PART 1: DEFINITIONS  TOTAL 10pts

DEFINITION 1  (2pts)

Given a propositional language $L_{\text{CON}}$ for $\text{CON} = C_1 \cup C_2$, where $C_1$ is the set of all unary connectives, and $C_2$ is the set of all binary connectives

1. (1pts) Write the definition of the set $F$ of all formulas of $L_{\text{CON}}$ for $C_1 = \{\neg, K\}$ and $C_2 = \{\cup\}$

**Solution**

$F \subseteq \mathcal{A}^*$ and $F$ is the smallest set for which the following conditions are satisfied

(1) $\text{VAR} \subseteq F$ - ATOMIC FORMULAS
(2) If $A \in F$, then $\neg A \in F$ and $KA \in F$
(3) If $A, B \in F$, then $(A \cup B) \in F$

2. (1pts) Write an example of 4 formulas, each of a different degree, of the language $L_{\{\neg, K, \cup\}}$

**Solution**

Here are, for example, formulas of the degree 0, 1, 2, 3, respectively

$a, \neg a, K\neg a, K(a \cup \neg b)$

DEFINITION 2.  (2pts)

Given the language $L_{\{\neg, K, \cup\}}$ and a $\text{M}$ truth assignment $v : \text{VAR} \rightarrow LV$, where $LV \neq \emptyset$ is the set of logical values of the extensional semantics $\text{M}$. Let $T \in LV$ be its distinguished logical value.

1. (1pts) We say that a function $v^*$ is the $\text{M}$ extension of $v$ to the set $F$ of the language $L_{\{\neg, K, \cup\}}$ if and only if the following conditions hold.

**Solution**

(i) for any $a \in \text{VAR}$, $v^*(a) = v(a)$; and

(ii) for any formulas $A, B \in F$,

\[ v^*(\neg A) = \neg v^*(A), \quad v^*(KA) = Kv^*(A), \quad \text{and} \quad v^*((A \cup B)) = \cup(v^*(A), v^*(B)) \]

We also use standard notation $v^*((A \cup B)) = v^*(A) \cup v^*(B)$

2. (1pts) We say that $\models_{\text{M}} A$ if and only if

**Solution**

$v^*(A) = T$ for all truth assignments $v : \text{VAR} \rightarrow LV$
DEFINITION 3 (4pts)

Given a language \( L \), and its extensional semantics \( M \).

1. (2pts) A formula \( A \in \mathcal{F} \) is called \( M \)-independent from a set \( \mathcal{G} \subseteq \mathcal{F} \) if and only if the sets \( \mathcal{G} \cup \{ A \} \) and \( \mathcal{G} \cup \{ \neg A \} \) are both \( M \)-consistent.

   I.e. when there are truth assignments \( v_1, v_2 \) such that \( v_1 \models_M \mathcal{G} \cup \{ A \} \) and \( v_2 \models_M \mathcal{G} \cup \{ \neg A \} \).

2. (2pts) Give an example of a set \( \mathcal{G} \subseteq \mathcal{F} \) and a formula \( A \in \mathcal{F} \) that is classically independent from \( \mathcal{G} \).

Solution

Here is a very simple example: \( \mathcal{G} = \{ a \} \) and \( A = b \).

Let \( v_1, v_2 \) be any truth assignments such that \( v_1(a) = T \), \( v_1(b) = T \) and \( v_2(a) = T \), \( v_2(b) = F \).

Obviously, \( \mathcal{G} \cup \{ A \} = \{ a, b \} \) and \( \mathcal{G} \cup \{ \neg A \} = \{ a, \neg b \} \).

DEFINITION 4 (2pts)

Given a proof system \( S = (L, \{ \neg, \cup \}, \mathcal{F}, \mathcal{LA}, \mathcal{R}) \). We write \( P_S = \{ A \in \mathcal{F} : \vdash_S A \} \) and \( T_M = \{ A \in \mathcal{F} : \models_M A \} \). The proof system \( S \) is complete under a semantics \( M \) if and only if the following condition holds.

Solution

\( P_S = T = \{ A \in \mathcal{F} : \models A \} \).

PART 2: PROBLEMS (75 pts)

QUESTION 1 (10pts)

1. (4pts) Give an example of three non-classical logics.

Solution

Intuitionistic Logic, Modal Logic S4, S5, and any of CS logics listed below.

2. (5pts) Give an example of two logics developed by computer scientists. Write one sentence description.

Solution

Dynamic logic (Harel 1979) which was created to facilitate the statement and proof of properties of programs.

Temporal Logics which were created for the specification and verification of concurrent programs Harel, Parikh, 1979, 1983 and for a specification of hardware circuits Halpern, Manna and Maszkowski, (1983).

Fuzzy logic, Many valued logics that were created and developed to describe reasoning with incomplete information.

