longrightarrow ((b \Rightarrow a) \Rightarrow (\neg a \Rightarrow b))$$

$$\mid (\rightarrow \Rightarrow)$$

$$(one\ choice)$$

$$(b \Rightarrow a)\ \longrightarrow\ (\neg a \Rightarrow b)$$

$$\bigwedge(\Rightarrow\longrightarrow)$$
$$(second\ of\ two\ choices)$$

$\longrightarrow\ (\neg a \Rightarrow b), b$	$a \longrightarrow\ (\neg a \Rightarrow b)$
$(\longrightarrow\Rightarrow)$	$\mid(\rightarrow\Rightarrow)$
$(one\ choice)$	$(one\ choice)$
$\neg a \longrightarrow\ b, b$	$a, \neg a \longrightarrow\ b$
$\mid(\neg \rightarrow)$	$\mid(\neg \rightarrow)$
$(one\ choice)$	$(one\ choice)$
$\longrightarrow\ a, b, b$	$a\ \longrightarrow\ a, b$
$non - axiom$	$axiom$

All possible trees end with an non-axiom leave whet proves that

$$\nvdash_{\mathbf{GL}} ((b \Rightarrow a) \Rightarrow (\neg a \Rightarrow b)).$$

PROBLEM 6

Let **GL** be the Gentzen style proof system for classical logic defined in chapter 6.

Prove, by constructing a **counter-model** defined by a proper decomposition tree that

$$\not\models ((a \Rightarrow (\neg b \cap c)) \Rightarrow (\neg b \Rightarrow (a \cup \neg c)))$$

Explain why your counter -model construction is valid

Solution

$$\mathbf{T}_{\rightarrow A}$$
$$\longrightarrow ((a \Rightarrow (\neg b \cap c)) \Rightarrow (\neg b \Rightarrow (a \cup \neg c)))$$
$$\mid(\rightarrow\Rightarrow)$$
$$(a \Rightarrow (\neg b \cap c)) \longrightarrow (\neg b \Rightarrow (a \cup \neg c))$$
$$\mid(\rightarrow\Rightarrow))$$
$$one\ of\ two\ choices$$
$$\neg b, (a \Rightarrow (\neg b \cap c)) \longrightarrow (a \cup \neg c))$$
$$\mid(\rightarrow \cup)$$
$$one\ of\ two\ choices$$

$$\neg b, (a \Rightarrow (\neg b \cap c)) \longrightarrow a, \neg c$$

$$one\ of\ two\ choices$$

$$\mid (\neg \rightarrow)$$

$$(a \Rightarrow (\neg b \cap c)) \longrightarrow a, \neg c, b$$

$$one\ of\ two\ choices$$

$$\mid (\rightarrow \neg)$$

$$c, (a \Rightarrow (\neg b \cap c)) \longrightarrow a, b$$

$$\bigwedge (\Rightarrow\longrightarrow)$$

$$c \longrightarrow a, a, b \qquad\qquad (\neg b \cap c) \longrightarrow a, b$$

$$non-axiom \qquad\qquad \mid (\cap \longrightarrow)$$

$$\neg b, c \longrightarrow a, b$$

$$\mid (\neg \longrightarrow)$$

$$c \longrightarrow b, a, b$$

$$non-axiom$$

The counter-model model determined by the non-axiom leaf $c \longrightarrow a, a, b$ is any truth assignment that evaluates it to F.

Observe that (we use a shorthand notation) $c \longrightarrow a, a, b = F$ if and only if $c = T$ and $a = F$ and $b = F$.

The counter-model model determined by the non-axiom leaf $c \longrightarrow b, a, b$ is any also any truth assignment that

$c = T$ and $a = F$ and $b = F$.

The counter -model construction is valid because of the **strong soundness** of **GL**.

PROBLEM 7

Let **LI** be the Gentzen system for intuitionistic logic as defined in chapter 7.