Non-monotonic logics were created by Mc Carthy (1985) and has been shown to be important in other areas. There are applications to logic programming, to planning and reasoning about action, and to automated diagnosis.
QUESTION 2 (10 pts)

Let $A$ be a formula

$(((a \land \neg c) \Rightarrow \neg b) \cup a) \Rightarrow (c \cup b))$.

1. (2pts) A language $L_{\text{CON}}$ to which the formula $A$ belongs is:

Solution: The language is $L_{\land, \neg, \Rightarrow}$.

2. (2pts) Determine the degree of $A$ and write down all its sub-formulas of the degree 2.

Solution: The degree of $A$ is 7. There is only one sub-formula of the degree 2: $(a \land \neg c)$.

3. (4pts) Determine whether $A \in T$. Use ”proof by contradiction” method and shorthand notation.

Solution: of the case $A \in T$.

Assume $(((a \land \neg c) \Rightarrow \neg b) \cup a) \Rightarrow (c \cup b)) = F$.

This is possible if and only if $((a \land \neg c) \Rightarrow \neg b) \cup a = T$ and $(c \cup b) = F$.

This gives as that $c = F, b = F$. We evaluate $((a \land \neg F) \Rightarrow \neg F) \cup a = T$. This is possible for $a = T$.

Any truth assignment such that $a = T, b = F, c = F$ is a counter-model for $A$, hence $A \notin T$.

4. (2pts) Determine whether $A \in C$. Use shorthand notation.

Solution: any truth assignment such that $a = T, b = T, c = F$ is a model for $A$, hence $A \notin C$.

This is not the only model.

PROBLEM 3 (15pts)

Given a language $L = L_{\land, \neg, \Rightarrow, \cup, \land}$. We a define an $M_4$ semantics as follows.

Logical values are $F, \bot_1, \bot_2, T$ and they are ordered: $F < \bot_1 < \bot_2 < T$

The connectives are defined as follows :

$\neg \bot_1 = \bot_1, \neg \bot_2 = \bot_2, \neg F = T, \neg T = F$.

For any $x, y \in \{F, \bot_1, \bot_2, T\}$, $x \land y = \text{min}\{x, y\}, x \lor y = \text{max}\{x, y\}$, and

$x \Rightarrow y = \begin{cases} 
\neg x \lor y & \text{if } x > y \\
T & \text{otherwise}
\end{cases}$

1. (5pts) Write Truth Tables for implication and negation.

Solution

<table>
<thead>
<tr>
<th>$\Rightarrow$</th>
<th>F</th>
<th>$\bot_1$</th>
<th>$\bot_2$</th>
<th>T</th>
</tr>
</thead>
<tbody>
<tr>
<td>F</td>
<td>T</td>
<td>T</td>
<td>T</td>
<td>T</td>
</tr>
<tr>
<td>$\bot_1$</td>
<td>T</td>
<td>T</td>
<td>T</td>
<td>T</td>
</tr>
<tr>
<td>$\bot_2$</td>
<td>$\bot_2$</td>
<td>$\bot_2$</td>
<td>T</td>
<td>T</td>
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<tr>
<td>T</td>
<td>F</td>
<td>$\bot_1$</td>
<td>$\bot_2$</td>
<td>T</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>$\neg$</th>
<th>F</th>
<th>$\bot_1$</th>
<th>$\bot_2$</th>
<th>T</th>
</tr>
</thead>
<tbody>
<tr>
<td>T</td>
<td>$\bot_1$</td>
<td>$\bot_2$</td>
<td>F</td>
<td></td>
</tr>
</tbody>
</table>
2. (5pts) Prove/disprove: $\not M_4((a \Rightarrow b) \Rightarrow (\neg a \cup b))$. Use shorthand notation.

**Solution**

Let $v$ be a truth assignment such that $v(a) = v(b) = \perp_1$.

We evaluate :

$v'((a \Rightarrow b) \Rightarrow (\neg a \cup b)) = ((\perp_1 \Rightarrow \perp_1) \Rightarrow (\neg \perp_1 \cup \perp_1)) = (T \Rightarrow (\perp_1 \cup \perp_1)) = (T \Rightarrow \perp_1) = \perp_1$.

This proves that $v$ is a counter-model for our formula and that

$\not M_4((a \Rightarrow b) \Rightarrow (\neg a \cup b))$

Observe that there are other counter-models.

For example, $v$ such that $v(a) = v(b) = \perp_2$ is also a counter model, as

$v'((a \Rightarrow b) \Rightarrow (\neg a \cup b)) = ((\perp_2 \Rightarrow \perp_2) \Rightarrow (\neg \perp_2 \cup \perp_2)) = (T \Rightarrow (\perp_2 \cup \perp_2)) = (T \Rightarrow \perp_2) = \perp_2$.

3. (5pts) Prove that the equivalence defining $\cup$ in terms of negation and implication in classical logic does not hold under $M_4$, i.e. prove that

$(A \cup B) \not M_4 (\neg A \Rightarrow B)$

**Solution**

Any $v$ such that $v'(A) = \perp_2$ and $v'(B) = \perp_1$ is a counter-model

This is not the only counter-model.

**QUESTION 4 (20pts)**

1. (10pts) Given a formula $A = ((a \cap \neg c) \Rightarrow (\neg a \cup b))$ of a language $L_{\{\neg, \cap, \cup, \Rightarrow\}}$.

   **Find** a formula $B$ of a language $L_{\{\neg, \Rightarrow\}}$, such that $A \equiv B$.

   **List** all proper logical equivalences defining respective connectives needed to be used at each step.

   **Solution** . $A = ((a \cap \neg c) \Rightarrow (\neg a \cup b))$ and $B = (\neg (a \Rightarrow c) \Rightarrow (a \Rightarrow b))$

   $((a \cap \neg c) \Rightarrow (\neg a \cup b)) \equiv ((a \cap \neg c) \Rightarrow (a \Rightarrow b)) \equiv (\neg (a \Rightarrow \neg c) \Rightarrow (a \Rightarrow b)) \equiv (\neg (a \Rightarrow c) \Rightarrow (a \Rightarrow b))$

   Equivalences used: 1. $(\neg A \cup B) \equiv (A \Rightarrow B)$, 2. $(A \cap B) \equiv (A \Rightarrow \neg B)$, 3. $\neg \neg A \equiv A$

2. (10pts) Write the Definition of Equivalence of Languages and use it to prove that $L_{\{\neg, \cap, \cup, \Rightarrow\}} \equiv L_{\{\neg, \Rightarrow\}}$.

**Definition**

We define the equivalence of languages as follows:

Given two languages: $L_1 = L_{\text{CON}_1}$ and $L_2 = L_{\text{CON}_2}$, for $\text{CON}_1 \neq \text{CON}_2$.

We say that they are logically equivalent, i.e. $L_1 \equiv L_2$ if and only if the following conditions C1, C2 hold.

C1: For every formula $A$ of $L_1$, there is a formula $B$ of $L_2$, such that $A \equiv B$.

C2: For every formula $C$ of $L_2$, there is a formula $D$ of $L_1$, such that $C \equiv D$. 

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Solution

We have to prove that \( L_{\langle \neg, \Rightarrow \rangle} \equiv L_{\langle \neg, \cap, \cup, \Rightarrow \rangle} \).

Condition \( C1 \) holds because \( \{ \neg, \Rightarrow \} \subseteq \{ \neg, \cap, \cup, \Rightarrow \} \).

Condition \( C2 \) holds because of the **Substitution Theorem** and because of the following logical equivalences

For any formulas \( A, B \)

\[
(A \cap B) \equiv \neg (A \Rightarrow \neg B) \quad \text{and} \quad (A \cup B) \equiv (\neg A \Rightarrow B)
\]

**QUESTION 5 (20pts)**

\( S \) is the following proof system:

\[
S = (L_{\langle \Rightarrow, \cup, \neg \rangle}, \mathcal{F}, LA, \{(r1), (r2)\})
\]

**Logical Axioms**

\( LA: (a \Rightarrow (a \cup b)) \), where \( a, b \in \text{VAR} \)

**Rules** of inference:

\[
(r1) \quad \frac{A; B}{(A \cup \neg B)}, \quad (r2) \quad \frac{A; (A \cup B)}{B}
\]

where \( A, B \in \mathcal{F} \)

1. **(15pts)** Write a formal proof of \( \neg (c \Rightarrow (c \cup a)) \) in \( S \), i.e. show that \( \vdash_S \neg (c \Rightarrow (c \cup a)) \)

**Solution**

The formal proof \( B_1, B_2, B_3, B_4 \) of \( \neg (c \Rightarrow (c \cup a)) \) in \( S \) is as follows

\( B_1: \ (c \Rightarrow (c \cup a)) \)

Axiom LA for \( a = c, b = a \)

\( B_2: \ (c \Rightarrow (c \cup a)) \)

Axiom LA for \( a = c, b = a \)

\( B_3: \ ((c \Rightarrow (c \cup a)) \cup \neg (c \Rightarrow (c \cup a))) \)

Rule \((r1)\) application to \( B_1 \) and \( B_2 \)

\( B_4: \ \neg (c \Rightarrow (c \cup a)) \)

Rule \((r2)\) application to \( B_1 \) and \( B_3 \)

2. **(5pts)** Does above point 1. prove that \( \models \neg (c \Rightarrow (c \cup a)) \)? Justify your answer

**Solution**

The system \( S \) is not sound. Consider rule \((r2)\).

Take any \( v \), such that it evaluates \( A = T \) and \( B = F \).

The premiss \( A \cup B \) of \((r2)\) is \( T \) and the conclusion \( B \) is \( F \).

Moreover, the proof \( B_1, B_2, B_3, B_4 \) of \( ((c \Rightarrow (c \cup a)) \cup \neg (c \Rightarrow (c \cup a))) \) used the rule \((r2)\) that is not sound.