Determine whether

$$\vdash_{\mathbf{LI}} \longrightarrow ((\neg a \cap \neg c) \Rightarrow \neg(a \cup c))$$

This means that you have to construct some, or all decomposition trees of

$$\longrightarrow ((\neg a \cap \neg c) \Rightarrow \neg(a \cup c))$$

If you find a decomposition tree such that all its leaves are axioms, you have a proof.

If all possible decomposition trees have a non-axiom leaf, the proof in **LI** does not exist.

Solution

Consider the following decomposition tree of

$$\longrightarrow ((\neg a \cap \neg c) \Rightarrow \neg(a \cup c))$$

T1

$$\longrightarrow ((\neg a \cap \neg c) \Rightarrow (\neg(a \cup c))$$

$$\mid (\longrightarrow\Rightarrow)$$

$$(\neg a \cap \neg c) \longrightarrow \neg(a \cup c)$$

$$\mid (\longrightarrow \neg)$$

$$(a \cup c), (\neg a \cap \neg c) \longrightarrow$$

$$\mid (exch \longrightarrow)$$

$$(\neg a \cap \neg c), (a \cup c) \longrightarrow$$

$$\mid (\cap \longrightarrow)$$

$$\neg a, \neg c, (a \cup c) \longrightarrow$$

$$\mid (\neg \longrightarrow)$$

$$\neg c, (a \cup c) \longrightarrow a$$

$$\mid (\longrightarrow weak)$$

$$\neg c, (a \cup c) \longrightarrow$$

$$\mid (\neg \longrightarrow)$$

$$(a \cup c) \longrightarrow c$$

$$\bigwedge (\cup \longrightarrow)$$

$$a \longrightarrow c \qquad\qquad c \longrightarrow c$$

$$non-axiom \qquad\qquad axiom$$

The tree **T1** has a non-axiom leaf, so it does not constitute a proof in **LI**. But this fact does not yet prove that proof doesn't exist, as the decomposition tree in **LI** is not always unique.

Let's consider now the following tree.

T2

$$\longrightarrow ((\neg a \cap \neg c) \Rightarrow (\neg(a \cup c))$$

$$\mid (\longrightarrow\Rightarrow)$$

$$(\neg a \cap \neg c) \longrightarrow \neg(a \cup c)$$

$$\mid (\longrightarrow \neg)$$

$$(a \cup c), (\neg a \cap \neg c) \longrightarrow$$

$$\mid (exch \longrightarrow)$$

$$(\neg a \cap \neg c), (a \cup c) \longrightarrow$$

$$\mid (\cap \longrightarrow)$$

$$\neg a, \neg c, (a \cup c) \longrightarrow$$

$$\mid (exch \longrightarrow)$$

$$\neg a, (a \cup c), \neg c \longrightarrow$$

$$\mid (exch \longrightarrow)$$

$$(a \cup c), \neg a, \neg c \longrightarrow$$

$$\bigwedge (\cup \longrightarrow)$$

$a, \neg a, \neg c \longrightarrow$	$c, \neg a, \neg c \longrightarrow$
$\mid (exch \longrightarrow)$	$\mid (exch \longrightarrow)$
$\neg a, a, \neg c \longrightarrow$	$c, \neg c, \neg a \longrightarrow$
$\mid (\neg \longrightarrow)$	$\mid (exch \longrightarrow)$
$a, \neg c \longrightarrow a$	$\neg c, c, \neg a \longrightarrow$
axiom	$\mid (\neg \longrightarrow)$
	$c, \neg a \longrightarrow c$
	axiom

All leaves of **T2** are axioms, what proves that **T2** is a proof of A and hence we proved that

$$\vdash_{\mathbf{LI}} \longrightarrow ((\neg a \cap \neg c) \Rightarrow \neg(a \cup c))$$

Observe that your FIRST tree is **T2**, you have found the PROOF, so there is **no need** to examine any other trees